Notes_2 – OUTLINE (chapters 7-9)

- Laplace transforms
- Transfer functions
- Inversion of Laplace transforms
- Dynamic response
- Intrinsic dynamics
- Stability of a dynamic system

Laplace Transforms

$f(t) \rightarrow f(s) = L\left[f(\mathrm{t})\right] = \int_0^{+\infty} f(\mathrm{t})\mathrm{e}^{-\mathrm{st}}\mathrm{dt}$ Definition of the Laplace transform

Exponential function

$e^{-at} \rightarrow \frac{1}{s+a}$ $e^{at} \rightarrow \frac{1}{s-a}$

Ramp function

$at \rightarrow \frac{a}{s^2}$

Sinusoidal function

$\sin(\omega t) \rightarrow \frac{\omega}{s^2+\omega^2}$

Step function

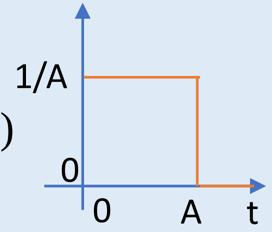
$A \rightarrow \frac{A}{s}$

Translated function

$f(t-t_0) \rightarrow e^{-st_0} f(\mathrm{s})$

Unit pulse function

$\delta_A(\mathrm{t})$



$\rightarrow \frac{1}{A}\frac{1-e^{-sA}}{s}$

Unit impulse function

$\delta(\mathrm{t}) \rightarrow 1$

Product of power law and exponential functions

$t^2 e^{-at} \rightarrow \frac{2}{(s+a)^3}$ $\frac{t^n e^{-at}}{n!} \rightarrow \frac{1}{(s+a)^{n+1}}$

Laplace transforms of derivatives

$\frac{d}{dt} f \rightarrow sf(s) - f(0)$ $\frac{d^2}{dt^2} f \rightarrow s^2 f(s) - sf(0) - f'(0)$ $\frac{d^n}{dt^n} f \rightarrow s^n f(s) - s^{n-1} f(0) - s^{n-2} f'(0) + \ldots - sf^{n-2}(0) - f^{n-1}(0)$

Laplace transforms of integrals

$$\int f(t)dt \rightarrow \frac{f(s)}{s}$$

Final-value theorem

$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sf(s)$

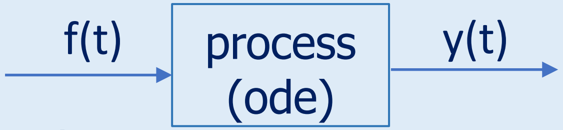
Initial-value theorem

$\lim_{t\to 0} f(t) = \lim_{s\to\infty} sf(s)$

Transfer functions – 1

The application of the Laplace transform to linear odes results into a system of algebraic equations which provide a relationship between the inputs and the outputs of a given process.

For a SISO system



described by the following edo equation written in terms of deviation variables (′ is eliminated)

$$a_n \frac{d^n}{dt^n} y + a_{n-1} \frac{d^{n-1}}{dt^{n-1}} y + \ldots + a_1 \frac{d}{dt} y + a_0 y = bf(t) \quad \text{with} \quad y(0) = \frac{d}{dt} y \bigg|_0 = \frac{d^2}{dt^2} y \bigg|_0 = \ldots = \frac{d^{n-1}}{dt^{n-1}} y \bigg|_0 = 0$$

The application of the Laplace transforms gives:

$$a_n s^n y(s) + a_{n-1} s^{n-1} y(s) + \ldots + a_1 s y(s) + a_0 y(s) = bf(s)$$

$$y(s)\left(a_n s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0\right) = bf(s)$$

$$\boxed{G(s) = \frac{output}{input} = \frac{y(s)}{f(s)} = \frac{b}{a_n s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0}}$$

Transfer function

$$y(s) = G(s) f(s)$$

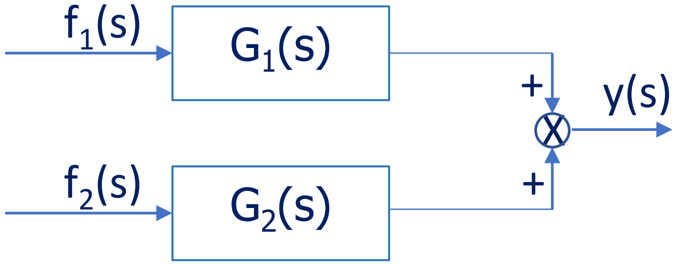
The transfer function G(s) relates in a simple algebraic form the output of a process to its input

Transfer functions – 2

For a process with two inputs and a single output

$$a_n\frac{d^n}{dt^n}y+a_{n-1}\frac{d^{n-1}}{dt^{n-1}}y+\ldots+a_1\frac{d}{dt}y+a_0y=b_1f_1(t)+b_2f_2(t)\quad +i.c.$$

$$y(s)=\frac{b_1}{a_ns^n+a_{n-1}s^{n-1}+\ldots+a_1s+a_0}f_1(s)+\frac{b_2}{a_ns^n+a_{n-1}s^{n-1}+\ldots+a_1s+a_0}f_2(s)$$

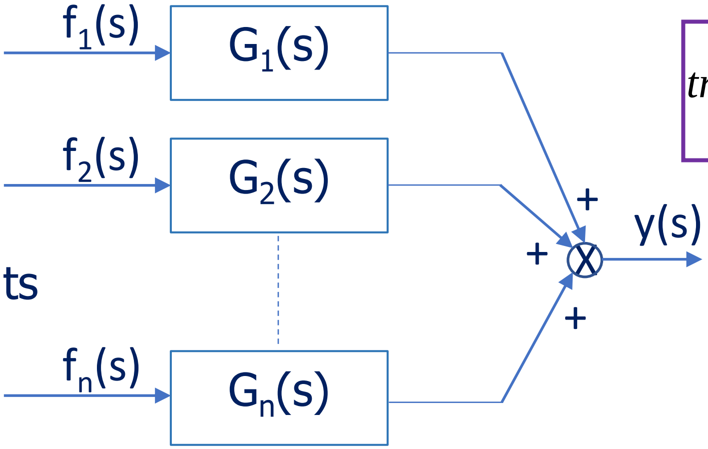


$$y(s)=G_1(s)\,f_1(s)+G_2(s)\,f_2(s)$$

$$\boxed{\begin{aligned}G_1(s)&=\frac{b_1}{a_ns^n+a_{n-1}s^{n-1}+\ldots+a_1s+a_0}\\ G_2(s)&=\frac{b_2}{a_ns^n+a_{n-1}s^{n-1}+\ldots+a_1s+a_0}\end{aligned}}$$

The transfer function $G_1(s)$ relates y(s) to $f_1(s)$ and $G_2(s)$ relates y(s) to $f_2(s)$. A similar procedure can be applied to any system with one output and several inputs

For n inputs



$$\boxed{transfer\ \ function\ G(s)=\frac{Laplace\ transform\ of\ \ the\ output\ (deviation\ \ form)}{Laplace\ transform\ of\ \ the\ input\ (deviation\ \ form)}}$$

Transfer functions – 3

Poles and zeros of a transfer function

$$G(s) = \frac{y(s)}{f(s)}$$

The transfer function G(s) is the ratio of two polynomials (with the exception of systems with dead time)

$$G(s) = \frac{Q(s)}{P(s)}$$

The roots of the polynomial Q(s) are called **zeros** of the transfer function (or zeros of the system).

The roots of the polynomial P(s) are called **poles** of the transfer function (or poles of the system).

The poles and zeros of a system determine the process dynamics and play a paramount role in the design of effective controllers.

For practical cases, the order m of the polynomial Q is ≤ of the order n of the polynomial P.

In summary the transfer function G(s)

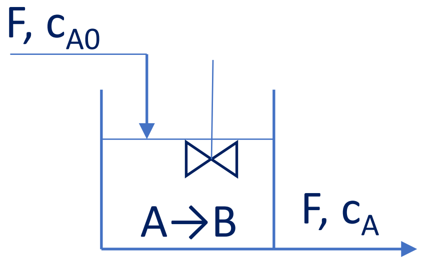
a) allows for the development of input-output models;

b) describes the dynamic behaviour y(s) when f(s) is known (to get y(t), the inversion of the Laplace transform should be made).

To get the transfer function on non linear systems, linearization is required (always deviation variables)

Transfer functions – Example #1

Isothermal CSTR with volume and volumetric flow rate at constant values



SISO

$$V\frac{d}{dt}c_A = Fc_{A0} - Fc_A - kVc_A \qquad + c_A(0) = 0$$

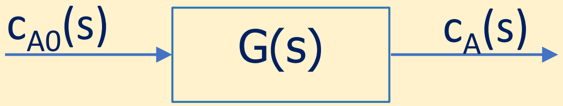
already in deviation variables

Laplace transform

$$sVc_A(s)+Fc_A(s)+kVc_A(s)=Fc_{A0}(s)$$

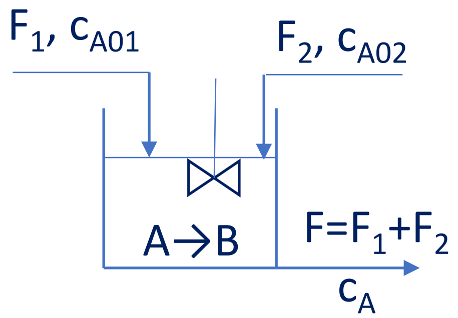
$$c_A(s)=\boxed{\frac{F}{Vs+F+kV}}c_{A0}(s)$$

G(s)



Transfer functions – Example #2

Isothermal CSTR (constant V and F)



MISO

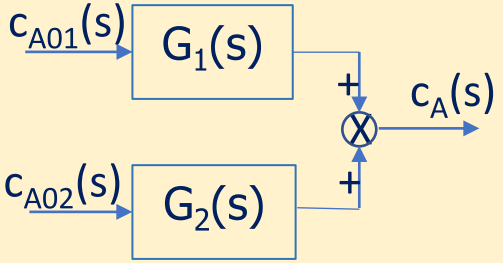
$$V\frac{d}{dt}c_A = F_1 c_{A01} + F_2 c_{A02} - (F_1+F_2)c_A - kVc_A \quad + c_A(0)=0$$ in deviation variables

Laplace transform

$$(Vs+F_1+F_2+kV)c_A(s) = F_1 c_{A01}(s) + F_2 c_{A02}(s)$$

$$c_A(s) = \boxed{\frac{F_1}{Vs+F_1+F_2+kV}}\,c_{A01}(s) + \boxed{\frac{F_2}{Vs+F_1+F_2+kV}}\,c_{A02}(s)$$

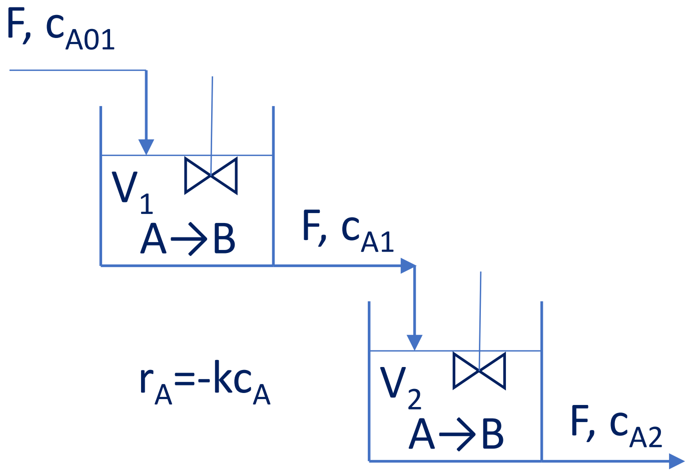
$G_1(s)$ $G_2(s)$



$$c_A(s) = G_1(s)c_{A01}(s) + G_2(s)c_{A02}(s) \qquad pole \rightarrow s = -\frac{F_1+F_2+kV}{V}$$

Transfer functions – Example #3

Two isothermal CSTRs (constant F, V_1 and V_2)



Transfer functions for a SIMO system

Linear odes in deviation variables

$$\begin{cases} V_1 \dfrac{d}{dt} c_{A1} = F\left(c_{A01} - c_{A1}\right) - kV_1 c_{A1} & c_{A1}(0) = 0 \\ V_2 \dfrac{d}{dt} c_{A2} = F\left(c_{A1} - c_{A2}\right) - kV_2 c_{A2} & c_{A2}(0) = 0 \end{cases}$$

in canonical form

$$\begin{cases} \tau_{p1} \dfrac{d}{dt} c_{A1} + c_{A1} = k_{p1} c_{A01} \\ \tau_{p2} \dfrac{d}{dt} c_{A2} + c_{A2} = k_{p2} c_{A1} \end{cases}$$

$$\tau_{p1} = \frac{V_1}{F + kV_1}; k_{p1} = \frac{F}{F + kV_1}$$

$$\tau_{p2} = \frac{V_2}{F + kV_2}; k_{p2} = \frac{F}{F + kV_2}$$

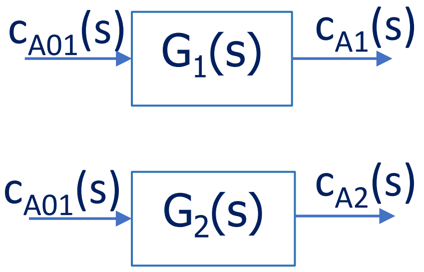
$$\begin{cases} \left(\tau_{p1} s + 1\right) c_{A1}(s) = k_{p1} c_{A01}(s) \\ \left(\tau_{p2} s + 1\right) c_{A2}(s) = k_{p2} c_{A1}(s) \end{cases}$$

$$c_{A1} = \frac{k_{p1}}{\tau_{p1} s + 1} c_{A01}$$

$$\boxed{G_1(s) = \frac{c_{A1}(s)}{c_{A10}(s)} = \frac{k_{p1}}{\tau_{p1} s + 1}}$$

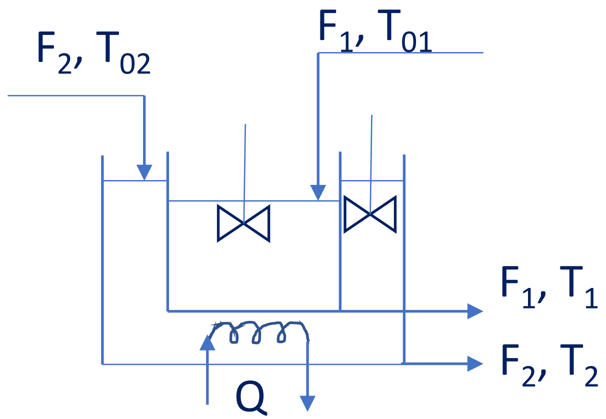
$$c_{A2}(s) = \frac{k_{p2}}{\tau_{p2} s + 1} c_{A1}(s) = \frac{k_{p2}}{\tau_{p2} s + 1} \frac{k_{p1}}{\tau_{p1} s + 1} c_{A01}(s)$$

$$\boxed{G_2(s) = \frac{c_{A2}(s)}{c_{A01}(s)} = \frac{k_{p1} k_{p2}}{\left(\tau_{p1} s + 1\right)\left(\tau_{p2} s + 1\right)}}$$



Transfer functions – Example #4 – 1

Stirred tank heater (negligible thermal inertia of the wall, adiabatic external wall, constant F and V and constant properties)



Transfer functions for a MIMO system

Linear odes in deviation variables

$$\begin{cases} V_1\rho_1 c_{p1}\dfrac{d}{dt}T_1 = F_1\rho_1 c_{p1}\left(T_{01}-T_1\right)+US\left(T_2-T_1\right) & T_1(0)=0 \\ V_2\rho_2 c_{p2}\dfrac{d}{dt}T_2 = F_2\rho_2 c_{p2}\left(T_{02}-T_2\right)+US\left(T_1-T_2\right)+Q & T_2(0)=0 \end{cases}$$

$$\begin{cases} V_1\rho_1 c_{p1}\dfrac{d}{dt}T_1 + \left(F_1\rho_1 c_{p1}+US\right)T_1 - UST_2 = F_1\rho_1 c_{p1}T_{01} \\ V_2\rho_2 c_{p2}\dfrac{d}{dt}T_2 + \left(F_2\rho_2 c_{p2}+US\right)T_2 - UST_1 = F_2\rho_2 c_{p2}T_{02}+Q \end{cases}$$

$$\tau_{p1}=\frac{V_1\rho_1 c_{p1}}{F_1\rho_1 c_{p1}+US},\ \alpha_1=\frac{US}{F_1\rho_1 c_{p1}+US},\ k_{p1}=\frac{F_1\rho_1 c_{p1}}{F_1\rho_1 c_{p1}+US}$$

$$\tau_{p2}=\frac{V_2\rho_2 c_{p2}}{F_2\rho_2 c_{p2}+US},\ \alpha_2=\frac{US}{F_2\rho_2 c_{p2}+US},\ k_{p2}=\frac{F_2\rho_2 c_{p2}}{F_2\rho_2 c_{p2}+US},\ k_{p3}=\frac{1}{F_2\rho_2 c_{p2}+US}$$

$$\begin{cases} \tau_{p1}\dfrac{d}{dt}T_1 + T_1 - \alpha_1 T_2 = k_{p1}T_{01} \\ \tau_{p2}\dfrac{d}{dt}T_2 + T_2 - \alpha_2 T_1 = k_{p2}T_{02} + k_{p3}Q \end{cases} \xrightarrow{\text{Laplace domain}} \begin{cases} \left(\tau_{p1}s+1\right)T_1(s) - \alpha_1 T_2(s) = k_{p1}T_{01}(s) \\ -\alpha_2 T_1(s) + \left(\tau_{p2}s+1\right)T_2(s) = k_{p2}T_{02}(s) + k_{p3}Q(s) \end{cases}$$

Transfer functions – Example #4 – 2

$$T_1(s)=\frac{\begin{vmatrix} k_{p1}T_{01} & -\alpha_1 \\ k_{p2}T_{02}+k_{p3}Q & (\tau_{p2}s+1) \end{vmatrix}}{\begin{vmatrix} (\tau_{p1}s+1) & -\alpha_1 \\ -\alpha_2 & (\tau_{p2}s+1) \end{vmatrix}},\quad T_2(s)=\frac{\begin{vmatrix} (\tau_{p1}s+1) & k_{p1}T_{01} \\ -\alpha_2 & k_{p2}T_{02}+k_{p3}Q \end{vmatrix}}{\begin{vmatrix} (\tau_{p1}s+1) & -\alpha_1 \\ -\alpha_2 & (\tau_{p2}s+1) \end{vmatrix}}$$

$$P(s)=(\tau_{p1}s+1)(\tau_{p2}s+1)-\alpha_1\alpha_2$$

$$T_1(s)=\frac{k_{p1}(\tau_{p2}s+1)T_{01}(s)}{P(s)}+\frac{\alpha_1k_{p2}T_{02}(s)}{P(s)}+\frac{\alpha_1k_{p3}Q(s)}{P(s)}$$

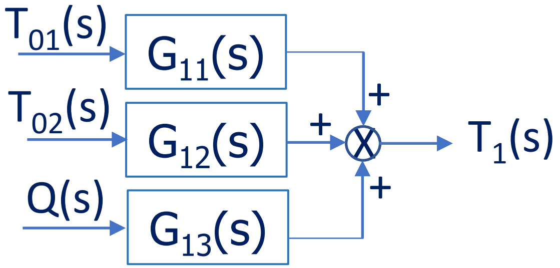
$$T_2(s)=\frac{\alpha_2k_{p1}T_{01}(s)}{P(s)}+\frac{k_{p2}(\tau_{p1}s+1)T_{02}(s)}{P(s)}+\frac{k_{p3}(\tau_{p1}s+1)Q(s)}{P(s)}$$

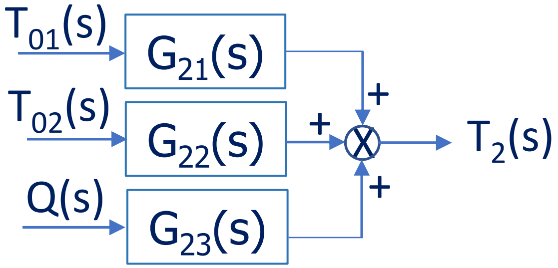
$$G_{11}(s)=\frac{T_1(s)}{T_{01}(s)}=\frac{k_{p1}(\tau_{p2}s+1)}{P(s)};\ G_{12}(s)=\frac{T_1(s)}{T_{02}(s)}=\frac{\alpha_1k_{p2}}{P(s)};\ G_{13}(s)=\frac{T_1(s)}{Q(s)}=\frac{\alpha_1k_{p3}}{P(s)}$$

$$G_{21}(s)=\frac{T_2(s)}{T_{01}(s)}=\frac{\alpha_2k_{p1}}{P(s)};\ G_{22}(s)=\frac{T_2(s)}{T_{02}(s)}=\frac{k_{p2}(\tau_{p1}s+1)}{P(s)};\ G_{23}(s)=\frac{T_2(s)}{Q(s)}=\frac{k_{p3}(\tau_{p1}s+1)}{P(s)}$$

$$T_1(s)=G_{11}(s)T_{01}(s)+G_{12}(s)T_{02}(s)+G_{13}(s)Q(s)$$

$$T_2(s)=G_{21}(s)T_{01}(s)+G_{22}(s)T_{02}(s)+G_{23}(s)Q(s)$$





Inversion of Laplace transforms – 1

(excluding systems with dead time)

Main steps:

a) Expand the Q(s)/P(s) (ratio of polynomials of order m and n respectively) (n≥m)

$\frac{Q(s)}{P(s)} = \frac{c_1}{r_1(s)} + \frac{c_2}{r_2(s)} + \ldots + \frac{c_n}{r_n(s)}$ where $r_1(s)$, $r_2(s)$,…,$r_n(s)$ are low-order polynomials (I and II order in general)

b) Compute the value of the constants C_1, C_2,…, C_n

c) Find the inverse Laplace transform of each partial fraction $L^{-1}\left[\frac{c_1}{r_1(s)}\right]$,…

In general

$P(s) = a_0 s^n + a_1 s^{n-1} + \ldots + a_{n-1} s + a_n = a_0 \prod_{k=1}^{n} (s - p_k)$ where p_k are the roots of the polynomial

Two main cases exist:

1) P(s) has n distinct roots, real or complex

2) P(s) has multiple roots

Dynamic response

In general the **dynamic response** of an output is given by:

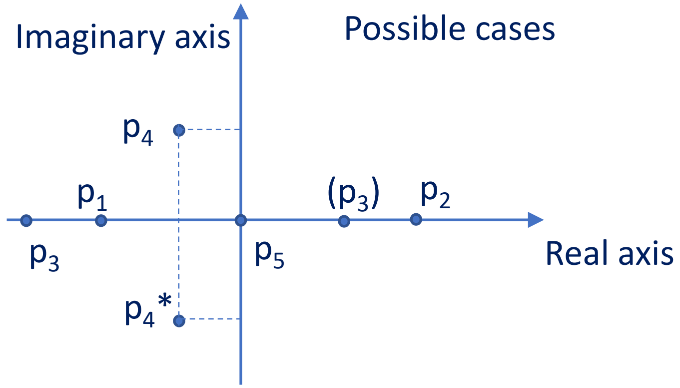
$$y(s) = G(s)\,f(s) = \frac{Q(s)}{P(s)}\,\frac{r(s)}{z(s)}$$ (excluding dead time)

The **"intrinsic" dynamics** of the system are the dynamics associated with the G(s): if we know where the poles of a system are located (in other words the roots of the polynomial P(s)), we can determine the quantitative characteristics of the system as caused by the model (or the transfer function G(s)).

Then we need to consider the effects caused by the input f(s).

$$G(s) = \frac{Q(s)}{P(s)} = \frac{Q(s)}{(s-p_1)(s-p_2)(s-p_3)^m(s-p_4)(s-p_4^*)(s-p_5)} \qquad (a_0 = 1)$$

where p_1, p_2, p_3, p_4, p_4^*, p_5 are the roots of P(s) (or the poles of the system as represented in the complex plane



Possible cases

p_1, p_2	real distinct poles
p_3	multiple real poles, repeated m times
p_4, p_4^*	complex conjugate poles
p_5	poles at the origin

Intrinsic dynamic – 1

The partial-fraction expansion of G(s) leads to:

$$G(s)=\frac{C_1}{s-p_1}+\frac{C_2}{s-p_2}+\left[\frac{C_{31}}{s-p_3}+\frac{C_{32}}{(s-p_3)^2}+\ldots+\frac{C_{3m}}{(s-p_3)^m}\right]+\frac{C_4}{s-p_4}+\frac{C_4{}^*}{s-p_4^*}+\frac{C_5}{s-p_5}$$

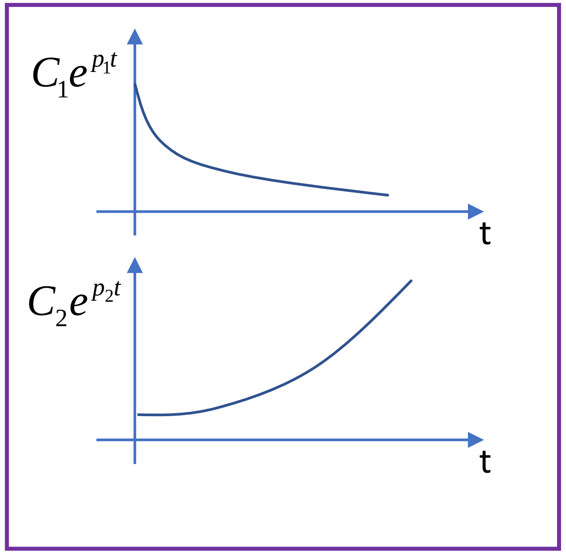
$$\boxed{L\left(\frac{t^n e^{-at}}{n!}\right)=\left(\frac{1}{s+a}\right)^{n+1}}$$

Please notice that this treatment concerns the “intrinsic” dynamics of the process

- Real distinc poles located on the real axis gives terms such as $C_1e^{p_1t}$ and $C_2e^{p_2t}$

Since $p_1<0$, the first term exponentially decays to zero, whereas as $p_2>0$, the second contribution grows exponentially to infinity with time.

Therefore, distinct poles on the negative real axis produce terms that decay to zero with time (**intrinsically stable contribution**), while real positive poles make the response of the system grow toward infinity with time (**intrinsically unstable contribution**)



Intrinsic dynamic – 2

- Multiple real poles such as p_3 repeated m times. Such terms give rise to contributions:

$$\left[C_{31}+\frac{C_{32}}{1!}t+\frac{C_{33}}{2!}t^2+\ldots+\frac{C_{3m}}{(\mathrm{m}-1)!}t^{m-1}\right]e^{p_3 t}$$

The term in brackets grows towards infinity with time

The behavior of the exponential term depends on the value of the pole p_3

$if\ \ p_3>0\ e^{p_3 t}\to\infty\ as\ t\to\infty$ $if\ \ p_3<0\ e^{p_3 t}\to 0\ as\ t\to\infty$ $if\ \ p_3=0\ e^{p_3 t}=1\ for\ all\ times$

Therefore, a real multiple pole gives rise to terms which either grow to infinity, if the pole is positive or zero (**intrinsically unstable contribution**) or decay to zero, if the pole is negative (**intrinsically stable contribution**)

- Complex conjugate poles such as p_4 and p_4* (complex poles always appear in conjugate pairs and never alone)

$$p_4=\alpha+j\beta$$
$$p_4^*=\alpha-j\beta$$

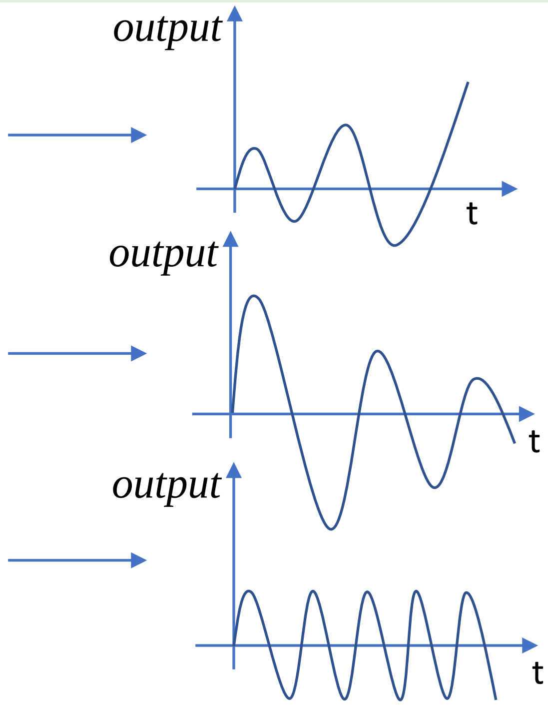
The conjugate pairs of the complex roots give rise to terms such as $e^{\alpha t}\sin(\beta t+\phi)$

The $\sin(\beta t+\phi)$ is a periodic oscillating function, while the behavior of $e^{\alpha t}$ depends on the value of the real part α

Intrinsic dynamic – 3

- If $\alpha > 0 \; then \; e^{\alpha t} \to \infty \; as \; t \to \infty$ and $e^{\alpha t}\sin\left(\beta t + \phi\right)$ grows to infinity in an oscillating manner with increasing amplitude
- If $\alpha < 0$, then $e^{\alpha t} \to 0 \; as \; t \to \infty$ and $e^{\alpha t}\sin\left(\beta t + \phi\right)$ decays to zero in an oscillating manner with an over-decreasing amplitude
- If $\alpha = 0, \; then \; e^{\alpha t} = 1$ for all times and $e^{\alpha t}\sin\left(\beta t + \phi\right) = \sin\left(\beta t + \phi\right)$

  which oscillates continously with constant amplitude



Therefore a pair of complex conjugate poles gives rise to oscillatory behavior whose amplitude may grow continously or remain unchanged for the real part of the poles positive or zero, respectively (**intrinsically unstable contributions**) or decay to zero if the real part of the poles is negative (**intrinsically stable contribution**)

- Poles at the origin

The pole p_5 is located at the origin of the complex plane ($p_5=0$). Therefore the contribution is C_5/s and after inversion it gives a constant term C_5 (**intrinsically unstable contribution**)

Stability of a dynamic system

The considerations previously made are general and applicable to any system. The qualitative characteristics of a system (a process represented by a mathematical model→G(s)) are known if the location of the poles of G(s) is known. For a particular input f(t) (f(s)), the additional related contributions should be considered in order to get the specific response.

Poles to the **right** of the imaginary axis give rise to terms which grow to infinity with time and are **unstable** (unbounded behavior).

A system is **stable** (with bounded behavior) if all the poles of the transfer function are located to the **left** of the imaginary axis.

The poles located at the origin or along the imaginary axis are also representative of an unstable behavior (systems lacking of self-regulating capability).

A dynamic system is considered to be ***stable*** *if* ***for every bounded input*** (it always remains between an upper and a lower limit), ***it produces a bounded output****, indipendently from the initial state*. Unbounded outputs exist only in theory and not in practice because all physical quantities are limited.