Facoltà di Ingegneria
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Game Theory and Analysis of Competitive Dynamics for Industrial Systems
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MATHEMATICAL MODELS OF ELECTRONIC JAMMING TECHNIQUES

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1. Theory of zero-sum matrix games
    - Mixed Strategies
    - Minimax Theorem
    - Relation with Linear Programming

2. Mathematical Models of Electronic Jamming Techniques

Theory of zero-sum matrix games

- A zero sum game is a game in which, at each terminal vertex, the payoff function's components add up to zero.
- A two-person zero-sum game is represented through an m x n matrix.
- The element a_{ij} is the payoff of player 1 if player 1 chooses the i-th strategy, assumed that player 2 chooses j-th strategy.

Since this matrix represents the different payoffs for player 1, it makes sense that player 1 will try to maximize the element a_{ij} which is chosen, while player 2 will try to minimize it, since that value is equivalent to player 2's loss.

$$\begin{array}{c|cccc} & P2_1 & P2_2 & \cdots & P2_n \\ \hline P1_1 & a_{11} & a_{12} & \cdots & a_{1n} \\ P1_2 & a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ P1_m & a_{m1} & a_{m2} & \cdots & a_{mn} \end{array}$$

Theory of zero-sum matrix games (2)

- Player 1 will choose the strategy i which allows to maximize his minimum payoff (maximin criterium).
- Player 2 will choose the strategy j which allows to minimize the maximum payoff of player 1 (minimax criterium).
- Strategies i and j are optimal strategies, respectively for player 1 and player 2, according to maximin and minimax criteria.
- The maximum minimum payoff for player 1 is called gain floor (lower value) of the game, i.e. maximum payoff player 1 will receive given player 2's attempt to minimize player 1's payoff: $\bar{v}_I = \max_i \min_j a_{ij}$.
- The minimum maximum payoff for player 1 is called loss ceiling (upper value) of the game, i.e. minimum loss player 2 can experience given player 1's attempt to maximize player 1's payoff: $\bar{v}_{II} = \min_j \max_i a_{ij}$.

Theory of zero-sum matrix games (3)

- A saddle point is an element in the matrix game that is both the largest in its column and the smallest in its row.
- Not all game matrices in pure strategies have saddle points, but if they do, they will clearly be the equilibrium strategies, since they both maximize player 1's payoff, and minimize player 2's loss.

An approach to obtain an equilibrium solution in matrix games that do not possess a saddle point and in which players act independently, is to enlarge the strategy spaces, so as to allow the players to base their decisions on the outcome of random events (e.g. games played over and over again, and the final outcome is determined by averaging the outcomes of individual plays).

Mixed Strategies

A mixed strategy is a probability distribution on the set of a player's pure strategies, equivalently it is a random variable whose values are the player's pure strategies.

Suppose players 1 and 2 are playing the matrix game $\mathbf{A}$, where $A = (a_{ij})$ is an $(m$ x $n)$ matrix. Let X denotes the set of m mixed strategies for player 1, and Y represents the set of n mixed strategies for player 2. Player 1 will choose the strategy $i \in X$ with a probability x_i, while player 2 will choose the strategy $i \in Y$ with probability y_i.

The mixed strategy of player 1 can be expressed as an m-vector, $x = (x_1, \ldots, x_m)$, such that $x_i \geq 0$ and $\sum_{i=1}^{m} x_i = 1$.

If player 1 chooses mixed strategy x, while player 2 chooses mixed strategy y, then the expected payoff can be written as $A(x, y) = \sum_{i=1}^{m} \sum_{j=1}^{n} x_i a_{ij} y_j$, or $A(x, y) = xAy^T$.

Mixed Strategies (2)

A vector $x^* \in X$ is called a mixed security strategy for $\mathbf{P1}$ if:

$$v_I = \max_{y \in Y}(x^* A y^T) \leq \max_{y \in Y}(x A y^T) \qquad x \in X$$

where v_I is known as the average security level of $\mathbf{P1}$ (or loss ceiling, upper value, of the game).

A vector $y^* \in Y$ is called a mixed security strategy for $\mathbf{P2}$ if:

$$v_{II} = \min_{x \in X}(x A y^{*T}) \geq \min_{x \in X}(x A y^T) \qquad y \in Y$$

where v_{II} is known as the average security level of $\mathbf{P2}$ (or gain floor, lower value, of the game).

A pair of strategies (x^*, y^*) is a saddle point for a matrix game $\mathbf{A}$, in mixed strategies, if:

$$x^* A y^T \leq x^* A y^{*T} \leq x A y^{*T} \qquad x \in X, \quad y \in Y$$

where $v = x^* A y^{*T}$ is the saddle point of the game.

Mixed Strategies (3)

In every matrix game:

- The average security level of each player is unique.
- There exists at least one mixed security strategy for each player.
- $v_{II} = v_I$.

In a matrix game $\mathbf{A}$, the average upper and lower values in a mixed strategy are given, respectively, by:

$$v_I = \min_X \max_Y (xAy^T) \qquad v_{II} = \max_Y \min_X (xAy^T)$$

Minimax Theorem

In any matrix game $\mathbf{A}$, the average security levels of the players in mixed strategies coincide, that is:

$$\min_X \max_Y (xAy^T) = \max_Y \min_X (xAy^T), \quad \text{i.e.}\, v_I = v_{II}$$

A corollary of the minimax theorem is that if $\mathbf{A}$ denote an (m x n) matrix game, then:

- $\mathbf{A}$ has always a saddle point in mixed strategy.
- A pair of mixed strategies provides a saddle point for $\mathbf{A}$ if, and only if, the first of these is a mixed security strategy for $\mathbf{P1}$, and the second one is a mixed security strategy for $P2$.
- v is uniquely given by $v = v_I = v_{II}$.
- in case of multiple saddle points, the mixed saddle point strategies possess the ordered interchangeability property.

Relation with Linear Programming

$$\begin{array}{c|cc|c}
 & P2_1 & P2_2 & \\
\hline
P1_1 & 3 & 1 & x_1 \\
P1_2 & 2 & 6 & x_2 \\
\hline
 & y_1 & y_2 &
\end{array}$$

This matrix has not any saddle point in pure strategies. If x_i and y_i represent the probabilities that $\mathbf{P1}$ and $\mathbf{P2}$ play strategy $P1_i$ and $P2_i$ respectively, the expected gain for $P1$ is $3x_1 + 2x_2$ if $\mathbf{P2}$ plays $P2_1$ and $x_1 + 6x_2$ if $\mathbf{P2}$ plays $P2_2$, while the expected loss for $\mathbf{P2}$ is $3y_1 + y_2$ if $\mathbf{P1}$ plays $P1_1$ and $2y_1 + 6y_2$ if $\mathbf{P1}$ plays $P1_2$.

The problem statement for $\mathbf{P1}$ is to choose the strategy (x_1, x_2) which guarantees the maximum gain, independently on $\mathbf{P2}$ strategy. It is a linear programming problem.

Relation with Linear Programming (2)

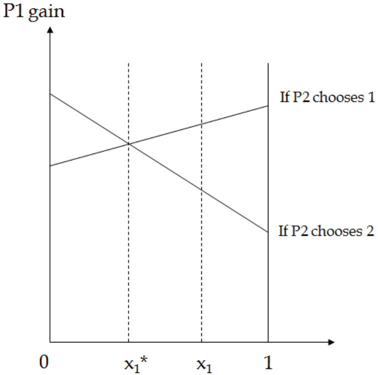
The two lines represent the expected gain for **P1** for both **P2** strategies, with x_1 and x_2 varying. The intersection x_1^* is the point where there is **P1** maximum gain independently on **P2** strategy, and is the solution of this system:

$$\begin{cases} 3x_1 + 2x_2 = x_1 + 6x_2 \\ x_1 + x_2 = 1 \\ x_1 \text{and} x_2 \geq 0 \end{cases}$$

That yields:

$$x1 = 2/3 \quad \text{and} \quad x2 = 1/3$$

Expected **P1** gain $= 8/3$.



Relation with Linear Programming (3)

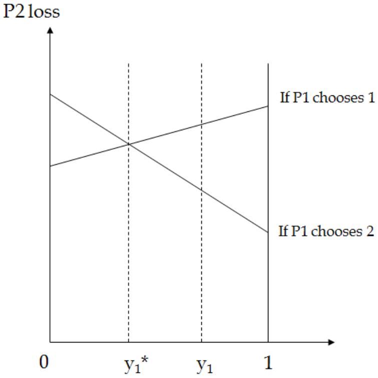
The same is valid for $\mathbf{P2}$, who wants to minimize the losses for every strategy of $\mathbf{P1}$. The intersection y_1^* is the point where there is $\mathbf{P2}$ minimum loss independently on $\mathbf{P1}$ strategy, and is the solution of this system:

$$\begin{cases} 3y_1 + y_2 = 2y_1 + 6y_2 \\ y_1 + y_2 = 1 \\ y_1 \text{and} y_2 \geq 0 \end{cases}$$

That yields:

$$y1 = 5/6 \quad \text{and} \quad y2 = 1/6$$

Expected $\mathbf{P2}$ loss $= 8/3$.

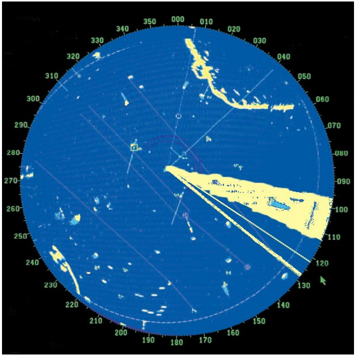


Relation with Linear Programming (4)

- Since $\mathbf{P1}$ maximum gain is equal to $\mathbf{P2}$ minimum loss, it is the saddle point for mixed strategies, and represents a probability distribution that assures both players the expected optimal result.
- It is easy to observe the relation of matrix games with linear programming, where the linear objective function to optimize is the expected gain for $\mathbf{P1}$ and the expected loss for $\mathbf{P2}$, and the constraints are linear.

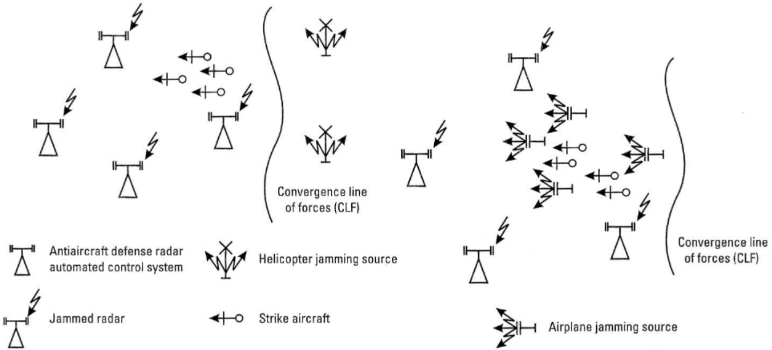
Noise Jamming

- Jamming: the intentional and deliberate transmission of signals for the purpose of interfering with, disturbing, exploiting, deceiving, masking, or otherwise degrading the reception of other signals that are used by radar systems (L. B. Van Brunt, *Applied ECM*, 1978, p. 29).
- A convenient classification of noise jamming is by the ratio of the jamming signal bandwidth to the acceptance bandwidth of the victim equipment (barrage jamming vs spot jamming vs sweep jamming).



it.wikipedia.org

Noise Jamming



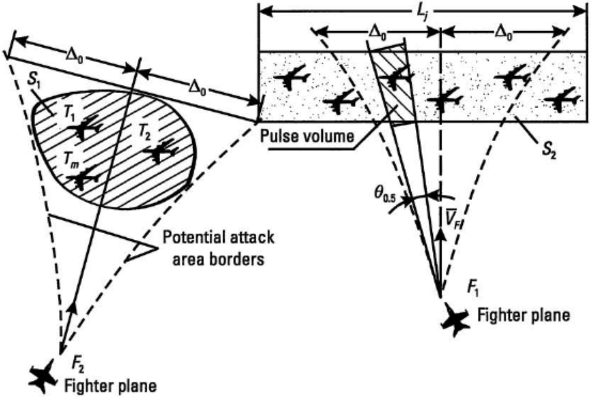
1. Methods of concealing airplanes and other targets from stationary (fixed) areas using jamming.
2. Methods of screening airplanes in battle formation using jamming.

Mathematical Models of Electronic Jamming Techniques

- One of the special features of attack using masking and deception jamming is its increasing sensitivity when compared to other types of armed warfare and to countermeasures by the target being jammed.
- The latter is due to the fact that the jamming radiation being considered does not cause irreversible destructive effects in the jamming target.
- This permits to take defensive countermeasures directly during the period of jamming action.
- It is necessary found the optimum course of action in order to resolve the conflict aspect of the problem (Solution → Game Theory).

Mathematical Models of Electronic Jamming Techniques

Let us assume that it is a possibility, with the help of an active jammer, to create an area S_1 in space within the bounds of which not a single aircraft T_1, T_2 and T_3 is detected.



Mathematical Models of Electronic Jamming Techniques

Let us consider that, besides the active active jammer, the offensive side has 2 strike craft (bombers) and AAD (Anti Aircraft Defence) has only 2 intercept fighters.
Let us find a way of dislocating the strike craft in the areas S_1 and S_2 so that the average number of attacks on each bomber is minimum.

- Offensive side (Airforce)
    1. A_1: both bombers are in area S_1;
    2. A_2: both bombers are in area S_2;
    3. A_3: one bomber is in area S_1 and the other is on area S_2.
- Defending side (AAD)
    1. B_1: both fighters (F_1 and F_2) are dispatched to area S_1;
    2. B_2: both fighters (F_1 and F_2) are dispatched to area S_2;
    3. B_3: one fighter (F_1) is dispatched to area S_1 and the other (F_2) to area S_2.

Mathematical Models of Electronic Jamming Techniques

The average number of attacks $\bar{n}$ on the bombers serves as the quantitative measure of the effectiveness of the actions of both sides.

$$\bar{n} = \bar{n}_1 + \bar{n}_2$$

It is possible to consider:

$$\bar{n}_1 = \frac{n_b}{n_b + n_j} n_f$$

$\frac{n_b}{n_b+n_j}$ is the probability that a bomber will be selected from the overall number of aircraft in the jamming area S_1; n_b : number of bombers in area S_1; n_j : number of jamming generators in the area S_1; n_f number of fighters dispatched to area S_1.
The situation is analogous for one bomber aircraft screened by passive jamming (area S_2) and dislocated in a random fashion in a jamming area of length L_j:

$$\bar{n}_2 \approx \frac{2\Delta_0}{L_j} n_b n_f$$

where Δ_0 is the maximum miss for the fighter selected according to overload conditions during homing time; $\frac{2\Delta_0}{L_j}$ is the probability of a bomber reaching the area of possible attacks.

Mathematical Models of Electronic Jamming Techniques

Let $\Delta_0 = 10$ km and $L_j = 100$ km.
We can put together a results matrix for the actions of both sides:

$$\begin{pmatrix} \mathbf{A} & \mathbf{B} & & \\ & B_1 & B_2 & B_3 \\ A_1 & 1.3 & 0 & 0.6 \\ A_2 & 0 & 0.8 & 0.4 \\ A_3 & 1 & 0.4 & 0.7 \end{pmatrix}$$

Mathematical Models of Electronic Jamming Techniques

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However, if we select action plan A_2 as the only one, then we conscientiously give the opponent the possibility of using plan B_2, which gives him an average number of attacks on the bombers equal to 0.8 (solution → select several plans, pure → mixed strategies).

Mathematical Models of Electronic Jamming Techniques

The structure of the solution must be:

$$S_A = \begin{pmatrix} A_1, & A_2, & A_3 \\ P_1, & P_2, & P_3 \end{pmatrix}$$

Given any of the opponent's action plans B_1, B_2, or B_3, the average number of attacks on the bombers will not exceed the number of attacks when the action plan is optimum, N (price of the game):

$$P_1\bar{n}_{11} + P_2\bar{n}_{21} + P_3\bar{n}_{31} \le N$$

$$P_1\bar{n}_{12} + P_2\bar{n}_{22} + P_3\bar{n}_{32} \le N$$

$$P_1\bar{n}_{13} + P_2\bar{n}_{23} + P_3\bar{n}_{33} \le N$$

In the general case, not all action plans should be used. The sum of frequencies P_1, P_2 and P_3 is equal to 1:

$$P_1 + P_2 + P_3 = 1$$

Mathematical Models of Electronic Jamming Techniques

Let us multiply all the numbers in matrix by 10, thus increasing by 10 times the price of the game ($N' = 10N$):

$$\left(\begin{array}{cccc} \mathbf{A} & \mathbf{B} & & \\ & B_1 & B_2 & B_3 \\ A_1 & 13 & 0 & 6 \\ A_2 & 0 & 8 & 4 \\ A_3 & 10 & 4 & 7 \end{array}\right)$$

Thus, the system of inequalities assumes the format:

$$13P_1 + 10P_3 \leq N'$$

$$8P_2 + 4P_3 \leq N'$$

$$6P_1 + 4P_2 + 7P_3 \leq N'$$

Mathematical Models of Electronic Jamming Techniques

Let us divide both parts of the last inequalities by N', and let:

$$\xi_1 = \frac{P_1}{N'}, \quad \xi_2 = \frac{P_2}{N'}, \quad \xi_3 = \frac{P_3}{N'}$$

we obtain the following equalities:

$$13\xi_1 + 10\xi_3 + z_1 = 1$$

$$8\xi_2 + 4\xi_3 + z_2 = 1$$

$$6\xi_1 + 4\xi_2 + 7\xi_3 + z_3 = 1$$

$$\xi_1 + \xi_2 + \xi_3 = \frac{1}{N'}$$

The task is now reduced to a problem of linear programming (determining ξ_1, ξ_2, ξ_3 where N' will be minimum.)

Mathematical Models of Electronic Jamming Techniques

Finally, we find:

$$N' = 4.95$$

and:

$$P_1 = 0.38 \qquad P_2 = 0.62$$

$$P_3 = 0 \qquad N \approx 0.5$$

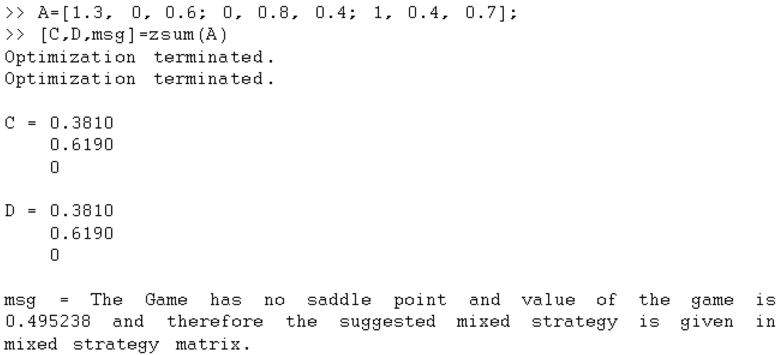
Thus, using plans A_1 and A_2 corresponding to that frequencies, we obtain a reduction in the number of attacks on bombers, independent of any possible countermeasures taken by the opponent in the given circumstances.
In an analogous manner, we obtain:

$$q_1 = 0.38 \qquad q_2 = 0.62$$

$$q_3 = 0 \qquad N \approx 0.5$$

Mathematical Models of Electronic Jamming Techniques

Simulation results using Matlab:



```
>> A=[1.3, 0, 0.6; 0, 0.8, 0.4; 1, 0.4, 0.7];
>> [C,D,msg]=zsum(A)
Optimization terminated.
Optimization terminated.

C = 0.3810
    0.6190
    0

D = 0.3810
    0.6190
    0

msg = The Game has no saddle point and value of the game is
0.495238 and therefore the suggested mixed strategy is given in
mixed strategy matrix.
```

Mathematical Models of Electronic Jamming Techniques

Each of the sides must know a priori all the pure strategies A_i B_j, and each value of the cost a_{ij}.

In conditions of armed conflict, measures for concealing operations play a decisive role. So, more meaningful results can be obtained using multi-step games with incomplete information (analogous to game with nature).

The side, who represents the nature, does not undertake deliberate countermeasures against the other side. If the side $\mathbf{B}$ represents *nature*, then the strategies B_j should be considered to be certain fixed states, and they be denoted by S_j. The choice by side $\mathbf{A}$ of strategy A_i, is determined by the evolving uncertainties of the states of *nature*. In order to account for possible losses, we introduce the concept of risk r_{ij}:

$$r_{ij} = \beta_j - a_{ij}$$

with

$$\beta_j = \max_j a_{ij}$$

Mathematical Models of Electronic Jamming Techniques

The determining of risk is a fairly complex problem. The task becomes much easier if we know the probabilities of the states of *nature*:

$$P(S_j), \qquad j = 1, \dots, n \qquad \sum_{j=1}^{n} P(S_j) = 1$$

It is desirable to orient ourselves toward the maximum of the mathematical expectation of winnings $\bar{a}_i$:

$$\bar{a}_i = \sum_{j=1}^{n} P(S_j) a_{ij}$$

and the average risk is:

$$\bar{r}_i = \sum_{j=1}^{n} P(S_j) r_{ij}$$

We choose the strategy A_i, where $\bar{r}_i$ attains the minimum.

Mathematical Models of Electronic Jamming Techniques

The distributions of probabilities $P(S_j)$ are not known. In such cases, a decision can be taken on the basis of other criteria.

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$$W = \max_i \min_j a_{ij}$$

player $\mathbf{A}$ receives winnings of no less than W.

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2. Savage's criterion of minimum risk:

$$\varepsilon = \min_i \max_j r_{ij}$$

   the risk will be no larger than ε.

Mathematical Models of Electronic Jamming Techniques

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   the risk will be no larger than ε.

3. Hurwitz's pessimism-optimism criterion:

$$H = \max_i \left(\chi \min_j a_{ij} + (1 - \chi) \max_j a_{ij} \right)$$

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