Introduction to the Geometry of Classical Dynamics

2nd edition

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Renato Grassini, *Introduction to the Geometry of Classical Dynamics*, 2nd edition, 2012.

Typeset using LaTeX.

Mathematics Subject Classification: 70H45, 58F05, 34A09

Keywords: smooth manifolds, implicit differential equations, d'Alembert's principle, Lagrangian and Hamiltonian dynamics

Preface

The aim of this paper is to lead (in most elementary terms) an undergraduate student of Mathematics or Physics from the historical Newtonian-d'Alembertian dynamics up to the border with the modern (*geometrical*) Lagrangian-Hamiltonian dynamics, without making any use of the traditional (*analytical*) formulation of the latter. [1]

Our expository method will in principle adopt a rigorously coordinate-free language, apt to gain – from the very historical formulation – the 'consciousness' (at an early stage) of the geometric structures that are 'intrinsic' to the very nature of classical dynamics. The coordinate formalism will be confined to the ancillary role of providing simple proofs for some geometric results (which would otherwise require more advanced geometry), as well as re-obtaining the *local* analytical formulation of the theory from the *global* geometrical one. [2]

The main conceptual tool of our approach will be the simple and general notion of *implicit differential equation*, which, treating an equation just as a 'locus' extracted from the tangent bundle of a manifold by some geometric or algebraic requirements, will directly allow us to capture the structural core underlying the evolution law of classical dynamics. [3]

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[1] Such an Introduction will cover the big gap existing in the current literature between the (empirical) elementary presentation of Newtonian-d'Alembertian dynamics and the (abstract) differential-geometric formulation of Lagrangian-Hamiltonian dynamics. Standard textbooks on the latter are [1][2][3][4], and typical research articles are [5][6][7][8][9][10].

[2] The differential-geometric techniques adopted in this paper – basically limited to smooth manifolds embedded in Euclidean affine spaces, like those treated in [11], but nevertheless providing the guidelines for the geometry of 'abstract' smooth manifolds – are listed in Appendix (whose reading is meant to preceed that of the main text). More advanced geometry can be found in the textbooks already quoted, as well as in a number of excellent introductions, e.g. [12][13][14][15].

[3] Research articles close to the spirit of this approach are, among others, [16][17] (on implicit differential equations) and [18][19][20][21][22] (on their role in advanced dynamics).

Contents

Chapter 1

From Newton to d'Alembert

First we shall recall the basic problem of classical particle dynamics, Newton's answer to the problem and d'Alembert's reformulation of the answer. Then the latter will be shown to correspond to an implicit differential equation on a Euclidean space.

1.1 The data

Classical particle dynamics basically deals with an empirical problem, whose data – in the *simplest* cases – can be described in mathematical terms as follows.

Configuration space

A *reference space* – 'mathematical extension' of a rigid body (e.g. the Earth, a vehicle, the Moon, a planet, the Sun, a star), meant to be carrying an uninfluential observer – is conceived as a 3-dimensional Euclidean affine space (usually denoted by $\mathcal{E}_3$ and modelled on a vector space E_3 with Euclidean metric " $\cdot$ ").

Time – ordering events on a graduated scale, meant to be independent of any observer – is conceived as a 1-dimensional, oriented, Euclidean affine space (isometrically identified with the oriented real line $\mathbb{R}$).

'Particle' is synonymous with 'point-like body', i.e. a body whose *position* in the chosen reference space $\mathcal{E}_3$ is conventionally defined as a single point of $\mathcal{E}_3$.

So, for an ordered system of ν particles, a position or *configuration* in $\mathcal{E}_3$ will be defined as a single point of the Cartesian power $\mathcal{E} := {\mathcal{E}_3}^{\nu}$ (Euclidean affine space, modelled on $E := {E_3}^{\nu}$, of dimension 3ν).

The given particle system may generally be subject in $\mathcal{E}_3$ to some time-independent, *holonomic* (i.e. positional) *constraints*, owing to which it is virtually allowed to occupy only the *admissible configurations* belonging to a region

$$Q \subset \mathcal{E}$$

Q will be assumed to be a smooth manifold embedded in $\mathcal{E}$.

The above manifold Q and its dimension $n := \dim(Q) \leq 3\nu$ are called *admissible configuration space* and *number of the degrees of freedom*, respectively, of the particle system in $\mathcal{E}_3$.

Generally, Q consists of all the points $\mathrm{p} \in \mathcal{E}$ satisfying some scalar inequalities $\{g_\alpha(\mathrm{p}) > 0\}_{\alpha=1,\ldots,\mu}$, called strict *one-sided constraints*, and/or equalities $\{f_\beta(\mathrm{p}) = 0\}_{\beta=1,\ldots,\kappa<3\nu}$, called *two-sided constraints.* As is known, under suitable hypotheses of regularity (continuity and differentiability) on the g_α's and f_β's, Q is a smooth manifold of $\mathcal{E}$, whose dimension $n = 3\nu - \kappa$ is given by the dimension of the Euclidean environment $\mathcal{E}$ minus the number of the two-sided constraints (in presence of strict one-sided constraints only, Q is an open – and then 3ν-dimensional – manifold of $\mathcal{E}$). [1]

Then, for any $\mathrm{p} \in Q$, the tangent vector space $T_{\mathrm{p}}Q$ is the set of vectors $\delta\mathrm{p} \in E$ satisfying the scalar equalities $\{d_{\mathrm{p}} f_\beta\,(\delta\mathrm{p}) = 0\}_{\beta=1,\ldots,\kappa}$. Such vectors can be regarded as *admissible* or *virtual displacements* (starting from $\mathrm{p} \in Q$), i.e. displacements virtually allowed by the constraints, since any 'small' $\delta\mathrm{p} \in T_{\mathrm{p}}Q$ takes – up to higher order infinitesimals – the point p belonging to Q, i.e. satisfying $g_\alpha(\mathrm{p}) > 0$ and $f_\beta(\mathrm{p}) = 0$, to a point $\mathrm{p}+\delta\mathrm{p}$ still belonging to Q i.e. satisfying $g_\alpha(\mathrm{p}+\delta\mathrm{p}) > 0$ by continuity and $f_\beta(\mathrm{p}+\delta\mathrm{p}) \approx f_\beta(\mathrm{p}) + d_{\mathrm{p}} f_\beta\,(\delta\mathrm{p}) = 0$ by differentiability.

Mass distribution

The response of the system to any internal or external influence, will generally depend on how 'massive' its particles are, the *inertial mass* of a particle being conceived as a positive scalar quantity.

The *inertial mass distribution* carried by the system will be denoted by

$$m := (m_1, \ldots, m_\nu)$$

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[1] We shall not consider non-strict one-sided constraints, which would give rise to a manifold Q with boundary (nor shall we consider time-dependent constraints, which would give rise to a manifold Q fibred over the time line).

Force field

The 'force' $-\delta\acute{\upsilon}\nu\alpha\mu\iota\varsigma-$ resultant of all the internal and/or external influences acting in $\mathcal{E}_3$ on the particle, is generally described as a vector-valued smooth map

$$\mathbf{F} : TQ \to E : (\mathrm{p}, \mathbf{v}) \longmapsto \mathbf{F}(\mathrm{p}, \mathbf{v}) = (\mathbf{F}_1(\mathrm{p}, \mathbf{v}), \ldots, \mathbf{F}_\nu(\mathrm{p}, \mathbf{v}))$$

defined on the space TQ of the *admissible* positions and velocities allowed by the constraints. [2]

If $\mathbf{F}$ is *positional*, i.e. its restriction to each fibre TQ is constant, then $\mathbf{F}$ can just be regarded as a smooth map

$$\mathbf{f} : Q \to E : \mathrm{p} \mapsto \mathbf{f}(\mathrm{p})$$

where $\mathbf{f}(\mathrm{p})$ denotes the constant value of $\mathbf{F}|_{T_\mathrm{p}Q}$.

Once assigned (on empirical grounds) such a 'law of force' $\mathbf{F}$, it will be said to be the *force field* acting in the reference space $\mathcal{E}_3$ on the particle system.

Mechanical system

The above (*empirical*) *mechanical system* will briefly be denoted by the triplet

$$\mathcal{S} := (Q, m, \mathbf{F})$$

1.2 The question

With reference to such a mechanical system $\mathcal{S}$, the basic problem of dynamics can be expressed as follows.

Smooth motions

The unknown of the problem is 'motion' $-\kappa\acute{\iota}\nu\eta\mu\alpha-$ whose matematical description is the concern of *kinematics.*

A *smooth motion* of the particle system in the reference space $\mathcal{E}_3$, is described as a smooth curve

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[2] As to the concept of (admissible) velocity, see the next section 1.2, *Smooth motions.*
We shall not consider time-dependent forces, which would later give rise to time-dependent differential equations.

$$\gamma : I \subset \mathbb{R} \to \mathcal{E} : t \mapsto \gamma(t) = \mathrm{p}(t)$$

of the Euclidean affine space $\mathcal{E}$, establishing a configuration $\mathrm{p}(t)$ at each *time* t of an open *time interval* $I \subset \mathbb{R}$. [3]

Along γ, owing to smoothness, the positions

$$\mathrm{p}(t) = (\mathrm{p}_1(t), \ldots, \mathrm{p}_\nu(t)) \in \mathcal{E}$$

of the particles, their *velocities*

$$\dot{\mathrm{p}}(t) = (\dot{\mathrm{p}}_1(t), \ldots, \dot{\mathrm{p}}_\nu(t)) \in E$$

and their *accelerations*

$$\ddot{\mathrm{p}}(t) = (\ddot{\mathrm{p}}_1(t), \ldots, \ddot{\mathrm{p}}_\nu(t)) \in E$$

(as well as higher-order derivatives) are all differentiable functions of time.

Remark that, along γ, the first derivative $\dot{\mathrm{p}}(t)$ is meant to be the *velocity* at time t, since it measures –up to higher order infinitesimals– the displacement

$$\mathrm{p}(t+1) - \mathrm{p}(t) = (\, \mathrm{p}_1(t+1) - \mathrm{p}_1(t), \ldots\ldots, \mathrm{p}_\nu(t+1) - \mathrm{p}_\nu(t)\,)$$

in the unit time interval $[t, t+1]$. In the same way, the second derivative $\ddot{\mathrm{p}}(t)$ is meant to be the *acceleration* at time t, since it measures –up to higher order infinitesimals– the increment of velocity

$$\dot{\mathrm{p}}(t+1) - \dot{\mathrm{p}}(t) = (\, \dot{\mathrm{p}}_1(t+1) - \dot{\mathrm{p}}_1(t), \ldots\ldots, \dot{\mathrm{p}}_\nu(t+1) - \dot{\mathrm{p}}_\nu(t)\,)$$

in the unit time interval $[t, t+1]$.

Clearly, any vector of E is the velocity (or the acceleration) of some smooth motion.

In particular, a smooth motion γ is an *admissible motion*, i.e. a motion virtually allowed by the constraints, if

$$\mathrm{Im}(\gamma) \subset Q$$

As a consequence, for every $\mathrm{p} \in Q$, any vector $\mathbf{v} \in T_{\mathrm{p}}Q$ can be regarded as an *admissible* or *virtual velocity*, i.e. a velocity virtually allowed by the constraints, since –by the very definition of tangent vector– it is the velocity (at a certain time) of an admissible motion passing (at that time) through p.

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[3] The restriction of a smooth curve to a closed subinterval of its domain of definition, is still said to be smooth.

Dynamically possible motions

Smooth *dynamics* basically deals with the 'time-evolution problem' – in the unknown γ – expressed by the following question:

> " For the above constrained point-mass system, what are the smooth motions – in the chosen reference space – that are *possible* under the action of the given $\delta\acute{\upsilon}\nu\alpha\mu\iota\varsigma$? "

Such motions will briefly be called the *dynamically possible motions* (DPMs) of $\mathcal{S}$ (whereas the smooth motions which would be possible in absence of force, i.e. $\mathbf{F} = 0$, will be said to be the *inertial motions* of $\mathcal{S}$).

1.3 The answer after Newton

After Newton, the answer to the above 'predictive' question is given by the following

Newton's law of constrained dynamics

A smooth motion

$$\gamma : I \subset \mathbb{R} \to \mathcal{E} : t \mapsto \gamma(t) = \mathrm{p}(t)$$

is a *DPM* of $\mathcal{S}$, iff, $\forall\, t \in I$,

$$\begin{aligned} &\mathrm{p}(t) \in Q \\ &m\, \ddot{\mathrm{p}}(t) = \mathbf{F}(\mathrm{p}(t), \dot{\mathrm{p}}(t)) + \Phi(t) \\ &\Phi(t) \in T^{\perp}_{\mathrm{p}(t)} Q \end{aligned}$$

The first condition [4] just exhibits the 'kinematical effects' of the constraints, which only allow motions living in Q .

The second condition [5] is the classical *Newton's law* with a right hand side encompassing the possible 'dynamical effects' of the contraints, expressed by an 'unknown' *constraint reaction* $\Phi(t) \in E$.

The third condition [6] expresses the only 'known' empirical requisite of the constraint reaction, which – in absence of any friction (tangent to Q) – is always orthogonal to Q.

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[4] Recall that $\mathrm{p}(t) \in Q\, , \forall\, t \in I \Longleftrightarrow \big(\mathrm{p}(t), \dot{\mathrm{p}}(t)\big) \in TQ\, , \forall\, t \in I$.

[5] Notice that, for any $\mu = (\mu_1, \ldots, \mu_\nu) \in \mathbb{R}^\nu$ and $\mathbf{w} = (\mathbf{w}_1, \ldots, \mathbf{w}_\nu) \in E$, we put $\mu\, \mathbf{w} := (\mu_1 \mathbf{w}_1, \ldots, \mu_\nu \mathbf{w}_n) \in E$.

[6] For any $\mathrm{p} \in Q$, $T^{\perp}_{\mathrm{p}} Q$ denotes the orthogonal complement in E of the tangent vector space $T_{\mathrm{p}} Q$.

1.4 d'Alembert's reformulation

After d'Alembert, the unknown constraint reaction can be 'cancelled' from Newton's law of constrained dynamics as follows.

d'Alembert's principle of virtual works

The last two of the above conditions can obviously be expressed in the form

$$m\,\ddot{\mathrm{p}}(t) - \mathbf{F}(\mathrm{p}(t), \dot{\mathrm{p}}(t)) \in T^{\perp}_{\mathrm{p}(t)}Q$$

that is,

$$\Big(\mathbf{F}(\mathrm{p}(t), \dot{\mathrm{p}}(t)) - m\,\ddot{\mathrm{p}}(t)\Big) \cdot \delta\mathrm{p} = 0\ , \quad \forall\, \delta\mathrm{p} \in T_{\mathrm{p}(t)}Q$$

called *d'Alembert's principle of virtual works* (since the inner product therein defines the *work* of *active force* $\mathbf{F}(\mathrm{p}(t), \dot{\mathrm{p}}(t))$ and *inertial force* $-m\ddot{\mathrm{p}}(t)$ along any virtual displacement $\delta\mathrm{p}$). [7]

So a smooth motion

$$\gamma : I \subset \mathbb{R} \to \mathcal{E} : t \mapsto \gamma(t) = \mathrm{p}(t)$$

is a *DPM* of $\mathcal{S}$, iff, $\forall\, t \in I$,

$$\begin{aligned}&\mathrm{p}(t) \in Q\\ &m\,\ddot{\mathrm{p}}(t) \cdot \delta\mathrm{p} = \mathbf{F}(\mathrm{p}(t), \dot{\mathrm{p}}(t)) \cdot \delta\mathrm{p}\,,\ \forall\, \delta\mathrm{p} \in T_{\mathrm{p}(t)}Q\end{aligned}$$

(*time-evolution law* $(\diamond\diamond)$).

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[7] In 3-dimensional vector formalism, d'Alembert's principle reads

$$\sum_{i=1}^{\nu} \Big(\mathbf{F}_i\big(\mathrm{p}(t), \dot{\mathrm{p}}(t)\big) - m_i \ddot{\mathrm{p}}_i(t)\Big) \cdot \delta\mathrm{p}_i = 0\ , \quad \forall\, \delta\mathrm{p} = (\delta\mathrm{p}_1, \ldots, \delta\mathrm{p}_\nu) \in T_{\mathrm{p}(t)}Q$$

Remark that, if Q is an open submanifold of $\mathcal{E}$ (absence of two-sided constraints), one has $T^{\perp}_{\mathrm{p}}Q = E^{\perp} = \{\mathbf{0}\}$ for all $\mathrm{p} \in Q$ (absence of constraint reaction) and then d'Alembert's principle takes the classical Newtonian form

$$m_i \ddot{\mathrm{p}}_i(t) = \mathbf{F}_i\big(\mathrm{p}(t), \dot{\mathrm{p}}(t)\big)\ , \quad i = 1, \ldots, \nu$$

1.5 d'Alembert's implicit equation

From the mathematical point of view, condition $(\diamond\diamond)$ shows that determining the *DPM*s of $\mathcal{S}$ is a *second-order* differential problem, whose unknown is a smooth curve of $\mathcal{E}$. It turns into a *first-order* differential problem, whose unknown is a smooth curve of $T\mathcal{E}$, as follows.

Tangent dynamically possible motions

If

$$\gamma : I \subset \mathbb{R} \to \mathcal{E} : t \mapsto \gamma(t) = \mathrm{p}(t)$$

is *DPM* of $\mathcal{S}$, its *tangent lift*

$$\dot\gamma : I \subset \mathbb{R} \to T\mathcal{E} : t \longmapsto \dot\gamma(t) = (\mathrm{p}(t), \dot{\mathrm{p}}(t))$$

will be called a *tangent dynamically possible motion* (*TDPM*) of $\mathcal{S}$.

*DPM*s and *TDPM*s bijectively correspond to one another, since the lift operator $\gamma \mapsto \dot\gamma$ is obviously inverted by the projection operator $\dot\gamma \mapsto \gamma$.
Through such a bijection, the problem of determining the *DPM*s proves to be naturally equivalent to that of determining the *TDPM*s.

Owing to $\diamond\diamond$, a smooth curve

$$c : I \subset \mathbb{R} \to T\mathcal{E} : t \longmapsto c(t) = (\mathrm{p}(t), \mathbf{v}(t))$$

is a *TDPM* of $\mathcal{S}$, iff, $\forall\, t \in I$,

$$\begin{aligned}
&\mathrm{p}(t) \in Q \\
&\dot{\mathrm{p}}(t) = \mathbf{v}(t) \\
&m\, \ddot{\mathrm{p}}(t) \cdot \delta\mathrm{p} = \mathbf{F}(\mathrm{p}(t), \dot{\mathrm{p}}(t)) \cdot \delta\mathrm{p}\,,\ \forall\ \delta\mathrm{p} \in T_{\mathrm{p}(t)}Q
\end{aligned}$$

that is,

$$\begin{aligned}
&(\mathrm{p}(t), \mathbf{v}(t)) \in TQ \\
&\dot{\mathrm{p}}(t) = \mathbf{v}(t) \\
&m\, \dot{\mathbf{v}}(t) \cdot \delta\mathrm{p} = \mathbf{F}(\mathrm{p}(t), \mathbf{v}(t)) \cdot \delta\mathrm{p}\,,\ \forall\ \delta\mathrm{p} \in T_{\mathrm{p}(t)}Q
\end{aligned}$$

(*time-evolution law* $(\diamond)$).

d'Alembert equation

Time-evolution law $(\diamond)$ will now be seen to correspond to a first-order implicit differential equation on $T\mathcal{E}$ (second-order on $\mathcal{E}$), namely *d'Alembert equation* [8]

$$\begin{aligned}\mathcal{D}_{d'Al} &:= \{(\mathrm{p},\mathbf{v};\mathbf{u},\mathbf{w}) \in TT\mathcal{E} \mid (\mathrm{p},\mathbf{v}) \in TQ\,,\ \mathbf{u}=\mathbf{v}\,,\ m\,\mathbf{w}-\mathbf{F}(\mathrm{p},\mathbf{v}) \in T_{\mathrm{p}}^{\perp}Q\,\} \\ &= \{(\mathrm{p},\mathbf{v};\mathbf{u},\mathbf{w}) \in TT\mathcal{E} \mid (\mathrm{p},\mathbf{v}) \in TQ\,,\ \mathbf{u}=\mathbf{v}\,,\ m\,\mathbf{w}\cdot\delta\mathrm{p}=\mathbf{F}(\mathrm{p},\mathbf{v})\cdot\delta\mathrm{p}\,, \\ &\qquad \forall\,\delta\mathrm{p} \in T_{\mathrm{p}}Q\,\} \subset T^2\mathcal{E}\end{aligned}$$

Proposition 1 *The* TDPM*s of* $\mathcal{S}$ *are the integral curves of* $\mathcal{D}_{d'Al}$ *(and then the* DPM*s are its base integral curves).*

Proof Recall that a smooth curve

$$c : I \subset \mathbb{R} \to T\mathcal{E} : t \longmapsto c(t) = (\mathrm{p}(t),\mathbf{v}(t))$$

is an integral curve of $\mathcal{D}_{d'Al}$, iff its tangent lift

$$\dot{c} : I \subset \mathbb{R} \to TT\mathcal{E} : t \longmapsto \dot{c}(t) = (\mathrm{p}(t),\mathbf{v}(t);\dot{\mathrm{p}}(t),\dot{\mathbf{v}}(t))$$

satisfies condition

$$\mathrm{Im}(\dot{c}) \subset \mathcal{D}_{d'Al}$$

that is, $\forall\, t \in I$,

$$\dot{c}(t) = (\mathrm{p}(t),\mathbf{v}(t);\dot{\mathrm{p}}(t),\dot{\mathbf{v}}(t)) \in \mathcal{D}_{d'Al}$$

which is exactly time-evolution $(\diamond)$, characterizing the *TDPM*s.

Alternatively, recall that a smooth motion

$$\gamma : I \subset \mathbb{R} \to T\mathcal{E} : t \mapsto \gamma(t) = \mathrm{p}(t)$$

is a base integral curve of $\mathcal{D}_{d'Al}$, iff its second tangent lift

$$\ddot{\gamma} : I \subset \mathbb{R} \to TT\mathcal{E} : t \longmapsto \ddot{\gamma}(t) = (\mathrm{p}(t),\dot{\mathrm{p}}(t);\dot{\mathrm{p}}(t),\ddot{\mathrm{p}}(t))$$

satisfies condition

$$\mathrm{Im}(\ddot{\gamma}) \subset \mathcal{D}_{d'Al}$$

that is, $\forall\, t \in I$,

$$\ddot{\gamma}(t) = (\mathrm{p}(t),\dot{\mathrm{p}}(t);\dot{\mathrm{p}}(t),\ddot{\mathrm{p}}(t)) \in \mathcal{D}_{d'Al}$$

which is exactly time-evolution $(\diamond\diamond)$, characterizing the *DPM*s. □

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[8] Recall that $T\mathcal{E} = \mathcal{E} \times E$ is a Euclidean affine space modelled on $E \times E$. Its tangent bundle is therefore $TT\mathcal{E} = (\mathcal{E} \times E) \times (E \times E)$.

Chapter 2

From d'Alembert to Lagrange

Dynamics is now a problem of *integration*, i.e. determination and/or qualitative analysis of the integral curves of d'Alembert equation (implicit differential equation on Euclidean space $T\mathcal{E}$). In this connection, the latter will be shown to be equivalent to a 'Lagrange equation' (implicit differential equation on manifold TQ), which will naturally be obtained and thoroughly discussed.

2.1 Integrable part of d'Alembert equation

As to the integration of $\mathcal{D}_{d'Al}$, the first step is to extract its *integrable part*, which is the region $\mathcal{D}^{(i)}_{d'Al} \subset \mathcal{D}_{d'Al}$ swept by the tangent lifts of all its integral curves (i.e. covered by the orbits of such lifts).

Restriction of d'Alembert equation

To this end, it is quite natural to start from Proposition 1, owing to which the integral curves (if any) of $\mathcal{D}_{d'Al}$ are constrained to live in TQ [1] and then their tangent lifts live in TTQ , that is to say, $\mathcal{D}^{(i)}_{d'Al} \subset TTQ$.

Hence

$$\mathcal{D}^{(i)}_{d'Al} \subset \mathcal{D}_{d'Al} \cap TTQ$$

The above result suggests focusing on the 'restriction' of $\mathcal{D}_{d'Al}$ obtained via intersection with TTQ , i.e. on the first-order implicit differential equation on TQ (second-order on Q)

$$\mathcal{D}_{Lagr} := \mathcal{D}_{d'Al} \ \cap \ TTQ$$

which will be called *Lagrange equation.*

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[1] See the first condition appearing in time-evolution law $(\diamond)$.

Note that Lagrange equation may be an effective restriction of d'Alembert equation (i.e. $\mathcal{D}_{Lagr} \subsetneqq \mathcal{D}_{d'Al}$) only in presence of two-sided constraints. In fact, in absence of two-sided constraints (when $\dim Q = \dim \mathcal{E}$), for any $(\mathrm{p}, \mathbf{v}; \mathbf{u}, \mathbf{w}) \in \mathcal{D}_{d'Al}$, we have $(\mathrm{p}, \mathbf{v}) \in TQ$, $\mathbf{u} = \mathbf{v}$ and $\mathbf{w} \in E = T^2_{(\mathrm{p},\mathbf{v})}Q$, that is, $(\mathrm{p}, \mathbf{v}; \mathbf{u}, \mathbf{w}) = (\mathrm{p}, \mathbf{v}; \mathbf{v}, \mathbf{w}) \in T^2Q \subset TTQ$ and then $(\mathrm{p}, \mathbf{v}; \mathbf{u}, \mathbf{w}) \in \mathcal{D}_{Lagr}$, which means $\mathcal{D}_{Lagr} = \mathcal{D}_{d'Al}$.

Extraction of the integrable part

Our aim now now is to show that, by extracting $\mathcal{D}_{Lagr}$ from $\mathcal{D}_{d'Al}$ via intersection of the latter with TTQ , we just obtain $\mathcal{D}^{(i)}_{d'Al} \neq \emptyset$.

To this end, we start with the following

Proposition 2 *If* $\mathcal{D}_{Lagr}$ *is integrable, then* $\mathcal{D}_{Lagr} = \mathcal{D}^{(i)}_{d'Al} \neq \emptyset$

Proof First remark that $\mathcal{D}_{d'Al}$ and $\mathcal{D}_{Lagr}$ are equivalent equations, i.e. they have the same integral curves, since inclusions

$$\mathcal{D}^{(i)}_{d'Al} \subset \mathcal{D}_{Lagr} \subset \mathcal{D}_{d'Al}$$

guarantee that condition $\mathrm{Im}(\dot{c}) \subset \mathcal{D}_{d'Al}$ (i.e. $\mathrm{Im}(\dot{c}) \subset \mathcal{D}^{(i)}_{d'Al}$) implies $\mathrm{Im}(\dot{c}) \subset \mathcal{D}_{Lagr}$ and, conversely, $\mathrm{Im}(\dot{c}) \subset \mathcal{D}_{Lagr}$ implies $\mathrm{Im}(\dot{c}) \subset \mathcal{D}_{d'Al}$, whence

$$\mathcal{D}^{(i)}_{d'Al} = \mathcal{D}^{(i)}_{Lagr}$$

Then recall that $\mathcal{D}_{Lagr}$ is integrable, if

$$\emptyset \neq \mathcal{D}_{Lagr} = \mathcal{D}^{(i)}_{Lagr}$$

Hence our claim. □

So the focal point is now to prove the integrability of $\mathcal{D}_{Lagr}$.

That will follow from the stronger property of $\mathcal{D}_{Lagr}$ being reducible to normal form, as will be shown in the sequel.

2.2 Lagrange equation

In order to prove the reducibility of Lagrange equation to normal form, we shall need to give a deeper insight into its algebraic formulation and the underlying geometric structures. Its integral curves will then be given a *global* characterization in terms of the above geometric structures (the traditional *local* characterization in coordinate formalism will finally be deduced).

Covector formulation

From the set-theoretical point of view, Lagrange equation can be expressed in the form

$$\mathcal{D}_{Lagr} = \{(\mathrm{p}, \mathbf{v}; \mathbf{u}, \mathbf{w}) \in TTQ \mid \mathbf{u} = \mathbf{v}\,,\ \ m\,\mathbf{w}\cdot\delta\mathrm{p} = \mathbf{F}(\mathrm{p}, \mathbf{v})\cdot\delta\mathrm{p}\,,\ \forall\,\delta\mathrm{p} \in T_{\mathrm{p}}Q\,\} \subset T^2Q$$

From the algebraic point of view, the condition on virtual works characterizing $\mathcal{D}_{Lagr}$ can be given the form of a *covector* equality, as will now be shown.

Associated with the mass distribution m, there is a semi-basic 1-form

$$[m] : T^2Q \to T^*Q : (\mathrm{p}, \mathbf{v}; \mathbf{v}, \mathbf{w}) \longmapsto (\mathrm{p}, [m](\mathrm{p}, \mathbf{v}, \mathbf{w}))$$

on T^2Q, called *covector inertial field*, whose value [2]

$$[m](\mathrm{p}, \mathbf{v}, \mathbf{w}) := (m\,\mathbf{w}) \cdot |_{T_{\mathrm{p}}Q} \in T^*_{\mathrm{p}}Q$$

at any $(\mathrm{p}, \mathbf{v}; \mathbf{v}, \mathbf{w}) \in T^2Q$ is, up to the sign, the virtual work of the inertial force $-\,m\,\mathbf{w}$.

Associated with the force field $\mathbf{F}$, there is a semi-basic 1-form

$$F : TQ \to T^*Q : (\mathrm{p}, \mathbf{v}) \longmapsto (\mathrm{p}, F(\mathrm{p}, \mathbf{v}))$$

on TQ, called *covector force field*, whose value

$$F(\mathrm{p}, \mathbf{v}) := \mathbf{F}(\mathrm{p}, \mathbf{v}) \cdot |_{T_{\mathrm{p}}Q} \in T^*_{\mathrm{p}}Q$$

at any $(\mathrm{p}, \mathbf{v}) \in TQ$ is the virtual work of the active force $\mathbf{F}(\mathrm{p}, \mathbf{v})$. [3]

Proposition 3 *Lagrange equation can be given the covector formulation*

$$\mathcal{D}_{Lagr} = \{(\mathrm{p}, \mathbf{v}; \mathbf{u}, \mathbf{w}) \in TTQ \mid \mathbf{u} = \mathbf{v}\,,\ \ [m](\mathrm{p}, \mathbf{v}, \mathbf{w}) = F(\mathrm{p}, \mathbf{v})\}$$

Proof Just notice that condition

$$m\,\mathbf{w} \cdot \delta\mathrm{p} = \mathbf{F}(\mathrm{p}, \mathbf{v}) \cdot \delta\mathrm{p}\,,\ \forall\,\delta\mathrm{p} \in T_{\mathrm{p}}Q$$

means

$$(m\,\mathbf{w}) \cdot |_{T_{\mathrm{p}}Q} = \mathbf{F}(\mathrm{p}, \mathbf{v}) \cdot |_{T_{\mathrm{p}}Q}$$

that is,

$$[m](\mathrm{p}, \mathbf{v}, \mathbf{w}) = F(\mathrm{p}, \mathbf{v})$$

That proves our claim. □

---

[2] Recall that, for any $\mathbf{u} \in E$, $\mathbf{u}\cdot \in E^*$. Then, for any $\mathrm{p} \in Q$, the restriction of $\mathbf{u}\cdot$ to $T_{\mathrm{p}}Q$ yields $\mathbf{u} \cdot |_{T_{\mathrm{p}}Q} \in T^*_{\mathrm{p}}Q$.

[3] The virtual work of a positional force field $\mathbf{f}$ can be regarded as an ordinary 1-form on Q, namely $f : \mathrm{p} \in Q \mapsto (\mathrm{p}, f(\mathrm{p})) \in T^*Q\,,\quad f(\mathrm{p}) := \mathbf{f}(\mathrm{p}) \cdot |_{T_{\mathrm{p}}Q} \in T^*_{\mathrm{p}}Q$.

Riemannian geodesic curvature field

The reducibility of $\mathcal{D}_{Lagr}$ to normal form requires that, for any choice of the data $(\mathrm{p}, \mathbf{v}) \in TQ$, the algebraic equation $[m](\mathrm{p}, \mathbf{v}, \mathbf{w}) = F(\mathrm{p}, \mathbf{v})$ should be uniquely solvable with respect to the unknown $\mathbf{w} \in T^2_{(\mathrm{p},\mathbf{v})}Q$. As the latter only appears in the left hand side of the algebraic equation, the above property is to be checked through a thorough investigation of $[m]$. The following geometric considerations –showing that such a semi-basic 1-form on T^2Q is the transformed of a suitable semi-basic 1-form on TQ through a distinguished semi-spray– will prove to be crucial.

Remark that $[m]$ is the semi-basic 1-form 'induced' on T^2Q by the Euclidean metric on E (positive definite, symmetric, linear map)

$$g_m : E \to E^* : \mathbf{u} \longmapsto g_m(\mathbf{u}) := (m\,\mathbf{u})\cdot$$

since, at any $(\mathrm{p}, \mathbf{v}; \mathbf{v}, \mathbf{w}) \in T^2Q$, its value is

$$[m](\mathrm{p}, \mathbf{v}, \mathbf{w}) = g_m(\mathbf{w})\Big|_{T_\mathrm{p}Q} \in T^*_\mathrm{p}Q$$

In the same way, there is a semi-basic 1-form

$$g : TQ \to T^*Q : (\mathrm{p}, \mathbf{v}) \longmapsto (\mathrm{p}, g_\mathrm{p}(\mathbf{v}))$$

induced on TQ by g_m, whose value at any $(\mathrm{p}, \mathbf{v}) \in TQ$ is

$$g_\mathrm{p}(\mathbf{v}) := g_m(\mathbf{v})\Big|_{T_\mathrm{p}Q} \in T^*_\mathrm{p}Q$$

The above Riemannian metric g on Q (positive definite, symmetric, vector bundle morphism) is characterized by the quadratic form –or *homogeneous quadratic Lagrangian function*–

$$K : TQ \to \mathbb{R} : (\mathrm{p}, \mathbf{v}) \mapsto K(\mathrm{p}, \mathbf{v}) := \frac{1}{2}\,\langle g_\mathrm{p}(\mathbf{v}) \mid \mathbf{v}\rangle = \frac{1}{2}\,m\,\mathbf{v}\cdot\mathbf{v}$$

(*kinetic energy* of the mechanical system).

Riemannian manifold (Q, K) carries a distinguished *Riemannian spray*

$$\Gamma_K : TQ \to T^2Q : (\mathrm{p}, \mathbf{v}) \mapsto (\mathrm{p}, \mathbf{v}; \mathbf{v}, \Gamma_K(\mathrm{p}, \mathbf{v}))$$

uniquely determined by the following

Proposition 4 *There exists one, and only one, semi-spray* Γ_K *on* TQ *such that* $g_m(\Gamma_K(\mathrm{p}, \mathbf{v}))\Big|_{T_\mathrm{p}Q} = 0$ *for all* $(\mathrm{p}, \mathbf{v}) \in TQ$.

Proof (*Unicity*) First remark that, for any $(\mathrm{p}, \mathbf{v}) \in TQ$, the map

$$\mathbf{w} \in T^2_{(\mathrm{p},\mathbf{v})}Q \overset{(\alpha)}{\longmapsto} g_m(\mathbf{w})\Big|_{T_\mathrm{p}Q} \in T^*_\mathrm{p}Q$$

is injective, since

$$g_m(\mathbf{w}_1)\Big|_{T_\mathrm{p}Q} = g_m(\mathbf{w}_2)\Big|_{T_\mathrm{p}Q}$$

(with $\mathbf{w}_1, \mathbf{w}_2 \in T^2_{(\mathrm{p},\mathbf{v})}Q$ and then $\mathbf{w}_1 - \mathbf{w}_2 \in T_\mathrm{p}Q$) implies

$$\begin{aligned} g_\mathrm{p}(\mathbf{w}_1 - \mathbf{w}_2) &= g_m(\mathbf{w}_1 - \mathbf{w}_2)\Big|_{T_\mathrm{p}Q} \\ &= g_m(\mathbf{w}_1)\Big|_{T_\mathrm{p}Q} - g_m(\mathbf{w}_2)\Big|_{T_\mathrm{p}Q} \\ &= 0 \end{aligned}$$

that is, recalling that $g_\mathrm{p} : T_\mathrm{p}Q \to T^*_\mathrm{p}Q$ is invertible,

$$\mathbf{w}_1 = \mathbf{w}_2$$

So, if there exists a vector in $T^2_{(\mathrm{p},\mathbf{v})}Q$ whose image through (α) is zero, it is unique.

(*Existence*) For any $(\mathrm{p}, \mathbf{v}) \in TQ$, put

$$\Gamma_K(\mathrm{p}, \mathbf{v}) := \mathbf{w} + \mathbf{u} \in T^2_{(\mathrm{p},\mathbf{v})}Q$$

with

$$\mathbf{w} \in T^2_{(\mathrm{p},\mathbf{v})}Q\ , \quad \mathbf{u} := -\, g_\mathrm{p}^{-1}\Big(g_m(\mathbf{w})\Big|_{T_\mathrm{p}Q} \Big) \in T_\mathrm{p}Q$$

The image of $\Gamma_K(\mathrm{p}, \mathbf{v})$ through (α) is zero, since

$$\begin{aligned} g_m(\Gamma_K(\mathrm{p}, \mathbf{v}))\Big|_{T_\mathrm{p}Q} &= g_m(\mathbf{w} + \mathbf{u})\Big|_{T_\mathrm{p}Q} \\ &= g_m(\mathbf{w})\Big|_{T_\mathrm{p}Q} + g_m(\mathbf{u})\Big|_{T_\mathrm{p}Q} \\ &= g_m(\mathbf{w})\Big|_{T_\mathrm{p}Q} + g_\mathrm{p}(\mathbf{u}) \\ &= g_m(\mathbf{w})\Big|_{T_\mathrm{p}Q} - g_m(\mathbf{w})\Big|_{T_\mathrm{p}Q} \\ &= 0 \end{aligned}$$

That proves our claim. □

Γ_K transforms g (semi-basic 1-form on TQ) into

$$[K] : T^2Q \to T^*Q : (\mathrm{p}, \mathbf{v}; \mathbf{v}, \mathbf{w}) \mapsto (\mathrm{p}, [K](\mathrm{p}, \mathbf{v}, \mathbf{w}))$$

(semi-basic 1-form on T^2Q) by putting, for any $(\mathrm{p}, \mathbf{v}; \mathbf{v}, \mathbf{w}) \in T^2Q$,

$$[K](\mathrm{p}, \mathbf{v}, \mathbf{w}) := g_{\mathrm{p}}(\mathbf{w} - \Gamma_K(\mathrm{p}, \mathbf{v})) \in T^*_{\mathrm{p}}Q$$

$[K]$ will be called *Riemannian geodesic curvature field.*

Actually $[K]$ does not differ from $[m]$, as is shown in the following

Proposition 5 *Lagrange equation, in covector formulation, reads*

$$\mathcal{D}_{Lagr} = \{(\mathrm{p}, \mathbf{v}; \mathbf{u}, \mathbf{w}) \in TTQ \mid \mathbf{u} = \mathbf{v}\,, \;\; [K](\mathrm{p}, \mathbf{v}, \mathbf{w}) = F(\mathrm{p}, \mathbf{v})\}$$

Proof Owing to Proposition 3, it will suffice to show that

$$[K] = [m]$$

To this end, note that, for any $(\mathrm{p}, \mathbf{v}; \mathbf{v}, \mathbf{w}) \in T^2Q$, from Proposition 4 we obtain

$$\begin{aligned} [K](\mathrm{p}, \mathbf{v}, \mathbf{w}) &:= g_{\mathrm{p}}(\mathbf{w} - \Gamma_K(\mathrm{p}, \mathbf{v})) \\ &= g_m(\mathbf{w} - \Gamma_K(\mathrm{p}, \mathbf{v}))\Big|_{T_{\mathrm{p}}Q} \\ &= g_m(\mathbf{w})\Big|_{T_{\mathrm{p}}Q} - g_m(\Gamma_K(\mathrm{p}, \mathbf{v}))\Big|_{T_{\mathrm{p}}Q} \\ &= g_m(\mathbf{w})\Big|_{T_{\mathrm{p}}Q} \\ &= [m](\mathrm{p}, \mathbf{v}, \mathbf{w}) \end{aligned}$$

That proves our claim. □

Normal form

The reducibility of Lagrange equation to normal form immediately follows from Proposition 5.

Consider the *vertical force field*

$$\Delta_F := g^{-1} \circ F : TQ \to TQ : (\mathrm{p}, \mathbf{v}) \mapsto (\mathrm{p}, \Delta_F(\mathrm{p}, \mathbf{v}))$$

$$\Delta_F(\mathrm{p}, \mathbf{v}) := g^{-1}_{\mathrm{p}}(F(\mathrm{p}, \mathbf{v})) \in T_{\mathrm{p}}Q$$

and the semi-spray

$$\Gamma := \Gamma_K + \Delta_F : TQ \to T^2Q : (\mathrm{p}, \mathbf{v}) \mapsto (\mathrm{p}, \mathbf{v}; \mathbf{v}, \Gamma(\mathrm{p}, \mathbf{v}))$$

$$\Gamma(\mathrm{p}, \mathbf{v}) := \Gamma_K(\mathrm{p}, \mathbf{v}) + \Delta_F(\mathrm{p}, \mathbf{v}) \in T^2_{(\mathrm{p}, \mathbf{v})}Q$$

Proposition 6 *Lagrange equation can be put in the normal form*

$$\mathcal{D}_{Lagr} = \operatorname{Im}\Gamma = \{(\mathrm{p}, \mathbf{v}; \mathbf{u}, \mathbf{w}) \in TTQ \mid \mathbf{u} = \mathbf{v}\,, \quad \mathbf{w} = \Gamma(\mathrm{p}, \mathbf{v})\}$$

Proof Just notice that covector equality

$$[K](\mathrm{p}, \mathbf{v}, \mathbf{w}) = F(\mathrm{p}, \mathbf{v})$$

that is,

$$g_{\mathrm{p}}(\mathbf{w} - \Gamma_K(\mathrm{p}, \mathbf{v})) = F(\mathrm{p}, \mathbf{v})$$

also reads

$$\mathbf{w} - \Gamma_K(\mathrm{p}, \mathbf{v}) = g_{\mathrm{p}}^{-1}(F(\mathrm{p}, \mathbf{v}))$$

or

$$\begin{aligned} \mathbf{w} &= \Gamma_K(\mathrm{p}, \mathbf{v})) + \Delta_F(\mathrm{p}, \mathbf{v}) \\ &= \Gamma(\mathrm{p}, \mathbf{v}) \end{aligned}$$

That proves our claim. □

Integral curves

The condition characterizing the integral curves of Lagrange equation can now be formulated as follows.

Let

$$c : I \subset \mathbb{R} \to TQ : t \mapsto c(t) = (\mathrm{p}(t), \mathbf{v}(t))$$

be a smooth curve of TQ and

$$\pi_Q \circ c : I \longrightarrow Q : t \mapsto (\pi_Q \circ c)(t) = \mathrm{p}(t)$$

its projection onto Q. [4]

Proposition 7 *c is an integral curve of $\mathcal{D}_{Lagr}$, iff*

$$(\pi_Q \circ c)^{\bullet} = c\,, \quad [K] \circ \dot{c} = F \circ c \qquad (\circ)$$

or, in normal form,

$$\dot{c} = \Gamma \circ c \qquad (\bullet)$$

---

[4] The tangent lift of $\pi_Q \circ c$ will be denoted by $(\pi_Q \circ c)^{\bullet}$.

Proof As is known, c is an integral curve of $\mathcal{D}_{Lagr}$, iff

$$\mathrm{Im}(\dot{c}) \subset \mathcal{D}_{Lagr}$$

that is to say, $\forall\, t \in I$,

$$(*) \qquad \dot{c}(t) = (\mathrm{p}(t), \mathbf{v}(t); \dot{\mathrm{p}}(t), \dot{\mathbf{v}}(t)) \in \mathcal{D}_{Lagr}$$

Owing to Proposition 5, condition $(*)$ reads

$$\dot{\mathrm{p}}(t) = \mathbf{v}(t)\ , \quad [K](\mathrm{p}(t), \mathbf{v}(t), \dot{\mathbf{v}}(t)) = F(\mathrm{p}(t), \mathbf{v}(t))$$

that is,

$$\begin{aligned} (\pi_Q \circ c)^{\boldsymbol{\cdot}}(t) &= (\mathrm{p}(t), \dot{\mathrm{p}}(t)) \\ &= (\mathrm{p}(t), \mathbf{v}(t)) \\ &= c(t) \end{aligned}$$

and

$$\begin{aligned} ([K] \circ \dot{c})(t) &= (\,\mathrm{p}(t),\ [K](\mathrm{p}(t), \mathbf{v}(t), \dot{\mathbf{v}}(t))\,) \\ &= (\,\mathrm{p}(t),\ F(\mathrm{p}(t), \mathbf{v}(t))\,) \\ &= (F \circ c)(t) \end{aligned}$$

That proves our first claim.

Owing to Proposition 6, condition $(*)$ also reads

$$\dot{\mathrm{p}}(t) = \mathbf{v}(t)\ , \quad \dot{\mathbf{v}}(t) = \Gamma(\mathrm{p}(t), \mathbf{v}(t))$$

that is,

$$\begin{aligned} \dot{c}(t) &= (\mathrm{p}(t), \mathbf{v}(t); \dot{\mathrm{p}}(t), \dot{\mathbf{v}}(t)) \\ &= (\mathrm{p}(t), \mathbf{v}(t); \mathbf{v}(t), \Gamma(\mathrm{p}(t), \mathbf{v}(t))) \\ &= (\Gamma \circ c)(t) \end{aligned}$$

That completes our proof. □

The condition characterizing the base integral curves of Lagrange equation will then be formulated as follows.

Let

$$\gamma : I \subset \mathbb{R} \to Q : t \mapsto \gamma(t) = \mathrm{p}(t)$$

be a smooth curve of Q and

$$[K] \circ \ddot{\gamma} : I \to T^*Q$$

its *Riemannian geodesic curvature*, whose vector (rather than covector) expression is the *covariant derivative* [5]

$$\frac{\nabla \dot{\gamma}}{dt} := g^{-1} \circ [K] \circ \ddot{\gamma} : I \to TQ$$

also defined by [6]

$$\frac{\nabla \dot{\gamma}}{dt} := \ddot{\gamma} - \Gamma_K \circ \dot{\gamma} : I \to TQ$$

Proposition 8 *γ is a base integral curve of $\mathcal{D}_{Lagr}$, iff*

$$[K] \circ \ddot{\gamma} = F \circ \dot{\gamma} \tag{$\circ\circ$}$$

or, in vector form,

$$\frac{\nabla \dot{\gamma}}{dt} = \Delta_F \circ \dot{\gamma}$$

or, in normal form,

$$\ddot{\gamma} = \Gamma \circ \dot{\gamma} \tag{$\bullet\bullet$}$$

---

[5] The covariant derivative is also related to an important geometric structure, called *Levi-Civita connection* of Riemannian manifold (Q, K).

[6] Notice that

$$g^{-1} \circ [K] \circ \ddot{\gamma} = \ddot{\gamma} - \Gamma_K \circ \dot{\gamma}$$

since

$$\begin{aligned} g^{-1} \circ [K] \circ \ddot{\gamma} : t \in I \quad & \xmapsto{\ddot{\gamma}} \quad \Big(\mathrm{p}(t), \dot{\mathrm{p}}(t); \dot{\mathrm{p}}(t), \ddot{\mathrm{p}}(t)\Big) \in T^2Q \\ & \xmapsto{[K]} \quad \Big(\mathrm{p}(t),\ g_{\mathrm{p}(t)}\,(\ddot{\mathrm{p}}(t) - \Gamma_K(\mathrm{p}(t), \dot{\mathrm{p}}(t)))\Big) \in T^*Q \\ & \xmapsto{g^{-1}} \quad \Big(\mathrm{p}(t),\ \ddot{\mathrm{p}}(t) - \Gamma_K(\mathrm{p}(t), \dot{\mathrm{p}}(t))\Big) \in TQ \end{aligned}$$

and

$$\begin{aligned} \ddot{\gamma} - \Gamma_K \circ \dot{\gamma} : t \in I \quad & \mapsto \quad \Big(\mathrm{p}(t), \dot{\mathrm{p}}(t); \dot{\mathrm{p}}(t), \ddot{\mathrm{p}}(t)\Big) - \Big(\mathrm{p}(t), \dot{\mathrm{p}}(t); \dot{\mathrm{p}}(t), \Gamma_K(\mathrm{p}(t), \dot{\mathrm{p}}(t))\Big) \\ & = \quad \Big(\mathrm{p}(t), \ddot{\mathrm{p}}(t) - \Gamma_K(\mathrm{p}(t).\dot{\mathrm{p}}(t))\Big) \in TQ \end{aligned}$$

Proof Recall that a base integral curve of Lagrange equation is the projection $\gamma = \pi_Q \circ c$ of an integral curve c.

Now, if γ is a base integral curve, conditions ($\circ$) imply $\dot{\gamma} = (\pi_Q \circ c)^{\boldsymbol{\cdot}} = c$, whence $\ddot{\gamma} = \dot{c}$, and then ($\circ\circ$). Conversely, if γ satisfies condition ($\circ\circ$), then it is obviously a base integral curve, namely the projection of $c := \dot{\gamma}$ satisfying conditions ($\circ$). Clearly, condition ($\circ\circ$) is equivalent to $g^{-1} \circ [K] \circ \ddot{\gamma} = g^{-1} \circ F \circ \dot{\gamma}$, which is the above mentioned vector formulation.

In the same way, through condition ($\bullet$), one can show that the base integral curves are characterized by ($\bullet\bullet$). Alternatively, remark that the vector formulation of condition ($\circ\circ$) also reads

$$\ddot{\gamma} - \Gamma_K \circ \dot{\gamma} = \Delta_F \circ \dot{\gamma}$$

which is condition ($\bullet\bullet$). □

From the dynamical point of view, some remarks are now in order.

For $F = 0$, the base integral curves –characterized by a vanishing Riemannian geodesic curvature $[K] \circ \ddot{\gamma} = 0$– coincide with the inertial motions of $\mathcal{S}$ (i.e. the motions which would be possible if $\mathbf{F}$ were zero).

The effect of a covector force field $F \neq 0$ is then that of deviating the *DPM*s of $\mathcal{S}$ from the inertial trend, by giving them a non-vanishing Riemannian geodesic curvature, namely $[K] \circ \ddot{\gamma} = F \circ \dot{\gamma}$.

Classical Lagrange equations

The scalar equations obtained, with the aid of a given chart on Q, by orderly equalling the components of the covector or vector-valued functions which appear in the left and right hand sides of ($\circ$) ($\circ\circ$) or ($\bullet$) ($\bullet\bullet$), are the classical 'Lagrange equations' of Analytical Dynamics. Clearly, such equations will prove to be only *locally* equivalent to the geometric Lagrange equation, in the sense that they will only characterize the integral curves of the latter whose projections live in the coordinate domain of the chart.

Preliminaries

Let

$$\xi : W \to \mathcal{U} : q \mapsto \xi(q)$$

be a chart of Q, expressing the points $\mathrm{p} \in \mathcal{U} \subset Q$ in function of coordinates $q \in W \subset \mathbb{R}^n$, with $n := \dim(Q)$.

To any smooth curve

$$\gamma : I \subset \mathbb{R} \to \mathcal{U} \subset Q : t \mapsto \mathrm{p} = \mathrm{p}(t)$$

living in the coordinate domain $\mathcal{U}$ of ξ, i.e. satisfying

$$\mathrm{p} = \mathrm{p}(t) \in \mathcal{U}$$

for all $t \in I$, there corresponds in ξ a smooth coordinate expression

$$q = q(t) \in W$$

related to γ by $\mathrm{p}(t) = \xi(q(t))$, also denoted

$$\mathrm{p} = \xi(q) \tag{1}$$

(dependence on time t is understood).

To the first tangent lift $c = \dot{\gamma}$, that is,

$$\mathrm{p} = \mathrm{p}(t) \in \mathcal{U} \ , \ \ \mathbf{v} = \mathbf{v}(t) \in T_{\mathrm{p}(t)}Q$$

with

$$\mathbf{v} = \dot{\mathrm{p}}$$

there corresponds in ξ a smooth coordinate expression

$$q = q(t) \in W \ , \ \ v = v(t) \in \mathbb{R}^n$$

related to $\dot{\gamma}$ by (1) and the time derivative of (1), i.e.

$$\mathbf{v} = \mathbf{v}(q, v) := v^k \frac{\partial \mathrm{p}}{\partial q^k}\Big|_q \tag{2}$$

where

$$v = \dot{q}$$

is the n-tuple of linear components of $\mathbf{v}$ in ξ. [7] Remark that, for all $h = 1, \dots, n$,

$$\frac{\partial \mathbf{v}}{\partial v^h}\Big|_{(q,v)} = \frac{\partial \mathrm{p}}{\partial q^h}\Big|_q \ , \quad \frac{\partial \mathbf{v}}{\partial q^h}\Big|_{(q,v)} = v^k \frac{\partial^2 \mathrm{p}}{\partial q^h \partial q^k}\Big|_q = \frac{d}{dt}\frac{\partial \mathrm{p}}{\partial q^h}\Big|_q \tag{3}$$

---

[7] Recall that a repeated index, in upper and lower position, denotes summation over $(1, \dots, n)$ and the above partial derivatives are a basis of $T_{\mathrm{p}}Q$.

To the second tangent lift $\dot{c} = \ddot{\gamma}$, that is,

$$\mathrm{p} = \mathrm{p}(t) \in \mathcal{U} \ , \quad \mathbf{v} = \mathbf{v}(t) \in T_{\mathrm{p}(t)}Q \ , \quad \mathbf{w} = \mathbf{w}(t) \in T^2_{(\mathrm{p}(t),\mathbf{v}(t))}Q$$

with

$$\mathbf{v} = \dot{\mathrm{p}} \ , \quad \mathbf{w} = \dot{\mathbf{v}} = \ddot{\mathrm{p}}$$

there corresponds in ξ a smooth coordinate expression

$$q = q(t) \in W \ , \quad v = v(t) \in \mathbb{R}^n \ , \quad w = w(t) \in \mathbb{R}^n$$

related to $\ddot{\gamma}$ by (1), (2) and the time derivative of (2), i.e.

$$\mathbf{w} = \mathbf{w}(q, v, w) := w^k \frac{\partial \mathrm{p}}{\partial q^k}\Big|_q + v^h v^k \frac{\partial^2 \mathrm{p}}{\partial q^h \partial q^k}\Big|_q$$

where

$$w = \dot{v} = \ddot{q}$$

is the n-tuple of affine components of $\mathbf{w}$ in ξ.

In the sequel, the components of $F \circ \dot{\gamma}$ in ξ will be denoted by

$$F_h(q, v) := (F \circ \dot{\gamma})_h = (F(\mathrm{p}, \mathbf{v}))_h = \left\langle F(\mathrm{p}, \mathbf{v}) \ \Big| \ \frac{\partial \mathrm{p}}{\partial q^h}\Big|_q \right\rangle = \mathbf{F}(\mathrm{p}, \mathbf{v}) \cdot \frac{\partial \mathrm{p}}{\partial q^h}\Big|_q$$

and the components of $\Delta_F \circ \dot{\gamma} = g^{-1} \circ F \circ \dot{\gamma}$ will be denoted by

$$F^i(q, v) := (\Delta_F \circ \dot{\gamma})^i = (\Delta_F(\mathrm{p}, \mathbf{v}))^i = (g_{\mathrm{p}}^{-1}(F(\mathrm{p}, \mathbf{v})))^i = g^{ih}(q)\,(F(\mathrm{p}, \mathbf{v}))_h = g^{ih}(q)\,F_h(q, v)$$

where $[g^{ih}(q)]$ is the inverse of the nonsingular matrix $[g_{hk}(q)]$ defined by

$$g_{hk}(q) := \left\langle g_{\mathrm{p}}\Big(\frac{\partial \mathrm{p}}{\partial q^h}\Big|_q\Big) \ \Big| \ \frac{\partial \mathrm{p}}{\partial q^k}\Big|_q \right\rangle = m\, \frac{\partial \mathrm{p}}{\partial q^h}\Big|_q \cdot \frac{\partial \mathrm{p}}{\partial q^k}\Big|_q$$

Lagrange equations

The above coordinate formalism will now be adopted for the characterization of the integral curves of $\mathcal{D}_{Lagr}$ whose projections (or base integral curves) live in the given coordinate domain.

Proposition 9 *A smooth curve* c *of* TQ *– with projection* $\gamma = \pi_Q \circ c$ *living in the coordinate domain of a chart* ξ *of* Q *– is an integral curve of* $\mathcal{D}_{Lagr}$ *, iff its natural coordinate expression* $(q, v) = (q(t), v(t))$ *in* ξ *satisfies equations*

$$\dot{q}^h = v^h$$

and the classical Lagrange equations

$$\frac{d}{dt}\frac{\partial K}{\partial v^h}\bigg|_{(q,v)} - \frac{\partial K}{\partial q^h}\bigg|_{(q,v)} = F_h(q, v)$$

or, in normal form, [8]

$$\dot{v}^i = -\left\{ \begin{matrix} i \\ jk \end{matrix} \right\}_q v^j v^k + F^i(q, v)$$

for all $h, i = 1, \ldots, n$ *.*

Proof It will suffice to prove that a smooth curve γ, living in the coordinate domain of ξ, is a base integral curve of $\mathcal{D}_{Lagr}$, iff its coordinate expression $q = q(t)$ in ξ, together with $v = v(t) = \dot{q}(t)$, satisfies Lagrange equations.

Recall that γ is a base integral curve of $\mathcal{D}_{Lagr}$, iff it satisfies (∘∘).

As γ lives in the coordinate domain of ξ, equation (∘∘) is equivalent to the n scalar equations obtained by orderly equalling the components in ξ of its left and right hand sides, i.e.

$$([K] \circ \ddot{\gamma})_h = (F \circ \dot{\gamma})_h$$

The components $F_h(q, v) := (F \circ \dot{\gamma})_h$ have already been shown in the preliminaries (where we have put $v = \dot{q}$).

The components $([K] \circ \ddot{\gamma})_h$ will now be evaluated.

---

[8] Here we encounter the *Christhoffel symbols* (of the Levi-Civita connection) associated with a Riemannian manifold, which are defined on the domain W of ξ by

$$\left\{ \begin{matrix} i \\ jk \end{matrix} \right\} := \frac{1}{2} g^{ih}(\partial_j g_{kh} + \partial_k g_{hj} - \partial_h g_{jk})$$

with

$$\partial_i g_{hk} := \frac{\partial g_{hk}}{\partial q^i}$$

To this end, by making use of (3) and usual rules of derivation, we obtain

$$\begin{aligned}
([K]\circ\ddot{\gamma})_h &= ([K](\mathrm{p},\mathbf{v},\mathbf{w}))_h \\
&= \left\langle [K](\mathrm{p},\mathbf{v},\mathbf{w}) \mid \left.\frac{\partial \mathrm{p}}{\partial q^h}\right|_q \right\rangle \\
&= m\mathbf{w}(q,v,w)\cdot \left.\frac{\partial \mathrm{p}}{\partial q^h}\right|_q \\
&= \frac{d}{dt}\left(m\mathbf{v}(q,v)\cdot \left.\frac{\partial \mathrm{p}}{\partial q^h}\right|_q \right) - m\mathbf{v}(q,v)\cdot \frac{d}{dt}\left.\frac{\partial \mathrm{p}}{\partial q^h}\right|_q \\
&= \frac{d}{dt}\left(m\mathbf{v}(q,v)\cdot \left.\frac{\partial \mathbf{v}}{\partial v^h}\right|_{(q,v)} \right) - m\mathbf{v}(q,v)\cdot \left.\frac{\partial \mathbf{v}}{\partial q^h}\right|_{(q,v)} \\
&= \frac{d}{dt}\left.\frac{\partial}{\partial v^h}\right|_{(q,v)} \left(\frac{1}{2}m\mathbf{v}\cdot\mathbf{v}\right) - \left.\frac{\partial}{\partial q^h}\right|_{(q,v)} \left(\frac{1}{2}m\mathbf{v}\cdot\mathbf{v}\right) \\
&= \frac{d}{dt}\left.\frac{\partial K}{\partial v^h}\right|_{(q,v)} - \left.\frac{\partial K}{\partial q^h}\right|_{(q,v)}
\end{aligned}$$

Hence our first claim.

Now recall that γ is a base integral curve of $\mathcal{D}_{Lagr}$, iff it satisfies (●●).

As γ lives in the coordinate domain of ξ, equation (●●) is equivalent to the n scalar equations obtained by orderly equalling the affine components in ξ of its left and right hand sides, i.e.

$$\ddot{\gamma}^i = (\Gamma\circ\dot{\gamma})^i$$

There we have

$$\begin{aligned}
\ddot{\gamma}^i &= w^i \\
&= \dot{v}^i
\end{aligned}$$

and

$$\begin{aligned}
(\Gamma\circ\dot{\gamma})^i &= (\Gamma_K\circ\dot{\gamma} + \Delta_F\circ\dot{\gamma})^i \\
&= (\Gamma_K\circ\dot{\gamma})^i + (\Delta_F\circ\dot{\gamma})^i \\
&= \Gamma^i_K(q,v) + F^i(q,v)
\end{aligned}$$

The components $F^i(q,v) := (\Delta_F\circ\dot{\gamma})^i$ have already been shown in the preliminaries.

The components $\Gamma^i_K(q,v) := (\Gamma_K\circ\dot{\gamma})^i$ will now be evaluated.

To this end we need the coordinate expression of K, which is given by

$$\begin{aligned} K(\mathrm{p},\mathbf{v}) &= \frac{1}{2}\left\langle\, g_{\xi(q)}\Big(v^h \frac{\partial \mathrm{p}}{\partial q^h}\Big|_q\Big) \;\Big|\; v^k \frac{\partial \mathrm{p}}{\partial q^k}\Big|_q \right\rangle \\ &= \frac{1}{2}\, g_{hk}(q)\, v^h v^k \end{aligned}$$

whence

$$\begin{aligned} \frac{\partial K}{\partial q^h}\Big|_{(q,v)} &= \frac{1}{2}(\partial_h g_{jk})_q\, v^j v^k \\ \frac{\partial K}{\partial v^h}\Big|_{(q,v)} &= g_{hk}(q) v^k \end{aligned}$$

and

$$\begin{aligned} \frac{\partial^2 K}{\partial v^h \partial q^k}\Big|_{(q,v)} &= (\partial_k g_{hj})_q\, v^j \\ \frac{\partial^2 K}{\partial v^h \partial v^k}\Big|_{(q,v)} &= g_{hk}(q) \end{aligned}$$

Owing to the above calculations, the components $([K]\circ\ddot{\gamma})_h$ can be more explicitly expressed in the form

$$\begin{aligned} ([K]\circ\ddot{\gamma})_h &= ([K](\mathrm{p},\mathbf{v},\mathbf{w}))_h \\ &= \frac{\partial^2 K}{\partial v^h \partial v^k}\Big|_{(q,v)} w^k + \frac{\partial^2 K}{\partial v^h \partial q^k}\Big|_{(q,v)} v^k - \frac{\partial K}{\partial q^h}\Big|_{(q,v)} \\ &= g_{hk}(q) w^k + (\partial_k g_{hj})_q\, v^j v^k - \frac{1}{2}(\partial_h g_{jk})_q\, v^j v^k \\ &= g_{hk}(q) w^k + \frac{1}{2}(\partial_k g_{hj})_q\, v^j v^k + \frac{1}{2}(\partial_j g_{hk})_q\, v^j v^k - \frac{1}{2}(\partial_h g_{jk})_q\, v^j v^k \\ &= g_{hk}(q) w^k + \frac{1}{2}(\partial_k g_{hj} + \partial_j g_{kh} - \partial_h g_{jk})_q\, v^j v^k \end{aligned}$$

whence

$$\begin{aligned} \left(\frac{\nabla\dot{\gamma}}{dt}\right)^i &= (g^{-1}\circ([K]\circ\ddot{\gamma}))^i \\ &= (g_{\mathrm{p}}^{-1}([K](\mathrm{p},\mathbf{v},\mathbf{w})))^i \\ &= g^{ih}(q)\,([K](\mathrm{p},\mathbf{v},\mathbf{w}))_h \\ &= w^i + \left\{ \begin{matrix} i \\ jk \end{matrix} \right\}_q v^j v^k \end{aligned}$$

As a consequence, the identities

$$\begin{aligned}\left(\frac{\nabla\dot\gamma}{dt}\right)^i &= (\ddot\gamma - \Gamma_K \circ \dot\gamma)^i \\ &= \ddot\gamma^i - (\Gamma_K \circ \dot\gamma)^i\end{aligned}$$

read

$$w^i + \left\{ \begin{matrix} i \\ jk \end{matrix} \right\}_q v^j v^k = w^i - \Gamma^i_K(q, v)$$

and then

$$\Gamma^i_K(q, v) = -\left\{ \begin{matrix} i \\ jk \end{matrix} \right\}_q v^j v^k$$

Hence our second claim. □

2.3 Euler-Lagrange equation

A special mention, for its primary role in both mathematical and theoretical physics, is to be given to the dynamics of a 'conservative system'.

Conservative system

Such a name refers to a system $\mathcal{S} = (Q, m, \mathbf{f})$ carrying a *conservative force field* $\mathbf{f}$, i.e. a positional force field whose virtual work

$$f = -dV$$

is an exact 1-form, deriving from a smooth *potential energy*

$$V : Q \to \mathbb{R}$$

(determined up to a locally constant function).

The attribute 'conservative' is due to the 'conservation law' of *mechanical energy*

$$\mathrm{E} := K + V : TQ \to \mathbb{R} : (\mathrm{p}, \mathbf{v}) \mapsto \mathrm{E}(\mathrm{p}, \mathbf{v}) := K(\mathrm{p}, \mathbf{v}) + V(\mathrm{p})$$

(kinetic energy *plus* potential energy) shown in the following

Proposition 10 *Along each integral curve* c *of Lagrange equation*

$$\mathcal{D}_{Lagr} = \{(\mathrm{p}, \mathbf{v}; \mathbf{u}, \mathbf{w}) \in TTQ \mid \mathbf{u} = \mathbf{v}\,,\ \ [K](\mathrm{p}, \mathbf{v}, \mathbf{w}) = -d_{\mathrm{p}}V\}$$

E *keeps constant, i.e.*

$$\mathrm{E} \circ c = \text{const.}$$

Proof If $c = (\mathrm{p}, \mathbf{v})$ denotes an integral curve (dependence on time t is understood) and $\mathbf{w} := \dot{\mathbf{v}}$, from $\dot{\mathrm{p}} = \mathbf{v}$ and $[K](\mathrm{p}, \mathbf{v}, \mathbf{w}) = -d_{\mathrm{p}}V$ it follows that

$$\begin{aligned}
\frac{d}{dt}(\mathrm{E} \circ c) &= \frac{d}{dt}\mathrm{E}(\mathrm{p}, \mathbf{v}) \\
&= \frac{d}{dt}K(\mathrm{p}, \dot{\mathrm{p}}) + \frac{d}{dt}V(\mathrm{p}) \\
&= \frac{d}{dt}K(\mathrm{p}, \mathbf{v}) + \frac{d}{dt}V(\mathrm{p}) \\
&= \frac{d}{dt}(\frac{1}{2}m\,\mathbf{v} \cdot \mathbf{v}) + \frac{d}{dt}V(\mathrm{p}) \\
&= m\,\mathbf{w} \cdot \mathbf{v} + \frac{d}{dt}V(\mathrm{p}) \\
&= \langle [K](\mathrm{p}, \mathbf{v}, \mathbf{w}) \mid \mathbf{v} \rangle + \langle d_{\mathrm{p}}V \mid \mathbf{v} \rangle \\
&= -\langle d_{\mathrm{p}}V \mid \mathbf{v} \rangle + \langle d_{\mathrm{p}}V \mid \mathbf{v} \rangle \\
&= 0
\end{aligned}$$

Hence our claim. □

Lagrangian geodesic curvature field

In conservative dynamics, the kinetic energy and the potential energy, which are the two ingredients 'generating' $\mathcal{D}_{Lagr}$, can be merged into a unique object as follows.

Define a new *non-homogeneous quadratic Lagrangian function* by putting

$$\mathbb{L} := K - V : TQ \to \mathbb{R} : (\mathrm{p}, \mathbf{v}) \mapsto \mathbb{L}(\mathrm{p}, \mathbf{v}) := K(\mathrm{p}, \mathbf{v}) - V(\mathrm{p})$$

(kinetic energy *minus* potential energy).

Associated with $\mathbb{L}$, there is a *Lagrangian geodesic curvature field* given by

$$[\mathbb{L}] := [K] + dV : T^2Q \to T^*Q : (\mathrm{p}, \mathbf{v}; \mathbf{v}, \mathbf{w}) \mapsto (\mathrm{p}, [L](\mathrm{p}, \mathbf{v}, \mathbf{w}))$$

$$[\mathbb{L}](\mathrm{p}, \mathbf{v}, \mathbf{w}) := [K](\mathrm{p}, \mathbf{v}, \mathbf{w}) + d_{\mathrm{p}}V \in T^*_{\mathrm{p}}Q$$

(Riemannian geodesic curvature field *minus* conservative field).

From the above definitions, it follows that

$$\mathcal{D}_{Lagr} = \mathbb{D}_{Eul-Lagr}$$

that is, $\mathcal{D}_{Lagr}$ does not differ from the *Euler-Lagrange equation*

$$\mathbb{D}_{Eul-Lagr} := \{(\mathrm{p}, \mathbf{v}; \mathbf{u}, \mathbf{w}) \in TTQ \mid \mathbf{u} = \mathbf{v}\,, \;\; [\mathbb{L}](\mathrm{p}, \mathbf{v}, \mathbf{w}) = 0\}$$

'generated' by $\mathbb{L}$.

Integral curves

The conditions characterizing the integral curves and the base integral curves of Euler-Lagrange equation will now be formulated.

Proposition 11 *A smooth curve c of TQ is an integral curve of $\mathbb{D}_{Eul-Lagr}$, iff*

$$(\pi_Q \circ c)^{\bullet} = c\,, \quad [\mathbb{L}] \circ \dot{c} = 0 \qquad (\diamond)$$

Proof The same as the proof of Proposition 7. □

Proposition 12 *A smooth curve γ of Q is a base integral curve of $\mathbb{D}_{Eul-Lagr}$, iff its* Lagrangian geodesic curvature *vanishes, i.e.*

$$[\mathbb{L}] \circ \ddot{\gamma} = 0 \qquad (\diamond\diamond)$$

Proof The same as the proof of Proposition 8. □

Classical Euler-Lagrange equations

The coordinate formalism will now be adopted for the characterization of the integral curves of $\mathbb{D}_{Eul-Lagr}$ whose projections live in the coordinate domain of a given chart of Q . Such a characterization, obtained by orderly equalling the components of the left and right hand sides of $(\diamond)$ $(\diamond\diamond)$, will result in the classical 'Euler-Lagrange equations' of Analytical Dynamics.

Proposition 13 *A smooth curve c of TQ – with projection $\gamma = \pi_Q \circ c$ living in the coordinate domain of a chart ξ of Q – is an integral curve of $\mathbb{D}_{Eul-Lagr}$, iff its natural coordinate expression $(q, v) = (q(t), v(t))$ in ξ satisfies equations*

$$\dot{q}^h = v^h$$

and the classical Euler-Lagrange equations

$$\frac{d}{dt} \frac{\partial \mathbb{L}}{\partial v^h}\Big|_{(q,v)} - \frac{\partial \mathbb{L}}{\partial q^h}\Big|_{(q,v)} = 0$$

for all $h = 1, \ldots, n$.

Proof It will suffice to prove that a smooth curve γ, living in the coordinate domain of ξ, is a base integral curve of $\mathbb{D}_{Eul-Lagr}$, iff its coordinate expression $q = q(t)$ in ξ, together with $v = v(t) = \dot{q}(t)$, satisfies Lagrange equations.

Recall that γ is a base integral curve of $\mathbb{D}_{Eul-Lagr}$, iff it satisfies $(\diamond\diamond)$.

As γ lives in the coordinate domain of ξ, equation $(\diamond\diamond)$ is equivalent to the n scalar equations obtained by equalling to zero the components in ξ of its left hand side, i.e.

$$([\mathbb{L}] \circ \ddot{\gamma})_h = 0$$

The above components are expressed, in coordinate formalism, by

$$\begin{aligned}
([\mathbb{L}] \circ \ddot{\gamma})_h &:= ([\mathbb{L}](\mathrm{p}, \mathbf{v}, \mathbf{w}))_h \\
&= \left\langle [\mathbb{L}](\mathrm{p}, \mathbf{v}, \mathbf{w}) \;\Big|\; \frac{\partial \mathrm{p}}{\partial q^h}\Big|_q \right\rangle \\
&= \left\langle [K](\mathrm{p}, \mathbf{v}, \mathbf{w}) \;\Big|\; \frac{\partial \mathrm{p}}{\partial q^h}\Big|_q \right\rangle + \left\langle d_{\mathrm{p}} V \;\Big|\; \frac{\partial \mathrm{p}}{\partial q^h}\Big|_q \right\rangle \\
&= \frac{d}{dt} \frac{\partial K}{\partial v^h}\Big|_{(q,v)} - \frac{\partial K}{\partial q^h}\Big|_{(q,v)} + \frac{\partial V}{\partial q^h}\Big|_q \\
&= \frac{d}{dt} \frac{\partial (K - V)}{\partial v^h}\Big|_{(q,v)} - \frac{\partial (K - V)}{\partial q^h}\Big|_{(q,v)} \\
&= \frac{d}{dt} \frac{\partial \mathbb{L}}{\partial v^h}\Big|_{(q,v)} - \frac{\partial \mathbb{L}}{\partial q^h}\Big|_{(q,v)}
\end{aligned}$$

That proves our claim. □

Remark For both a homogeneous and a non-homogeneous quadratic Lagrangian function $\mathbb{L}$, the semi-basic 1-form

$$[\mathbb{L}] : T^2 Q \to T^* Q$$

is a map which takes a vector $\mathbf{w} \in T^2_{(\mathrm{p},\mathbf{v})} Q$ to the covector $[\mathbb{L}](\mathrm{p}, \mathbf{v}, \mathbf{w}) \in T^*_{\mathrm{p}} Q$ characterized, in any chart near p, by components

$$([\mathbb{L}](\mathrm{p}, \mathbf{v}, \mathbf{w}))_h = \frac{\partial^2 \mathbb{L}}{\partial v^h \partial v^k}\Big|_{(q,v)} w^k + \frac{\partial^2 \mathbb{L}}{\partial v^h \partial q^k}\Big|_{(q,v)} v^k - \frac{\partial \mathbb{L}}{\partial q^h}\Big|_{(q,v)}$$

Such a coordinate technique can be used to define $[\mathbb{L}]$ for an *arbitrary* Lagrangian function $\mathbb{L} : U \subset TQ \to \mathbb{R}$ (smooth on an open subset U of TQ), since –also in this case– the value $[\mathbb{L}](\mathrm{p}, \mathbf{v}, \mathbf{w})$ (for any $(\mathrm{p}, \mathbf{v}) \in U$), defined by the above components $([\mathbb{L}](\mathrm{p}, \mathbf{v}, \mathbf{w}))_h$ in a given chart near p, turns out to be an 'invariant' covector, i.e. it does not depend on the choice of the chart. [9]

---

[9] Actually, through higher geometric methods, $[\mathbb{L}]$ can be given a 'coordinate-free' definition.

Chapter 3

From Lagrange to Hamilton

Two mathematical features of conservative dynamics (of crucial importance both in mathematical and theoretical physics) will now be shown.

On the one hand, Euler-Lagrange equation – supported by the *velocity phase space* TQ – will be taken into a symplectic arena – the *momentum phase space* T^*Q – supporting an equivalent equation, which (introduced with the aid of the canonical symplectic structure of T^*Q) directly arises in a very special normal form (*Hamilton's differential equation*).

On the other hand, Euler-Lagrange equation will be taken into the wide range of variational theories, where it will prove to be equivalent to the variational principle of stationary action (*Hamilton's variational principle*).

3.1 Legendre transformation

The classical transition from velocity to momentum phase space is provided by the well known map which takes any *virtual velocity* $(\mathrm{p}, \mathbf{v}) \in TQ$ onto the corresponding *kinetic momentum* $(\mathrm{p}, m\,\mathbf{v} \cdot |_{T_\mathrm{p}Q}) \in T^*Q$.

Lagrangian function and Legendre transformation

The above map is indeed the vector bundle isomorphism

$$g : TQ \to T^*Q : (\mathrm{p}, \mathbf{v}) \mapsto (\mathrm{p}, g_\mathrm{p}(\mathbf{v}))\ , \quad g_\mathrm{p}(\mathbf{v}) := m\,\mathbf{v} \cdot |_{T_\mathrm{p}Q}$$

owing to which the configuration space of the system has been given the structure of a Riemannian manifold (Q, K) . [1]

---

[1] See section 2.2, *Riemannian geodesic curvature field.*

Such an isomorphism is also said to be the *Legendre transformation* determined by Lagrangian function $\mathbb{L} = K - V$ (in terms of which it can be expressed [2]) and is denoted by

$$F\mathbb{L} : TQ \to T^*Q : (\mathrm{p}, \mathbf{v}) \mapsto (\mathrm{p}, F_{\mathrm{p}}\mathbb{L}(\mathbf{v})) \,, \quad F_{\mathrm{p}}\mathbb{L}(\mathbf{v}) := g_{\mathrm{p}}(\mathbf{v})$$

The fact of $F\mathbb{L}$ being a bundle morphism, is expressed by

$$\pi^*_Q \circ F\mathbb{L} = \pi_Q$$

Legendre transformation in coordinate formalism

Recall that T^*Q is a $2n$-dimensional manifold, where each element (p, π) is completely characterized, in a chart ξ of Q, by two n-tuples of natural coordinates (q, p), namely the coordinates $q = (q^1, \dots, q^n)$ of $\mathrm{p} = \xi(q)$, and the components $p = (p_1, \dots, p_n)$ of π, given by $p_h = \left\langle \pi \mid \frac{\partial \mathrm{p}}{\partial q^h}\Big|_q \right\rangle$.

In natural coordinate formalism, Legendre transformation

$$(\mathrm{p}, \mathbf{v}) \mapsto (\mathrm{p}, \pi) \,, \quad \pi = F_{\mathrm{p}}\mathbb{L}(\mathbf{v})$$

is expressed by [3]

$$(q, v) \mapsto (q, p) \,, \quad p_h = g_{hk}(q)\, v^k = \frac{\partial \mathbb{L}}{\partial v^h}\Big|_{(q,v)} \tag{1}$$

The inverse transformation

$$(\mathrm{p}, \pi) \mapsto (\mathrm{p}, \mathbf{v}) \,, \quad \mathbf{v} = (F_{\mathrm{p}}\mathbb{L})^{-1}(\pi)$$

is then expressed by

$$(q, p) \mapsto (q, v) \,, \quad v^h = g^{hk}(q)\, p_k =: v^h(q, p)$$

---

[2] See the next subsection *Legendre transformation in coordinate formalism.*

[3] Recall that

$$p_h = \left\langle g_{\mathrm{p}}(\mathbf{v}) \mid \frac{\partial \mathrm{p}}{\partial q^h}\Big|_q \right\rangle = \left\langle g_{\xi(q)} \left(\frac{\partial \mathrm{p}}{\partial q^k}\Big|_q \right) \mid \frac{\partial \mathrm{p}}{\partial q^h}\Big|_q \right\rangle v^k = g_{hk}(q)\, v^k = \frac{\partial K}{\partial v^h}\Big|_{(q,v)} = \frac{\partial \mathbb{L}}{\partial v^h}\Big|_{(q,v)}$$

Remark Such a coordinate technique can be used to define a *Legendre morphism* $F\mathbb{L}$ for an *arbitrary* Lagrangian function $\mathbb{L} : U \subset TQ \to T^*Q$ (smooth on an open subset U of TQ), since –also in this case– the value $\pi = F_{\mathrm{p}}\mathbb{L}(\mathbf{v}) \in T^*_{\mathrm{p}}Q$ (for any $(\mathrm{p}, \mathbf{v}) \in U$), defined by the above components $p_h = \frac{\partial \mathbb{L}}{\partial v^h}\Big|_{(q,v)}$ in a given chart near p, turns out to be an 'invariant' covector, i.e. it does not depend on the choice of the chart. [4]

$\mathbb{L}$ is then said to be a *regular* or a *singular* Lagrangian function, according to whether $F\mathbb{L}$ is injective or not.

3.2 Hamilton's differential equation

The symplectic formulation of conservative dynamics will now be given.

Energy and Hamiltonian function

Legendre transformation determines a transition from the energy associated with the Lagrangian function on TQ to a 'Hamiltonian function' on T^*Q, as follows.

Recall that the energy function associated with the Lagrangian function $\mathbb{L} = K - V$ is $\mathrm{E} = K + V = 2K - \mathbb{L}$.

Now put

$$\begin{aligned} \langle F\mathbb{L} \mid \mathrm{id}_{\mathrm{TQ}} \rangle : (\mathrm{p}, \mathbf{v}) \in TQ \longmapsto \langle F_{\mathrm{p}}\mathbb{L}(\mathbf{v}) \mid \mathrm{id}_{\mathrm{TQ}}(\mathbf{v}) \rangle &= \langle F_{\mathrm{p}}\mathbb{L}(\mathbf{v}) \mid \mathbf{v} \rangle \\ &= \langle g_{\mathrm{p}}(\mathbf{v}) \mid \mathbf{v} \rangle \\ &= 2K(\mathrm{p}, \mathbf{v}) \in \mathbb{R} \end{aligned}$$

that is,

$$\langle F\mathbb{L} \mid \mathrm{id}_{\mathrm{TQ}} \rangle = 2K$$

The energy function can then be expressed in the form

$$\mathrm{E} = \langle F\mathbb{L} \mid \mathrm{id}_{\mathrm{TQ}} \rangle - \mathbb{L} \, : \, TQ \to \mathbb{R}$$

---

[4] Actually, through higher geometric methods, $F\mathbb{L}$ can be given a 'coordinate-free' definition.

The *Hamiltonian function*

$$H := \mathrm{E} \circ (F\mathbb{L})^{-1} \, : \, T^*Q \to \mathbb{R}$$

is the 'pull back' of E by $(F\mathbb{L})^{-1}$, that is,

$$\begin{aligned} H(\mathrm{p}, \pi) &= \mathrm{E}\Big((F\mathbb{L})^{-1}(\mathrm{p}, \pi)\Big) \\ &= \mathrm{E}\Big(\mathrm{p}, (F_\mathrm{p}\mathbb{L})^{-1}(\pi)\Big) \\ &= \mathrm{E}(\mathrm{p}, \mathbf{v}) \\ &= \langle F_\mathrm{p}\mathbb{L}(\mathbf{v}) \mid \mathbf{v}\rangle - \mathbb{L}(\mathrm{p}, \mathbf{v}) \\ &= \langle \pi \mid \mathbf{v}\rangle - \mathbb{L}(\mathrm{p}, \mathbf{v}) \, , \qquad \mathbf{v} = (F_\mathrm{p}\mathbb{L})^{-1}(\pi) \end{aligned}$$

for all $(\mathrm{p}, \pi) \in T^*Q$.

Coordinate formalism

In natural coordinate formalism, the Hamiltonian function

$$(\mathrm{p}, \pi) \mapsto H(\mathrm{p}, \pi) = \langle \pi \mid \mathbf{v}\rangle - \mathbb{L}(\mathrm{p}, \mathbf{v}) \, , \qquad \mathbf{v} = (F_\mathrm{p}\mathbb{L})^{-1}(\pi)$$

is expressed by

$$(q, p) \mapsto H(q, p) = p_k \, v^k - \mathbb{L}(q, v) \, , \quad v = v(q, p)$$

Hence, owing to (1),

$$\frac{\partial H}{\partial q^h}\Big|_{(q,p)} = p_k \, \frac{\partial v^k}{\partial q^h}\Big|_{(q,p)} - \frac{\partial \mathbb{L}}{\partial v^k}\Big|_{(q,v)} \frac{\partial v^k}{\partial q^h}\Big|_{(q,p)} - \frac{\partial \mathbb{L}}{\partial q^h}\Big|_{(q,v)} = -\frac{\partial \mathbb{L}}{\partial q^h}\Big|_{(q,v)} \tag{2}$$

and

$$\frac{\partial H}{\partial p_h}\Big|_{(q,p)} = v^h + p_k \, \frac{\partial v^k}{\partial p_h}\Big|_{(q,p)} - \frac{\partial \mathbb{L}}{\partial v^k}\Big|_{(q,v)} \frac{\partial v^k}{\partial p_h}\Big|_{(q,p)} = v^h \tag{3}$$

Symplectic structure and Hamiltonian vector field

The canonical symplectic structure of T^*Q will now be taken into consideration.

Let

$$\omega : TS \longrightarrow T^*S : (\pi, X) \mapsto (\pi, \omega_\pi(X))$$

be the canonical symplectic structure of cotangent bundle $S := T^*Q$. [5]

As ω is a vector bundle isomorphism of TS onto T^*S, it transforms the differential

$$dH : S \longrightarrow T^*S : \pi \mapsto (\pi, d_\pi H)$$

of Hamiltonian function H into the *Hamiltonian vector field*

$$X_H := \omega^{-1} \circ dH : S \longrightarrow TS : \pi \mapsto (\pi, X_H(\pi)), \quad X_H(\pi) = \omega_\pi^{-1}(d_\pi H)$$

Coordinate formalism

Recall that, on the $2n$-dimensional manifold S, the canonical symplectic structure ω is characterized, in any natural chart, by the $2n \times 2n$ matrix of components

$$\begin{bmatrix} \omega_{(1)(1)} & \omega_{(1)(2)} \\ \omega_{(2)(1)} & \omega_{(2)(2)} \end{bmatrix}$$

with

$$\begin{aligned} \omega_{(1)(1)} &= \omega_{(2)(2)} = 0 && (n \times n \ \text{zero matrix}) \\ \omega_{(1)(2)} &= -\,\omega_{(2)(1)} = \delta && (n \times n \ \text{identity matrix}) \end{aligned}$$

For any $\pi \in S$, let

$$\begin{bmatrix} d_\pi H_{(1)} & d_\pi H_{(2)} \end{bmatrix}$$

and

$$\begin{bmatrix} X_H^{(1)}(\pi) \\ X_H^{(2)}(\pi) \end{bmatrix}$$

be the $2n$-tuples of components of $d_\pi H \in T^*_\pi S$ and $X_H(\pi) \in T_\pi S$, respectively.

---

[5] Any ‘point’ $(\mathrm{p}, \pi) \in T^*Q$ is now referred to as $\pi \in S$.

The linear mapping

$$d_\pi H = \omega_\pi(X_H(\pi))$$

is then expressed by [6]

$$d_\pi H_{(\alpha)} = -\,\omega_{(\alpha)(\beta)} X_H^{(\beta)}(\pi)$$

that is,

$$d_\pi H_{(1)} = -\,X_H^{(2)}(\pi)\,, \quad d_\pi H_{(2)} = X_H^{(1)}(\pi)$$

As

$$d_\pi H_{(1)} = \left[\frac{\partial H}{\partial q^h}\Big|_{(q,p)}\right]\,, \quad d_\pi H_{(2)} = \left[\frac{\partial H}{\partial p_h}\Big|_{(q,p)}\right]$$

we obtain

$$X_H^{(1)}(\pi) = \left[\frac{\partial H}{\partial p_h}\Big|_{(q,p)}\right]\,, \quad X_H^{(2)}(\pi) = \left[-\frac{\partial H}{\partial q^h}\Big|_{(q,p)}\right] \tag{4}$$

Cotangent dynamically possible motions

Legendre transformation $F\mathbb{L} : TQ \to T^*Q$ also determines a transition from smooth curves $c : I \subset \mathbb{R} \to TQ$ of velocity phase space to smooth curves $F\mathbb{L} \circ c : I \subset \mathbb{R} \to T^*Q$ of momentum phase space.

If c is an integral curve of $\mathbb{D}_{Eul-Lagr}$, i.e. a *TDPM* of mechanical system $\mathcal{S}$, then $F\mathbb{L} \circ c$ will be said to be a *cotangent dynamically possible motion* (*CDPM*) of $\mathcal{S}$. As $F\mathbb{L}$ is an isomorphism, *TDPM*s and *CDPM*s bijectively correspond to one another.

Through such a bijection, the problem of determining the *TDPM*s proves to be naturally equivalent to that of determining the *CDPM*s.

---

[6] Summation over $(\beta) = (1), (2)$ is understood.

Hamilton equation

A very lucky characterization of the *CDPM*s (yielding a great many results concerning their qualitative analysis) is given in the following

Proposition 14 *The* CDPM*s are the integral curves of* Hamilton equation

$$\mathbb{D}_{Ham} := \mathrm{Im}(X_H)$$

Proof Let

$$k : I \subset \mathbb{R} \to T^*Q \ , \quad c : I \subset \mathbb{R} \to TQ$$

be smooth curves of T^*Q and TQ, respectively, corresponding to each other through Legendre transformation

$$k = F\mathbb{L} \circ c$$

and then projecting down onto the same smooth curve

$$\pi_Q^* \circ k = \pi_Q^* \circ F\mathbb{L} \circ c = \pi_Q \circ c$$

of the base manifold Q.

We shall prove that

$$\mathrm{Im}(\dot{c}) \subset \mathbb{D}_{Eul-Lagr} \quad \Longleftrightarrow \quad \mathrm{Im}(\dot{k}) \subset \mathbb{D}_{Ham}$$

that is,

$$(\lozenge) \qquad (\pi_Q \circ c)^{\cdot} = c \ , \ [\mathbb{L}] \circ \dot{c} = 0 \quad \Longleftrightarrow \quad \dot{k} = X_H \circ k \qquad (\blacklozenge)$$

Let $t_* \in I$ and choose a chart near $\pi_Q^*(k(t_*)) = \pi_Q(c(t_*))$.

In a suitably small open neighbourhood of t_*, the natural coordinates

$$(q, p) = (q(t), p(t))$$

of $k = k(t)$ are related to the natural coordinates

$$(q, v) = (q(t), v(t))$$

of $c = c(t)$ by (1), i.e.

$$p_h(t) = \frac{\partial \mathbb{L}}{\partial v^h}\Big|_{(q(t),\, v(t))}$$

As a consequence, conditions $(\Diamond)$ at time t_*, that is,

$$\dot{q}^h(t_*) = v^h(t_*)\,, \qquad \frac{d}{dt}\Big|_{t_*} \frac{\partial \mathbb{L}}{\partial v^h}\Big|_{(q,v)} - \frac{\partial \mathbb{L}}{\partial q^h}\Big|_{\left(q(t_*),\,v(t_*)\right)} = 0$$

(with $h = 1, \ldots, n$) are equivalent to equations

$$\dot{q}^h(t_*) = v^h(t_*)\,, \quad \dot{p}_h(t_*) = \frac{\partial \mathbb{L}}{\partial q^h}\Big|_{\left(q(t_*),\,v(t_*)\right)}$$

which, owing to (2) and (3), do not differ from equations

$$\dot{q}^h(t_*) = \frac{\partial H}{\partial p_h}\Big|_{\left(q(t_*),\,p(t_*)\right)}\,, \quad \dot{p}_h(t_*) = -\frac{\partial H}{\partial q^h}\Big|_{\left(q(t_*),\,p(t_*)\right)}$$

orderly equalling, owing to (4), [7] the components of the left and right hand sides of $(\blacklozenge)$ at time t_*.

That proves our claim. $\square$

Remark It is important to remark the 'second-order' character of the above Hamilton equation, exhibeted by the fact that any integral curve k of $\mathbb{D}_{Ham}$ – image through $F\mathbb{L}$ of a smooth curve $c = (\pi_Q \circ c)^{\bullet}$ – is the *Legendre lift* $F\mathbb{L} \circ \dot{\gamma}$ of its own projection or *base integral curve* $\gamma = \pi_Q^* \circ k$, since

$$\begin{aligned} k &= F\mathbb{L} \circ c \\ &= F\mathbb{L} \circ (\pi_Q \circ c)^{\bullet} \\ &= F\mathbb{L} \circ (\pi_Q^* \circ k)^{\bullet} \\ &= F\mathbb{L} \circ \dot{\gamma} \end{aligned}$$

The base integral curves, bijectively related to the integral curves, are then characterized by condition

$$\mathrm{Im}(F\mathbb{L} \circ \dot{\gamma})^{\bullet} \subset \mathbb{D}_{Ham}$$

that is,

$$(F\mathbb{L} \circ \dot{\gamma})^{\bullet} = X_H \circ (F\mathbb{L} \circ \dot{\gamma})$$

---

[7] Also recall that $\begin{bmatrix} \dot{q}^h(t_*) \\ \dot{p}_h(t_*) \end{bmatrix}$ are the components of $\dot{k}(t_*)$.

Classical Hamilton equations

The scalar equations

$$\dot{q}^h = \frac{\partial H}{\partial p_h}\Big|_{(q,p)} \, , \quad \dot{p}_h = -\frac{\partial H}{\partial q^h}\Big|_{(q,p)}$$

obtained, with the aid of a given chart on Q, by orderly equalling the components of the left and right hand side of condition

$$\dot{k} = X_H \circ k$$

and therefore characterizing , in coordinate formalism, the integral curves of $\mathbb{D}_{Ham}$ whose projections live in the coordinate domain of the chart, are the classical *Hamilton equations* of Analytical Dynamics.

3.3 Hamilton's variational principle

The variational formulation of conservative dynamics will finally be given.

Variational calculus

Let

$$\gamma : I \subset \mathbb{R} \to Q : t \mapsto \mathrm{p}(t)$$

be a smooth curve of Q.

A *smooth variaton of* γ *with fixed end-points in a closed subinterval of* I, is a smooth map

$$\chi : (-\epsilon, \epsilon) \times J \to Q : (s,t) \mapsto \mathrm{p}(s,t)$$

(with $\epsilon > 0$ and $J \subset I$) satisfying

$$\mathrm{p}(0,t) = \mathrm{p}(t) \, , \quad \forall \, t \in J$$

and, at the end-points of a closed interval $[t_1, t_2] \subset J$,

$$\mathrm{p}(s,t_1) = \mathrm{p}(t_1) \, , \; \mathrm{p}(s,t_2) = \mathrm{p}(t_2) \, , \quad \forall \, s \in (-\epsilon, \epsilon)$$

χ can be thought of as a one-parameter family

$$\{\, \chi_s : J \to Q : t \mapsto \mathrm{p}_s(t) := \mathrm{p}(s,t) \,\}_{s\in(-\epsilon,\epsilon)}$$

of 'varied' curves near $\gamma|_J = \chi_o$ with fixed end-points in $[t_1,t_2] \subset J \subset I$, whose tangent lifts $\{\, \dot\chi_s \,\}_{s\in(-\epsilon,\epsilon)}$ are all included in

$$\dot\chi : (-\epsilon,\epsilon)\times J \to TQ : (s,t) \mapsto \left(\mathrm{p}(s,t), \frac{\partial \mathrm{p}}{\partial t}\Big|_{(s,t)} \right)$$

χ also define a one-parameter family

$$\{\, \chi_t : (-\epsilon,\epsilon) \to Q : s \mapsto \mathrm{p}_t(s) := \mathrm{p}(s,t) \,\}_{t\in J}$$

of 'isocronous' curves, whose tangent vectors at the points of $\gamma|_J$ are the values of

$$\chi'_o : J \to TQ : t \mapsto \left(\mathrm{p}(t), \frac{\partial \mathrm{p}}{\partial s}\Big|_{(0,t)} \right)$$

Now consider, in a closed subinterval $[t_1,t_2]$ of I, the *action* of γ, i.e. the integral [8]

$$\mathcal{I}_\gamma := \int_{t_1}^{t_2} (\mathbb{L}\circ\dot\gamma)\, dt$$

For any smooth variation χ of γ with fixed end-points in $[t_1,t_2]$, the action of χ is then the real-valued function

$$\mathcal{I}_\chi := \int_{t_1}^{t_2} (\mathbb{L}\circ\dot\chi)\, dt : (-\epsilon,\epsilon) \to \mathbb{R} : s \mapsto \mathcal{I}_{\chi_s} := \int_{t_1}^{t_2} (\mathbb{L}\circ\dot\chi_s)\, dt$$

At $s=0$, the value of $\mathcal{I}_\chi$ is $\mathcal{I}_\gamma$ and its derivative

$$\delta_\gamma \mathcal{I}_\chi := \frac{d\mathcal{I}_\chi}{ds}\Big|_0$$

is called the *first variation* of $\mathcal{I}_\chi$ at γ.

---

[8] All of the following considerations and results of variational theory, hold true for an arbitrary smooth Lagrangian function as well (on this matter, recall the final Remark of section 2.3).

The calculus of the first variation

$$\delta_\gamma \mathcal{I}_\chi := \frac{d}{ds}\Big|_0 \int_{t_1}^{t_2} (\mathbb{L} \circ \dot{\chi})\, dt = \int_{t_1}^{t_2} \frac{\partial (\mathbb{L} \circ \dot{\chi})}{\partial s}\Big|_{s=0} dt$$

exhibits the following interesting result.

Proposition 15 *For every smooth variation* χ *of* γ *with fixed end-points in a closed sub-interval* $[t_1, t_2]$ *of* I *, the first variation of* $\mathcal{I}_\chi$ *at* γ *is related to the Lagrangian geodesic curvature of* γ *by*

$$\delta_\gamma \mathcal{I}_\chi = -\int_{t_1}^{t_2} \langle [\mathbb{L}] \circ \ddot{\gamma} \mid \chi_o' \rangle \, dt$$

Proof Let $t_* \in J$.

Consider a chart ξ near $\mathrm{p}(t_*) \in \mathrm{Im}(\gamma)$.

In a suitably small neighbourhood of $(0, t_*) \in (-\epsilon, \epsilon) \times J$, χ takes values in the coordinate domain of ξ (by continuity) and then can be given a local coordinate expression $(q^h(s,t))$.

As a consequence, $\dot{\chi}$ will have local coordinate expression $\left(q^h(s,t), \frac{\partial q^h}{\partial t}\Big|_{(s,t)}\right)$,

whose value at $s = 0$ is the local coordinate expression $(q, v) = (q(t), v(t))$ (with $v = \dot{q}$) of $\dot{\gamma}$.

Finally, $\chi_o'(t)$ will have components $\chi_o'{}^h = \frac{\partial q^h}{\partial s}\Big|_{s=0}$, that is, $\chi_o'{}^h(t) = \frac{\partial q^h}{\partial s}\Big|_{(0,t)}$.

With the aid of ξ , we obtain

$$\begin{aligned}
\frac{\partial (\mathbb{L} \circ \dot{\chi})}{\partial s}\Big|_{(0,t_*)} &= \frac{\partial \mathbb{L}}{\partial q^h}\Big|_{(q(t_*),v(t_*))} \frac{\partial q^h}{\partial s}\Big|_{(0,t_*)} + \frac{\partial \mathbb{L}}{\partial v^h}\Big|_{(q(t_*),v(t_*))} \frac{\partial^2 q^h}{\partial s \partial t}\Big|_{(0,t_*)} \\
&= \frac{\partial \mathbb{L}}{\partial q^h}\Big|_{(q(t_*),v(t_*))} \frac{\partial q^h}{\partial s}\Big|_{(0,t_*)} + \\
&\quad \frac{d}{dt}\Big|_{t_*} \left(\frac{\partial \mathbb{L}}{\partial v^h}\Big|_{(q,v)} \frac{\partial q^h}{\partial s}\Big|_{s=0} \right) - \left(\frac{d}{dt}\Big|_{t_*} \frac{\partial \mathbb{L}}{\partial v^h}\Big|_{(q,v)} \right) \frac{\partial q^h}{\partial s}\Big|_{(0,t_*)} \\
&= \left(\frac{\partial \mathbb{L}}{\partial q^h}\Big|_{(q(t_*),v(t_*))} - \frac{d}{dt}\Big|_{t_*} \frac{\partial \mathbb{L}}{\partial v^h}\Big|_{(q,v)} \right) \chi_o'{}^h(t_*) + \frac{d}{dt}\Big|_{t_*} \left(\frac{\partial \mathbb{L}}{\partial v^h}\Big|_{(q,v)} \chi_o'{}^h \right)
\end{aligned}$$

that is,

$$\frac{\partial(\mathbb{L}\circ\dot{\chi})}{\partial s}\Big|_{(0,t_*)} = -\,\langle[\mathbb{L}]\circ\ddot{\gamma}\mid\chi'_o\rangle(t_*) + \frac{d}{dt}\Big|_{t_*}\langle F\mathbb{L}\circ\dot{\gamma}\mid\chi'_o\rangle$$

So, on the whole J, we have

$$\frac{\partial(\mathbb{L}\circ\dot{\chi})}{\partial s}\Big|_{s=0} = -\,\langle[\mathbb{L}]\circ\ddot{\gamma}\mid\chi'_o\rangle + \frac{d}{dt}\,\langle F\mathbb{L}\circ\dot{\gamma}\mid\chi'_o\rangle$$

As $\chi'_o(t_1)=0$ and $\chi'_o(t_2)=0$, we also have

$$\int_{t_1}^{t_2}\frac{d}{dt}\langle F\mathbb{L}\circ\dot{\gamma}\mid\chi'_o\rangle\,dt = \langle F\mathbb{L}\circ\dot{\gamma}\mid\chi'_o\rangle(t_2) - \langle F\mathbb{L}\circ\dot{\gamma}\mid\chi'_o\rangle(t_1) = 0$$

Hence our claim. □

Geodesic curves and Hamilton's principle

A smooth curve $\gamma : I\subset\mathbb{R}\to Q$ is said to be a *geodesic curve* of $(Q,\mathbb{L})$, if it satisfies Hamilton's *variational principle of stationary action*, owing to which, for every smooth variation χ of γ with fixed end-points in a closed subinterval of I, $\mathcal{I}_\chi$ is required to be stationary at γ, that is to say,

$$\delta_\gamma\mathcal{I}_\chi = 0\ ,\quad \forall\chi$$

The above variational principle is completely equivalent to Euler-Lagrange equation, as is shown in the following

Proposition 16 *The geodesic curves of* $(Q,\mathbb{L})$ *are the base integral curves of* $\mathbb{D}_{Eul-Lagr}$.

Proof If $\gamma : I\subset\mathbb{R}\to Q$ is a base integral curve of $\mathbb{D}_{Eul-Lagr}$, i.e.

$$[\mathbb{L}]\circ\ddot{\gamma} = 0$$

we clearly obtain, for every smooth variation χ of γ with fixed end-points in a closed subinterval $[t_1,t_2]$ of I,

$$\int_{t_1}^{t_2}\langle[\mathbb{L}]\circ\ddot{\gamma}\mid\chi'_o\rangle\,dt = 0$$

and hence, owing to Proposition 14, γ is a geodesic curve of $(Q,\mathbb{L})$.

Conversely, if $\gamma : t \in I \subset \mathbb{R} \mapsto \mathrm{p}(t) \in Q$ is not a base integral curve of $\mathbb{D}_{Eul-Lagr}$, say

$$([\mathbb{L}] \circ \ddot{\gamma})(t_*) \neq 0 \,, \quad t_* \in I$$

we shall prove the existence of a smooth variation χ of γ, with fixed end-points in a closed subinterval $[t_1, t_2]$ of I, such that

$$\int_{t_1}^{t_2} \langle [\mathbb{L}] \circ \ddot{\gamma} \mid \chi'_o \rangle \, dt \neq 0$$

which means, owing to Proposition 14, that γ is not a geodesic curve of $(Q, \mathbb{L})$.

To this end, consider a chart with coordinate domain

$$\mathcal{U} \ni \mathrm{p}(t_*)$$

Owing to our hypothesis, at least one of the components $([\mathbb{L}] \circ \ddot{\gamma})_h$ in the above chart is non-null at t_*, say

$$([\mathbb{L}] \circ \ddot{\gamma})_1 \, (t_*) > 0$$

By continuity, there exists a suitably small open interval $J \subset I$ containing t_* such that, for all $t \in J$,

$$\mathrm{p}(t) \in \mathcal{U} \,, \quad ([\mathbb{L}] \circ \ddot{\gamma})_1 \, (t) > 0$$

Now, for each $s \in (-\epsilon, \epsilon)$ $-$with a suitably small $\epsilon > 0-$ and each $t \in J$, we consider the point $\mathrm{p}(s, t) \in \mathcal{U}$ of coordinates

$$\begin{array}{ll} q^1(s,t) & := q^1(t) + s(cos(t - t_*) - cos\,\alpha) \\ q^2(s,t) & := q^2(t) \\ \dots & \dots \\ q^n(s,t) & := q^n(t) \end{array}$$

where $(q^1(t), \dots, q^n(t))$ is the n-tuple of coordinates of $\mathrm{p}(t)$ and

$$0 < \alpha < \frac{\pi}{2} \quad \text{s.t.} \quad [t_* - \alpha, t_* + \alpha] \subset J$$

By doing so, we define a map

$$\chi : (-\epsilon, \epsilon) \times J \to Q : (s, t) \mapsto \mathrm{p}(s, t)$$

which is immediately seen to be a smooth variation of γ with fixed end-points in

$$[t_1, t_2] := [t_* - \alpha, t_* + \alpha]$$

The components of χ'_o are , for all $t \in J$,

$$\begin{aligned} {\chi'_o}^1(t) &= cos(t - t_*) - cos\,\alpha \\ {\chi'_o}^2(t) &= 0 \\ \dots \quad & \dots \\ {\chi'_o}^n(t) &= 0 \end{aligned}$$

and, for all $t \in (t_1, t_2)$,

$${\chi'_o}^1(t) > 0$$

Hence

$$\int_{t_1}^{t_2} \langle [\mathbb{L}] \circ \ddot{\gamma} \mid \chi'_o \rangle \, dt = \int_{t_1}^{t_2} ([\mathbb{L}] \circ \ddot{\gamma})_h \, {\chi'_o}^h \, dt = \int_{t_1}^{t_2} ([\mathbb{L}] \circ \ddot{\gamma})_1 \, {\chi'_o}^1 \, dt \; > 0$$

which is our claim. □

Geodesic curves of a Riemannian manifold

For $\mathbb{L} = K$, the above variational theory characterizes the geodesic curves of Riemannian manifold (Q, K) through condition

$$[K] \circ \ddot{\gamma} = 0$$

Owing to the positive definiteness of K , from the conservation law of energy $\mathrm{E} = K$ it follows that, on the one hand, a geodesic curve γ satisfying

$$K \circ \dot{\gamma} = \text{const.} = 0$$

is a ‘motion’ in Q which degenerates into a *state of rest* (whose velocity vanishes) and, on the other hand, a *non-degenerate* geodesic curve is a *uniform motion* in Q (whose velocity has non-null constant norm), that is,

$$K \circ \dot{\gamma} = \text{const.} > 0$$

The geometric meaning of non-degenerate geodesic curves will now be shown.

Let us consider one more Lagrangian function, namely

$$\Lambda := \sqrt{2K}$$

which is clearly smooth on $TQ \setminus K^{-1}(0)$. [9]

If $\gamma : t \in I \mapsto \mathrm{p}(t) \in Q$ is a smooth curve satisfying

$$\operatorname{Im}(\dot\gamma) \subset TQ \setminus K^{-1}(0)$$

(i.e. $\dot{\mathrm{p}}(t) \neq 0$ for all $t \in I$) and $[t_1, t_2] \subset I$, the integral

$$\mathcal{L}_\gamma := \int_{t_1}^{t_2} (\Lambda \circ \dot\gamma)\, dt$$

defines the *length* of the arc of γ with end-points $(\mathrm{p}(t_1), \mathrm{p}(t_2))$. [10]

By applying the variational theory to Λ, [11] we obtain that that γ is a curve of *stationary length*

$$\delta_\gamma \mathcal{L}_\chi = 0\,, \quad \forall \chi$$

iff

$$[\Lambda] \circ \ddot\gamma = 0$$

Proposition 17 *γ is a non-degenerate geodesic curve of (Q, K), iff it is a uniform motion of stationary length.*

Proof Follows from the fact that, if γ is a uniform motion, say $K \circ \dot\gamma = \kappa > 0$, then

$$[\Lambda] \circ \ddot\gamma = \frac{1}{\sqrt{2\kappa}}\, [K] \circ \ddot\gamma$$

---

[9] Remark that, in any natural chart of TQ, Λ admits partial derivatives

$$\frac{\partial \Lambda}{\partial q^h}\Big|_{(q,v)} = \frac{1}{\sqrt{2K(q,v)}} \frac{\partial K}{\partial q^h}\Big|_{(q,v)}\,, \quad \frac{\partial \Lambda}{\partial v^h}\Big|_{(q,v)} = \frac{1}{\sqrt{2K(q,v)}} \frac{\partial K}{\partial v^h}\Big|_{(q,v)}$$

which are not defined at (q, v) with $v = 0$ (i.e. at $(\mathrm{p}, \mathbf{v}) \in K^{-1}(0)$).

[10] For any $dt \neq 0$, $\Lambda(\mathrm{p}(t), \dot{\mathrm{p}}(t))\, dt = \sqrt{\langle g_{\mathrm{p}(t)}(d\mathrm{p}) \mid d\mathrm{p} \rangle} > 0$ defines the length, in the given Riemannian metric, of the 'infinitesimal arc' $d\mathrm{p} := \dot{\mathrm{p}}(t) dt$.

[11] See footnote [8].

In order to prove the above statement, just notice that, in coordinate formalism, we have [12]

$$\begin{aligned}
([\Lambda]\circ\ddot{\gamma})_h &= \frac{d}{dt}\frac{\partial\Lambda}{\partial v^h}\Big|_{(q,v)} - \frac{\partial\Lambda}{\partial q^h}\Big|_{(q,v)} \\
&= \frac{d}{dt}\left(\frac{1}{\sqrt{2\kappa}}\frac{\partial K}{\partial v^h}\Big|_{(q,v)}\right) - \frac{1}{\sqrt{2\kappa}}\frac{\partial K}{\partial q^h}\Big|_{(q,v)} \\
&= \frac{1}{\sqrt{2\kappa}}\left(\frac{d}{dt}\frac{\partial K}{\partial v^h}\Big|_{(q,v)} - \frac{\partial K}{\partial q^h}\Big|_{(q,v)}\right) \\
&= \frac{1}{\sqrt{2\kappa}}\,([K]\circ\ddot{\gamma})_h
\end{aligned}$$

which is our claim. □

---

[12] See the partial derivatives of Λ evaluated in footnote [9].

Chapter 4

Concluding remarks

Some comments on the previous results and some perspectives on further developments, now follow.

4.1 Inertia and force

The general theory exposed in chapter 2, has shown that the mathematical model actually underlying classical dynamics is given by the triplet

$$\mathcal{M} := (Q, K, F)$$

which still deserves the name of (*geometrical*) *mechanical system.*

The corresponding Lagrange equation

$$\mathcal{D}_{Lagr} = \{(\mathrm{p}, \mathbf{v}; \mathbf{v}, \mathbf{w}) \in TTQ \mid \mathbf{u} = \mathbf{v}\,, \;\; [K](\mathrm{p}, \mathbf{v}, \mathbf{w}) = F(\mathrm{p}, \mathbf{v})\}$$

is then the *dynamics* of $\mathcal{M}$, the *DPM*s of $\mathcal{M}$ being defined as the base integral curves γ of $\mathcal{D}_{Lagr}$ and hence characterized by condition

$$[K] \circ \ddot{\gamma} = F \circ \dot{\gamma}$$

If $F = 0$, the *DPM*s γ of $\mathcal{M}$ – characterized by a Riemannian geodesic curvature $[K] \circ \ddot{\gamma}$ identically vanishing – coincide with the geodesic curves of Riemannian manifold (Q, K), which can therefore be called *inertial motions* in (Q, K).

If $F \neq 0$, the *DPM*s γ of $\mathcal{M}$ – characterized by a Riemannian geodesic curvature $[K] \circ \ddot{\gamma}$ differing from zero as much as is imposed by the covector force $F \circ \dot{\gamma}$ – are then to be called *forced motions* in (Q, K), since they appear to be perturbed with respect to the above inertial trend.

So K and F seem to correspond to the *empirical* notions of 'inertia' (defining inertial motions) and 'force' (defining forced motions), respectively.

4.2 Gauge transformations

However the above notions of inertia and force are *not* uniquely determined by dynamics. In fact, there exist 'gauge transformations' of $\mathcal{M}$, which alter the geometrical ingredient K and the dynamical ingredient F without altering the dynamics itself, as will now be shown.

For any real-valued smooth function $V : Q \to \mathbb{R}$, consider the *gauge transformation*

$$K \mapsto \mathbb{L} := K - V\ , \quad F \mapsto \mathbb{F} := F + dV$$

On the one hand, such a transformation gives rise to a new kind of (*geometrical*) *mechanical system*

$$\mathbb{M} := (Q, \mathbb{L}, \mathbb{F})$$

with *Lagrange equation*

$$\mathbb{D}_{Lagr} = \{(\mathrm{p}, \mathbf{v}; \mathbf{v}, \mathbf{w}) \in T^2Q \mid \mathbf{u} = \mathbf{v}\ ,\ [\mathbb{L}](\mathrm{p}, \mathbf{v}, \mathbf{w}) = \mathbb{F}(\mathrm{p}, \mathbf{v})\}$$

and *DPM*s – the base integral curves of $\mathbb{D}_{Lagr}$ – characterized by condition

$$[\mathbb{L}] \circ \ddot{\gamma} = \mathbb{F} \circ \dot{\gamma}$$

If $\mathbb{F} = 0$, the *DPM*s γ of $\mathbb{M}$ – characterized by a Lagrangian geodesic curvature $[\mathbb{L}] \circ \ddot{\gamma}$ identically vanishing – coincide with the geodesic curves of Lagrangian manifold $(Q, \mathbb{L})$, which – generalizing the Riemannian case – will be called *inertial motions* in $(Q, \mathbb{L})$.

If $\mathbb{F} \neq 0$, the *DPM*s γ of $\mathbb{M}$ – characterized by a Lagrangian geodesic curvature $[\mathbb{L}] \circ \ddot{\gamma}$ differing from zero as much as is imposed by the covector force $\mathbb{F} \circ \dot{\gamma}$ – are then to be called *forced motions* in $(Q, \mathbb{L})$, since they appear to be perturbed with respect to the above inertial trend.

On the other hand, from the very definitions – for all $(\mathrm{p}, \mathbf{v}; \mathbf{v}, \mathbf{w}) \in T^2Q$ – of

$$[\mathbb{L}](\mathrm{p}, \mathbf{v}, \mathbf{w}) := [K](\mathrm{p}, \mathbf{v}, \mathbf{w}) + d_{\mathrm{p}}V\ , \quad \mathbb{F}(\mathrm{p}, \mathbf{v}) := F(\mathrm{p}, \mathbf{v}) + d_{\mathrm{p}}V$$

it follows that

$$[\mathbb{L}](\mathrm{p}, \mathbf{v}, \mathbf{w}) = \mathbb{F}(\mathrm{p}, \mathbf{v}) \iff [K](\mathrm{p}, \mathbf{v}, \mathbf{w}) = F(\mathrm{p}, \mathbf{v})$$

whence

$$\mathbb{D}_{Lagr} = \mathcal{D}_{Lagr}$$

So transition from $\mathcal{M}$ to $\mathbb{M}$ just leads on different specifications of the *conventional* notions of inertia and force, without altering the *observable* class of *DPM*s.

4.3 Geometrizing physical fields

From the physical point of view, the meaning of the above kind of gauge transformations can be illustrated as follows.

Think of V as the potential energy of a conservative component of the $\delta\acute{\upsilon}\nu\alpha\mu\iota\varsigma$ $\mathbf{F}$, that is to say, put

$$\mathbf{F} = \vec{\mathbb{F}} + \mathbf{f}$$

$\mathbf{f}$ being a conservative field with virtual work

$$f = -dV$$

Now look at the ingredients $\mathbb{L}$ and $\mathbb{F}$ of the mechanical system $\mathbb{M}$ obtained through the gauge transformation generated by the potential energy V of $\mathbf{f}$.

On the one hand, $\mathbb{F} = F - f$ is the virtual work of $\vec{\mathbb{F}} = \mathbf{F} - \mathbf{f}$, since −for all $(\mathrm{p}, \mathbf{v}) \in TQ$−

$$\begin{aligned}
\mathbb{F}(\mathrm{p}, \mathbf{v}) &= F(\mathrm{p}, \mathbf{v}) + d_{\mathrm{p}}V \\
&= F(\mathrm{p}, \mathbf{v}) - f(\mathrm{p}) \\
&= \mathrm{F}(\mathrm{p}, \mathbf{v}) \cdot \Big|_{T_{\mathrm{p}}Q} - \mathrm{f}(\mathrm{p}) \cdot \Big|_{T_{\mathrm{p}}Q} \\
&= \Big(\mathrm{F}(\mathrm{p}, \mathbf{v}) - \mathrm{f}(\mathrm{p})\Big) \cdot \Big|_{T_{\mathrm{p}}Q} \\
&= \vec{\mathbb{F}}(\mathrm{p}, \mathbf{v}) \cdot \Big|_{T_{\mathrm{p}}Q}
\end{aligned}$$

and then the conservative field does not appear any more in the dynamical ingredient $\vec{\mathbb{F}}$ of $\mathbb{M}$.

On the other hand, $\mathbb{L}$ embodies $-V$, that is to say, the conservative field is encompassed in the geometrical ingredient of $\mathbb{M}$ as a *potential* function, whose effect is that of transforming the 'natural' Riemannian geometry K of the particle system into a 'perturbed' Lagrangian geometry $\mathbb{L} = K - V$.

That anticipates, in a sense, Einstein's idea of 'geometrizing physical fields'.

4.4 Geometrical dynamics

From the mathematical point of view, gauge transformations suggest a generalized *geometrical dynamics.*

In such a theory, a *mechanical system* will be conceived as an arbitrary triplet $\mathbb{M}$ consisting of a smooth manifold Q equipped with a (regular or a singular) Lagrangian function $\mathbb{L}$ and a semibasic 1-form $\mathbb{F}$ (both smooth on an open subset of TQ).

The *dynamics* of $\mathbb{M}$ will then be defined by the corresponding Lagrange equation $\mathbb{D}_{Lagr}$ (which will or will not be reducible to normal form, according to whether $\mathbb{L}$ is a regular or a singular Lagrangian function, respectively).

The global analysis of $\mathbb{D}_{Lagr}$, its Hamiltonian formulation (or Hamilton-Dirac formulation, in the general case of $\mathbb{L}$ being a singular Lagrangian) and further generalizations, as well as applications to classical and relativistic dynamics, are the object of (part of) the current research work on *geometrical dynamics.*

Chapter 5

Appendix : Introduction to the geometry of smooth manifolds

The geometry of smooth manifolds embedded in Euclidean affine spaces, is the main topic of this Appendix. Tangent bundles and differential equations, cotangent bundles and differential forms, will be made to fall into its range. The above geometry will finally be re-read, so as to lead to the modern approach to smooth manifolds (which does away with any reference to embeddings into Euclidean environments).

5.1 Advanced calculus on Euclidean affine spaces

The fundamentals of differential calculus on Euclidean affine spaces will first be recalled. [1]

Euclidean affine spaces

A *Euclidean affine space* $\mathcal{E}$, modelled on a Euclidean vector space E, is a non-empty set on which the vector space acts through a 'plus map'

$$+ : \mathcal{E} \times E \longrightarrow \mathcal{E} : (\mathrm{p}, \mathbf{v}) \mapsto \mathrm{q} = \mathrm{p} + \mathbf{v}$$

satisfying – for all $\mathrm{p} \in \mathcal{E}$ and $\mathbf{v}, \mathbf{w} \in E$ – the properties [2]

---

[1] Real vector spaces and topological spaces are the prerequisites for reading this section.

[2] The properties listed below are naturally suggested by empirical geometry, where any vector $\mathbf{v}$ (represented by an oriented segment with an arbitrary origin) takes its origin p to its end-point $\mathrm{p} + \mathbf{v}$ and the sum of vectors is defined by the parallelogram rule.

$$\begin{aligned}
&(\mathrm{p}+\mathbf{v})+\mathbf{w}=\mathrm{p}+(\mathbf{v}+\mathbf{w})\ ,\quad \mathrm{p}+\mathbf{0}=\mathrm{p}\\
&\mathbf{v}\neq\mathbf{0}\implies \mathrm{p}+\mathbf{v}\neq\mathrm{p}\\
&\mathcal{E}=\mathrm{p}+E:=\{\mathrm{p}+\mathbf{v}\}_{\mathbf{v}\in E}
\end{aligned}$$

From the above properties it follows that, for any two points p and q of $\mathcal{E}$, there exists a unique vector

$$\mathbf{v}=\mathrm{q}-\mathrm{p}$$

linking p to q, that is,

$$\mathrm{q}=\mathrm{p}+\mathbf{v}$$

The *dimension* of $\mathcal{E}$ is defined by putting

$$\dim(\mathcal{E}):=\dim(E)$$

We shall assume

$$m:=\dim(\mathcal{E})>0$$

The Euclidean metric structure (or *scalar product*) in E will be meant to be a linear map [3]

$$\cdot\ : E\to E^{*}:\mathbf{u}\mapsto\mathbf{u}\,\cdot$$

satisfying the property of *symmetry* (or *commutativity*), i.e. −for all $\mathbf{u},\mathbf{v}\in E$−

$$\mathbf{u}\cdot\mathbf{v}=\mathbf{v}\cdot\mathbf{u}$$

and the property of *positive definiteness*, i.e. −for all $\mathbf{v}\neq\mathbf{0}\in E$−

$$\mathbf{v}\cdot\mathbf{v}>0$$

which in turn implies the property of *nondegenerateness* (saying that the above linear map is an isomorphism of E onto E^{*}).

The main metric concept arising from the scalar product is that of *Euclidean norm* or *Euclidean length* $|\mathbf{v}|$ of a vector $\mathbf{v}\in E$, defined by putting

$$|\mathbf{v}|:=\sqrt{\mathbf{v}^2}$$

(where $\mathbf{v}^2:=\mathbf{v}\cdot\mathbf{v}$).

---

[3] For any real vector space $\mathbb{V}$, the dual vector space (whose elements are the linear maps of $\mathbb{V}$ in the space $\mathbb{R}$ of real numbers) will be denoted by $\mathbb{V}^{*}$.

The Euclidean length in E naturally leads to the *Euclidean distance*

$$d(\mathrm{p},\mathrm{q}) := |\mathrm{q}-\mathrm{p}|$$

between any two points $\mathrm{p},\mathrm{q} \in \mathcal{E}$, owing to which $\mathcal{E}$ is given the structure of a *metric space.*

The *open balls*

$$\mathcal{B}^r_{\mathrm{p}_o} := \{\mathrm{p} \in \mathcal{E} \mid d(\mathrm{p}_o,\mathrm{p}) < r\}$$

(for any *centre* $\mathrm{p}_o \in \mathcal{E}$ and any *radius* $r > 0$) defined by the metric structure, are in turn a *basis* of the *Euclidean topology* of $\mathcal{E}$, i.e. the Hausdorff topology $\tau_{\mathcal{E}}$ containing all the *open subsets* obtained from the union of open balls (and then the open balls themselves) plus the empty subset.

Remark Let $\mathcal{E}_1,\ldots,\mathcal{E}_\nu$ be Euclidean affine spaces, modelled on Euclidean vector spaces $E_1,\ldots,E_\nu$, respectively.

Recall that the Cartesian product

$$E := E_1 \times \cdots \times E_\nu$$

is a Euclidean vector space (whose dimension is the sum of the dimensions) with the operations defined by putting – for all $a, b \in \mathbb{R}$ and $\mathbf{u} = (\mathbf{u}_1,\ldots,\mathbf{u}_\nu)$, $\mathbf{v} = (\mathbf{v}_1,\ldots,\mathbf{v}_\nu) \in E$ –

$$a\,\mathbf{u} + b\,\mathbf{v} := (a\,\mathbf{u}_1 + b\,\mathbf{v}_1,\ldots,a\,\mathbf{u}_\nu + b\,\mathbf{v}_\nu)$$

and

$$\mathbf{u}\cdot\mathbf{v} := \sum_{i=1}^{\nu} \mathbf{u}_i\cdot\mathbf{v}_i$$

Then the Cartesian product

$$\mathcal{E} := \mathcal{E}_1 \times \cdots \times \mathcal{E}_\nu$$

turns into a Euclidean affine space modelled on E, if the latter is let to act on the former by putting – for all $\mathrm{p} = (\mathrm{p}_1,\ldots,\mathrm{p}_\nu) \in \mathcal{E}$ and $\mathbf{v} = (\mathbf{v}_1,\ldots,\mathbf{v}_\nu) \in E$ –

$$\mathrm{p} + \mathbf{v} := (\mathrm{p}_1 + \mathbf{v}_1,\ldots,\mathrm{p}_\nu + \mathbf{v}_\nu)$$

whence – for all $\mathrm{p} = (\mathrm{p}_1,\ldots,\mathrm{p}_\nu), \mathrm{q} = (\mathrm{q}_1,\ldots,\mathrm{q}_\nu) \in \mathcal{E}$ –

$$\mathrm{q} - \mathrm{p} := (\mathrm{q}_1 - \mathrm{p}_1,\ldots,\mathrm{q}_\nu - \mathrm{p}_\nu)$$

Notice that the Cartesian products $\{\mathcal{U}_1 \times \cdots \times \mathcal{U}_\nu\}_{\mathcal{U}_1\in\tau_{\mathcal{E}_1},\ldots,\mathcal{U}_\nu\in\tau_{\mathcal{E}_\nu}}$ could be shown to be another basis of Euclidean topology $\tau_{\mathcal{E}}$.

Also notice that a Euclidean vector space $\mathbb{V}$ turns into a Euclidean affine space modelled on itself, if the *vector space* $E := \mathbb{V}$ is thought of as acting on the *set* $\mathcal{E} := \mathbb{V}$ through the sum operation of $\mathbb{V}$.

In particular, $\mathbb{R}$ (and then $\mathbb{R}^n$ for any integer $n > 1$) can be viewed as both a Euclidean vector space and a Euclidean affine space.

Affine maps

Let $\mathcal{A}$ and $\mathcal{E}$ be Euclidean affine spaces (modelled on vector spaces A and E, respectively).

An *affine map* of $\mathcal{A}$ in $\mathcal{E}$ is a map

$$f : \mathcal{A} \to \mathcal{E}$$

for which there exists a (uniquely determined) linear map

$$F : A \to E$$

such that – for one (and then every) $x_o \in \mathcal{A}$ and all $v \in A$ –

$$f(x_o + v) = f(x_o) + F(v)$$

or – for one (and then every) $x_o \in \mathcal{A}$ and all $x \in \mathcal{A}$ –

$$f(x) - f(x_o) = F(x - x_o)$$

(i.e. the increment $f(x) - f(x_o)$ of f is a linear function of the increment $x - x_o$ given to the value x_o of its argument).

A *Euclidean affine map* is then an affine map which preserves the metric, i.e. – fo all $x, y \in \mathcal{A}$ –

$$d(f(x), f(y)) = d(x, y)$$

An *affine isomorphism* (resp. *affine isometry*) is a bijective affine map (resp. bijective Euclidean affine map).

In particular, for an m-dimensional Euclidean affine space $\mathcal{E}$, an affine isomorphism (resp. affine isometry) $\varphi : \mathbb{R}^m \to \mathcal{E}$ – whose linear part is a linear isomorphism (resp. linear isometry) $\Phi : \mathbb{R}^m \to E$ – corresponds to a *Cartesian system* (resp. *orthogonal Cartesian system*) given by the origin [4]

$$\mathrm{p}_o = \varphi(0) \in \mathcal{E}$$

and the basis (resp. orthonormal basis) [5]

$$\mathbf{e}_i = \Phi(\delta_i) \in E\,, \quad i = 1, \ldots, m$$

---

[4] Here 0 will denote the zero element of $\mathbb{R}^m$.

[5] $\delta = (\delta_i)_{i=1,\ldots,m}$ – with $\delta_i = (\delta_i^j)_{j=1,\ldots,m}$ (δ_i^j being Kronecker's symbol) – will denote the orthonormal *natural basis* of $\mathbb{R}^m$.

owing to which any point [6]

$$\begin{aligned} \mathrm{p} &= \varphi(x) \\ &= \varphi(0 + x^i \delta_i) \\ &= \varphi(0) + x^i\, \Phi(\delta_i) \\ &= \mathrm{p}_o + x^i\, \mathbf{e}_i \ \in \ \mathcal{E} \end{aligned}$$

can be epressed in function of its *Cartesian coordinates* (resp. *orthogonal Cartesian coordinates*)

$$(x^i)_{i=1,\dots,m} = x = \varphi^{-1}(\mathrm{p}) \in \mathbb{R}^m$$

Continuous maps

Let $\mathcal{A}$ and $\mathcal{E}$ be Euclidean affine spaces (modelled on vector spaces A and E, respectively) and

$$f : X \subset \mathcal{A} \to \mathcal{E}$$

a map defined on a subset X of $\mathcal{A}$.

Cauchy's definition of

$$\lim_{x \to x_o} f(x) = \mathrm{p}_o$$

(*limit* $\mathrm{p}_o \in \mathcal{E}$ of f at an accumulation point $x_o \in \mathcal{A}$ of X) is given by

$$\forall\, \epsilon > 0,\ \exists\, \delta > 0 : x \in X,\ 0 < |x - x_o| < \delta \implies |f(x) - \mathrm{p}_o| < \epsilon$$

and is easily seen to correspond to the well known topological definition [7]

$$\forall\, \mathcal{U}_{\mathrm{p}_o} \in \tau_{\mathcal{E}},\ \exists\, \mathcal{U}_{x_o} \in \tau_{\mathcal{A}} : f\Big((X \cap \mathcal{U}_{x_o}) - \{x_o\}\Big) \subset \mathcal{U}_{\mathrm{p}_o}$$

As a consequence, the classical definition of *continuity of* f *at a point* $x_o \in X$, given by the requirement of x_o not being an accumulation point of X or by

$$\lim_{x \to x_o} f(x) = f(x_o)$$

if x_o is an accumulation point of X, corresponds to the topological definition

$$\forall\, \mathcal{U}_{f(x_o)} \in \tau_{\mathcal{E}},\ \exists\, \mathcal{V}_{x_o} \in \tau_X : f(\mathcal{V}_{x_o}) \subset \mathcal{U}_{f(x_o)}$$

where $\tau_X := \{\mathcal{V} \subset X \ /\ \mathcal{V} = X \cap \mathcal{U}\,,\, \mathcal{U} \in \tau_{\mathcal{A}}\}$ is the *subspace topology* that X inherits from $\mathcal{A}$.

---

[6] Summation symbol over an index will be understood, whenever the index appears both in upper and in lower position.

[7] A symbol like $\mathcal{U}_{\mathrm{p}_o}$ will denote an *open neighbourhood* of a point $\mathrm{p}_o \in \mathcal{E}$ (i.e. an open subset $\mathcal{U}_{\mathrm{p}_o} \in \tau_{\mathcal{E}}$ containing p_o).

Hence the well known result saying that f is *continuous* (i.e. continuous at every point of its domain of definition), iff

$$f^{-1}(\mathcal{U}) \in \tau_X \,, \; \forall \mathcal{U} \in \tau_{\mathcal{E}}$$

An affine map between Euclidean affine spaces can be shown to be continuous and then an affine isomorphism is a *homeomorphism* (i.e. continuous together with its inverse).

Differentiable maps

Let $\mathcal{A}$ and $\mathcal{E}$ be Euclidean affine spaces (modelled on vector spaces A and E, respectively) and

$$f : X \subset \mathcal{A} \to \mathcal{E}$$

a map defined on a subset X of $\mathcal{A}$.

Differentiation is a process of 'infinitesimal linearization', which – if applicable – allows f, restricted to an 'arbitrarily small' open subset of X, to be replaced – up to 'higher-order infinitesimals' – by the restriction of a suitable affine map.

Namely, f is said to be *differentiable at a point* $x_o \in X$, if

(i) x_o is an internal point of X, [8]

(ii) there exists an affine map $g : \mathcal{A} \to \mathcal{E}$, such that the difference $\psi := f - g|_X : x \in X \mapsto \psi(x) = f(x) - g(x) \in E$ vanishes at x_o, i.e.

$$|\psi(x_o)| = 0$$

and is *higher-order infinitesimal* at x_o, i.e. [9]

$$\lim_{x \to x_o} \frac{|\psi(x)|}{|x - x_o|} = 0$$

If f is differentiable at x_o, the affine map g is uniquely determined, since – owing to (ii) – its value at x_o is given by $f(x_o)$ and its linear part $d_{x_o} f : A \to E$, called *differential* of f at x_o, will be shown to take the value

$$d_{x_o} f\,(u) = \lim_{a \to 0} \frac{f(x_o + a u) - f(x_o)}{a} \qquad (*)$$

---

[8] That means that there exists an open ball $\mathcal{B}^r_{x_o} \in \tau_{\mathcal{A}}$ s.t. $\mathcal{B}^r_{x_o} \subset X$.

As a consequence, if f is *differentiable*, i.e. differentiable at each point $x_o \in X$, its domain of definition X is an open subset of $\mathcal{A}$.

[9] Recall that an internal point $x_o \in X$ is an accumulation point of X and $X - \{x_o\}$, and that the higher-order infinitesimality condition implies the ordinary infinitesimality condition $\lim_{x \to x_o} \psi(x) = 0$.

at any normal vector $u \in A$ (i.e. $|u| = 1$). [10]

If f is differentiable at x_o, then – in an 'arbitrarily small' open ball $\mathcal{B}^r_{x_o}$ contained in X – the difference ψ between the given map f and the affine map g is negligible

$$\psi|_{\mathcal{B}^r_{x_o}} = f|_{\mathcal{B}^r_{x_o}} - g|_{\mathcal{B}^r_{x_o}} \approx 0$$

So, for all $x \in \mathcal{B}^r_{x_o} \subset X$, one has – up to higher-order infinitesimals –

$$f(x) \approx g(x)$$

that is,

$$f(x) \approx f(x_o) + d_{x_o}f\,(x - x_o)$$

(i.e. $f|_{\mathcal{B}^r_{x_o}}$ can be replaced – up to higher-order infinitesimals – by the restriction of an affine map) or

$$f(x) - f(x_o) \approx d_{x_o}f\,(x - x_o)$$

(i.e. the increment $f(x) - f(x_o)$ of f is a linear function – up to higher-order infinitesimals – of the increment $x - x_o$ given to the value x_o of its argument).

That expresses the announced process of 'infinitesimal linearization'.

Clearly, an affine map $f : \mathcal{A} \to \mathcal{E}$, with linear part $F : A \to E$, is differentiable and $d_{x_o}f = F$ for all $x_o \in \mathcal{A}$ (in particular $d_{x_o}f = 0$, if f is constant).

Now notice that the right hand side of ($*$) is the classical *directional derivative* of f at x_o along the oriented direction of the normal vector $u \in A$. So equality ($*$), which will now be proved, means that differentiability implies derivability:

Proposition 18 *If f is differentiable at x_o, then the derivative of f at x_o along the oriented direction of any normal vector $u \in A$ exists and equals $d_{x_o}f\,(u)$.*

Proof On the one hand, for any $a \in (-r, r) \setminus \{0\}$, we have

---

[10] In the right hand side of ($*$), a is meant to vary in $(-r, r) \setminus \{0\}$ with $r > 0$ s.t. $\mathcal{B}^r_{x_o} \subset X$. Equality ($*$) completely determines the differential, since (owing to linearity) $d_{x_o}f$ vanishes at the zero vector of A and, at any non-null vector $v = |v|u \in A$ (with $|u| = 1$), takes the value $d_{x_o}f\,(v) = |v|\, d_{x_o}f(u)$.

$$\begin{aligned}\left|\frac{f(x_o+au)-f(x_o)}{a}-d_{x_o}f\,(u)\right| &= \left|\frac{(f(x_o+au)-f(x_o))-d_{x_o}f\,(au)}{a}\right| \\ &= \left|\frac{f(x_o+au)-(f(x_o)+d_{x_o}f\,(au))}{a}\right| \\ &= \left|\frac{f(x_o+au)-g(x_o+au)}{a}\right| \\ &= \left|\frac{\psi(x_o+au)}{a}\right| \\ &= \frac{|\psi(x_o+au)|}{|a|} \\ &= \frac{|\psi(x_o+au)|}{|(x_o+au)-x_o|}\end{aligned}$$

On the other hand, as ψ is higher-order infinitesimal at x_o, for any $\epsilon > 0$ there exists a suitably small $\delta > 0$, say $\delta < r$, such that, whenever $0 < |(x_o + au) - x_o| < \delta$, we have

$$\frac{|\varphi(x_o+au)|}{|(x_o+au)-x_o|} < \epsilon$$

So, for any $\epsilon > 0$ there exists a suitably small $\delta > 0$, say $\delta < r$, such that, whenever $0 < |a| < \delta$, we have

$$\left|\frac{f(x_o+au)-f(x_o)}{a}-d_{x_o}f\,(u)\right| < \epsilon$$

That proves our claim. □

Differentiability also implies continuity:

Proposition 19 *If f is differentiable at x_o, then it is continuous at x_o.*

Proof From $f(x) = g(x) + \varphi(x)$ for all $x \in X$, it follows that

$$\lim_{x\to x_o} f(x) = \lim_{x\to x_o} g(x) + \lim_{x\to x_o} \varphi(x)$$

Now recall that g is continuous and φ is infinitesimal at x_o, i.e.

$$\lim_{x\to x_o} g(x) = g(x_o) = f(x_o)\,, \quad \lim_{x\to x_o} \varphi(x) = 0$$

Hence

$$\lim_{x\to x_o} f(x) = f(x_o)$$

That proves our claim. □

Now we give the list (but not the proof) of the basic rules of differentiation.

Local character If $f : X \subset \mathcal{A} \to \mathcal{E}$ is differentiable at $x_o \in X$, so is its restriction $f|_{\mathcal{V}_{x_o}}$ to any open neighbourhood $\mathcal{V}_{x_o} \in \tau_{\mathcal{A}}$ contained in X and

$$d_{x_o}(f|_{\mathcal{V}_{x_o}}) = d_{x_o} f$$

Additivity If $f : X \subset \mathcal{A} \to \mathcal{E}$ and $\psi : X \subset \mathcal{A} \to E$ are differentiable at $x_o \in S$, so is their sum $f + \psi : X \subset \mathcal{A} \to \mathcal{E}$ and [11]

$$d_{x_o}(f + \psi) = d_{x_o} f + d_{x_o} \psi$$

Leibniz rule If $f : X \subset \mathcal{A} \to \mathbb{R}$ and $h : X \subset \mathcal{A} \to \mathbb{R}$ are differentiable at $x_o \in S$, so is their product $fh : X \subset \mathcal{A} \to \mathbb{R}$ and [12]

$$d_{x_o}(fh) = f(x_o)\, d_{x_o} h + h(x_o)\, d_{x_o} f$$

Chain rule If $f : X \subset \mathcal{A} \to \mathcal{H}$ is differentiable at $x_o \in X$ and $h : Y \subset \mathcal{H} \to \mathcal{E}$ is differentiable at $f(x_o) \in f(X) \subset Y$, so is their composite $h \circ f : X \subset \mathcal{A} \to \mathcal{E}$ at x_o and

$$d_{x_o}(h \circ f) = d_{f(x_o)} h \circ d_{x_o} f$$

Remark If

$$\begin{aligned} f \;&:\; X \subset \mathcal{A} \to \mathcal{E} := \mathcal{E}_1 \times \cdots \times \mathcal{E}_\nu \\ &:\; x \mapsto f(x) = (f_1(x), \ldots, f_\nu(x)) \end{aligned}$$

is differentiable at $x_o \in X$, so is each projection $f_i : x \in X \subset \mathcal{A} \mapsto f_i(x) \in \mathcal{E}_i$ and

$$\begin{aligned} d_{x_o} f \;&:\; A \to E := E_1 \times \cdots \times E_\nu \\ &:\; v \mapsto d_{x_o} f(v) = (d_{x_o} f_1(v), \ldots, d_{x_o} f_\nu(v)) \end{aligned}$$

Hint Let $pr_i : \mathcal{E} \to \mathcal{E}_i$ (resp. $pr_i : E \to E_i$) be the projection of $\mathcal{E}$ onto its i-th factor $\mathcal{E}_i$ (resp. the projection of E onto its i-th factor E_i) and then apply the chain rule to $f_i = pr_i \circ f$.

---

[11] Linear maps like $d_{x_o} f : A \to E$ and $d_{x_o} \psi : A \to E$, can naturally be summed.

[12] Recall that $d_{x_o} f$, $d_{x_o} h$ and $d_{x_o}(fh)$ belong to A^*.
Remark that Leibniz rule extends to any kind of product between vector-valued maps.

C^∞ *differentiable maps*

We shall now introduce differentiability of higher order for a kind of map

$$\xi : W \subset \mathbb{R}^n \to \mathcal{E} : q \;\mapsto\; \mathrm{p} = \xi(q)$$

defined on an open subset W of $\mathbb{R}^n$ [13] and taking its values in a Euclidean affine space $\mathcal{E}$ (ξ is said to be a *parametrization* of the subset $\mathrm{Im}(\xi)$ of $\mathcal{E}$).

If ξ is differentiable (i.e. differentiable at every point of its domain of definition), its differential at any $q = (q^1, \ldots, q^n) \in W$ is a linear map

$$\begin{aligned} d_q\xi : \mathbb{R}^n \to E : \delta q = \delta q^h \delta_h \mapsto \delta \mathrm{p} &= d_q\xi\,(\delta q) \\ &= (d_q\xi\,(\delta_h))\delta q^h \\ &= \left.\frac{\partial p}{\partial q^h}\right|_q \delta q^h \;\approx\; \xi(q + \delta q) - \xi(q) \end{aligned}$$

whose image

$$\mathrm{Im}(d_q\xi) = \mathrm{Span}\left(\left.\frac{\partial p}{\partial q^h}\right|_q\right)_{h=1,\ldots,n}$$

is spanned by the directional derivatives

$$\left.\frac{\partial \mathrm{p}}{\partial q^h}\right|_q := \lim_{a \to 0} \frac{\xi(q + a\delta_h) - \xi(q)}{a} = d_q\xi\,(\delta_h)$$

which, if q is let to vary in W, determine the n-tuple of maps

$$\frac{\partial \mathrm{p}}{\partial q^h} : W \subset \mathbb{R}^n \to E : q \mapsto \left.\frac{\partial \mathrm{p}}{\partial q^h}\right|_q$$

called *first-order partial derivatives* of ξ .

The above maps, if differentiable, give rise to the $n \times n$ matrix of *second-order partial derivatives* of ξ , [14] i.e.

$$\frac{\partial^2 \mathrm{p}}{\partial q^k \partial q^h} : W \subset \mathbb{R}^n \to E : q \mapsto \left.\frac{\partial^2 \mathrm{p}}{\partial q^k \partial q^h}\right|_q := \left.\frac{\partial}{\partial q^k}\right|_q \frac{\partial \mathrm{p}}{\partial q^h}$$

By iterating such a procedure, under further hypotheses of differentiability, one can obtain higher-order partial derivatives of ξ .

---

[13] The case $n = 1$ will be treated in detail in the next subsection.

[14] Such a matrix, owing to a classical result of Analysis, is symmetric.

ξ is said to be C^∞ *differentiable*, if it is differentiable and admits differentiable partial derivatives of any order (clearly, such a map is continuous and admits continuous partial derivatives of any order).

Owing to the local character of differentiability, ξ is C^∞ *differentiable* iff so is its restiction to an open neighbourhood of each point of its domain.

Elementary operations (such as sum, product, composition) preserve C^∞ differentiability.

An affine mapping

$$\xi : \mathbb{R}^n \to \mathcal{E} : q \mapsto \mathrm{p} = \mathrm{p}_o + q^h \, \mathbf{e}_h$$

(where $\mathrm{p}_o := \xi(0)$ and $\mathbf{e}_h = \Xi(\delta_h)$, $\Xi : \mathbb{R}^n \to E$ being the linear part of ξ) is C^∞, with constant first-order derivatives

$$\frac{\partial \mathrm{p}}{\partial q^h} = \mathbf{e}_h$$

and vanishing higher-order derivatives.

Remark If

$$\begin{aligned} \xi \quad &: \quad W \subset \mathbb{R}^n \to \mathcal{E} := \mathcal{E}_1 \times \cdots \times \mathcal{E}_\nu \\ &: \quad q \mapsto \mathrm{p}(q) = (\mathrm{p}_1(q), \ldots, \mathrm{p}_\nu(q)) \end{aligned}$$

is C^∞, so is each projection $\xi_i : q \in W \subset \mathbb{R}^n \mapsto \mathrm{p}_i(q) \in \mathcal{E}_i$ and, for all $q \in W$,

$$\left.\frac{\partial \mathrm{p}}{\partial q^h}\right|_q = \left(\left.\frac{\partial \mathrm{p}_1}{\partial q^h}\right|_q, \ldots, \left.\frac{\partial \mathrm{p}_\nu}{\partial q^h}\right|_q \right) \in E := E_1 \times \cdots E_\nu$$

(and so on for higher-order derivatives).

In particular, if

$$\xi : W \subset \mathbb{R}^n \to \mathbb{R}^m : q \mapsto x(q) = x^i(q) \, \delta_i$$

is a C^∞ map taking its values in $\mathbb{R}^m$, then

$$\left.\frac{\partial x}{\partial q^h}\right|_q = \left.\frac{\partial x^i}{\partial q^h}\right|_q \delta_i \in \mathbb{R}^m$$

(and so on for higher-order derivatives).

As a consequence, the differential $d_q\xi : \mathbb{R}^n \to \mathbb{R}^m$ operates by means of the *Jacobian matrix* $\left[\left.\frac{\partial x^i}{\partial q^h}\right|_q \right]$ of ξ, since, for any $v = v^h \delta_h \in \mathbb{R}^n$,

$$\begin{aligned} w^i\delta_i = w &= d_q\xi(v) \\ &= v^h\, d_q\xi(\delta_h) \\ &= v^h\, \frac{\partial x}{\partial q^h}\Big|_q \\ &= v^h\, \left(\frac{\partial x^i}{\partial q^h}\Big|_q\, \delta_i\right) \\ &= \left(v^h\, \frac{\partial x^i}{\partial q^h}\Big|_q\right)\, \delta_i \end{aligned}$$

that is,

$$w^i = \frac{\partial x^i}{\partial q^h}\Big|_q\, v^h$$

C^∞ ***differentiable curves***

We shall now specialize our considerations to functions of one real variable (with values in an arbitrary Euclidean affine space), whose role is crucial in the next sections (as well as in the main text).

Let

$$\gamma : I \subset \mathbb{R} \to \mathcal{E} : t \mapsto \mathrm{p} = \gamma(t) = \mathrm{p}(t)$$

be a *parametrized curve* of a Euclidean affine space $\mathcal{E}$, defined on an open interval (connected open subset) $I \subset \mathbb{R}$ (and then describing, if continuous, a connected *orbit* $\mathrm{Im}(\gamma)$ in $\mathcal{E}$).

If γ is differentiable, its differential at any $t \in I$ is a linear map

$$\begin{aligned} d_t\gamma : \mathbb{R} \to E : dt \mapsto d\mathrm{p} &= d_t\gamma\,(dt) \\ &= (d_t\gamma\,(1))dt \\ &= \frac{d\mathrm{p}}{dt}\Big|_t dt \quad \approx \quad \mathrm{p}(t+dt) - \mathrm{p}(t) \end{aligned}$$

whose image

$$\mathrm{Im}(d_t\gamma) = \mathrm{Span}\left(\frac{d\mathrm{p}}{dt}\Big|_t\right)$$

is spanned by the derivative

$$\frac{d\mathrm{p}}{dt}\Big|_t := \lim_{a\to 0} \frac{\mathrm{p}(t+a) - \mathrm{p}(t)}{a} = d_t\gamma\,(1)$$

which, if t is let to vary in I, determines the parametrized curve of E

$$\frac{d\mathrm{p}}{dt} : I \to E : t \mapsto \left.\frac{d\mathrm{p}}{dt}\right|_t =: \dot{\mathrm{p}}(t)$$

called *first-order derivative* of γ.

The above parametrized curve, if differentiable, gives rise – at any $t \in I$ – to

$$\left.\frac{d^2\mathrm{p}}{dt^2}\right|_t := \left.\frac{d\dot{\mathrm{p}}}{dt}\right|_t := \lim_{a\to 0} \frac{\dot{\mathrm{p}}(t+a) - \dot{\mathrm{p}}(t)}{a}$$

which in turn determines the parametrized curve of E

$$\frac{d^2\mathrm{p}}{dt^2} : I \to E : t \mapsto \left.\frac{d^2\mathrm{p}}{dt^2}\right|_t =: \ddot{\mathrm{p}}(t)$$

called *second-order derivative* of γ.

By iterating such a procedure, under further hypotheses of differentiability, one can obtain higher-order derivatives of γ.

γ is a C^∞ *differentiable curve*, if it is differentiable and admits differentiable derivatives of any order.

Remark If

$$\begin{aligned} \gamma \quad &: \quad I \subset \mathbb{R} \to \mathcal{E} := \mathcal{E}_1 \times \cdots \times \mathcal{E}_\nu \\ &: \quad t \mapsto \mathrm{p}(t) = (\mathrm{p}_1(t), \ldots, \mathrm{p}_\nu(t)) \end{aligned}$$

is C^∞, so is each projection $\gamma_i : t \in I \subset \mathbb{R} \mapsto \mathrm{p}_i(t) \in \mathcal{E}_i$ and, for all $t \in I$,

$$\dot{\mathrm{p}}(t) = (\dot{\mathrm{p}}_1(t), \ldots, \dot{\mathrm{p}}_\nu(t)) \in E := E_1 \times \cdots E_\nu$$

(and so on for higher-order derivatives).

In particular, if

$$q(t) = q^h(t)\, \delta_h \in \mathbb{R}^n$$

is a C^∞ curve of $\mathbb{R}^n$, then

$$\dot{q}(t) = \dot{q}^h(t)\, \delta_h \in \mathbb{R}^n$$

(and so on for higher-order derivatives).

Coordinate expression Let

$$\gamma : I \subset \mathbb{R} \to \mathcal{E} : t \mapsto \mathrm{p}(t)$$

be a parametrized curve of $\mathcal{E}$, a piece of whose orbit

$$\mathrm{Im}(\gamma|_{I_*}) \subset \mathcal{U}$$

(image of an open subinterval $I_* \subset I$ by γ) lives in a subset

$$\mathcal{U} = \mathrm{Im}(\xi) \subset \mathcal{E}$$

parametrized by an injective C^∞ map

$$\xi : W \subset \mathbb{R}^n \to \mathcal{E} : q \mapsto \xi(q)$$

and whose local *coordinate expression* (through $\xi^{-1} : \mathcal{U} \to W$)

$$\gamma_{\xi} := \xi^{-1} \circ \gamma|_{I_*} : I_* \to \mathbb{R}^n : t \mapsto q(t) := \xi^{-1}(\mathrm{p}(t))$$

is a C^∞ curve of $\mathbb{R}^n$.

From

$$\gamma|_{I_*} = \xi \circ \gamma_{\xi} : I_* \to \mathcal{E} : t \mapsto \mathrm{p}(t) = \xi(q(t))$$

we infer that $\gamma|_{I_*}$ is a C^∞ curve of $\mathcal{E}$ and then, for any $t \in I_*$,

$$\begin{aligned}
\dot{\mathrm{p}}(t) &= d_t\gamma(1) \\
&= d_t\gamma|_{I_*}(1) \\
&= (d_t(\xi \circ \gamma_{\xi}))(1) \\
&= (d_{q(t)}\xi \circ d_t\gamma_{\xi})(1) \\
&= d_{q(t)}\xi\,(d_t\gamma_{\xi}\,(1)) \\
&= d_{q(t)}\xi\,(\dot{q}(t))
\end{aligned}$$

or (recalling that $q(t) = q^h(t)\,\delta_h$ implies $\dot{q}(t) = \dot{q}^h(t)\,\delta_h$)

$$\begin{aligned}
\dot{\mathrm{p}}(t) &= d_{q(t)}\xi\,(\dot{q}^h(t)\delta_h) \\
&= (d_{q(t)}\xi\,(\delta_h))\dot{q}^h(t) \\
&= \left.\frac{\partial \mathrm{p}}{\partial q^h}\right|_{q(t)} \dot{q}^h(t)
\end{aligned}$$

5.2 Smooth manifolds embedded in a Euclidean affine space

We shall now be concerned with the differential-topological properties of smooth manifolds embedded in a Euclidean affine space, a kind of well behaved subspaces including familiar 'loci' such as smooth curves and surfaces (which 'infinitesimally' resemble straight lines and planes, respectively).

Flat manifolds

We start with the study of 'flat manifolds' (like straight lines and planes).

Let $\mathcal{E}$ be a Euclidean affine space (modelled on a vector space E).

A non-empty subset $\mathcal{A} \subset \mathcal{E}$ is said to be an n-dimensional *flat manifold* embedded in $\mathcal{E}$ (n being a positive integer), if it is the orbit of a point $\mathrm{p}_o \in \mathcal{E}$ under the affine action of an n-dimensional vector subspace $A \subset E$, i.e.

$$\mathcal{A} = \mathrm{p}_o + A := \{\mathrm{p}_o + \mathrm{v}\}_{\mathrm{v} \in A}$$

(clearly, $\mathrm{p}_o \in \mathcal{A}$).

Such an $\mathcal{A} \subset \mathcal{E}$ is usually called an n-dimensional *affine subspace* of $\mathcal{E}$, owing to to the fact that it inherits the structure of an affine space from $\mathcal{E}$, modelled on the vector subspace $A \subset E$. [15]

Special names are adopted for $\mathcal{A}$ in the following cases: $\mathcal{A}$ is said to be a *straight line*, a *plane* or (in the case $\dim(\mathcal{E}) > 3$) a *hyperplane*, according to whether $\dim(\mathcal{A}) = 1$, $\dim(\mathcal{A}) = 2$ or $\dim(\mathcal{A}) = \dim(\mathcal{E}) - 1$, respectively.

Check that

(i) $\mathcal{A} = \mathrm{p} + A$, for all $\mathrm{p} \in \mathcal{A}$.

(ii) $\dim(\mathcal{A}) \leq \dim(\mathcal{E})$, the equality holding iff $\mathcal{A} = \mathcal{E}$.

(iii) $\mathcal{A}$ is naturally Euclidean.

(iv) In $\mathcal{A}$, meant as both a Euclidean affine space of its own and a topological subspace of $\mathcal{E}$, the Euclidean topology coincides with the subspace topology. [16]

---

[15] The above 'inheritance' means that, if $+$ denotes the affine structure of $\mathcal{E}$, the restriction $+|_{\mathcal{A} \times A}$ takes values in $\mathcal{A}$ and defines an affine action of A on $\mathcal{A}$.

Clearly, any non-empty subset $\mathcal{A} \subset \mathcal{E}$ satisfying –with a vector subspace $A \subset E$– the latter condition, is a flat manifold.

[16] As to (iv), remark that any open ball of $\mathcal{A}$ is the intersection of $\mathcal{A}$ with the open ball of $\mathcal{E}$ with the same centre and the same radius.

An n-dimensional affine subspace $\mathcal{A}$ can *globally* be given an n-dimensional affine parametrization, as follows.

Let $\varphi : \mathbb{R}^n \to \mathcal{A}$ be a Cartesian system in $\mathcal{A}$, with linear part $\Phi : \mathbb{R}^n \to A$. The composite

$$\xi := \iota_{\mathcal{A}} \circ \varphi : \mathbb{R}^n \to \mathcal{A} \hookrightarrow \mathcal{E}$$

is an injective affine map of $\mathbb{R}^n$ into $\mathcal{E}$, satisfying

$$\mathrm{Im}(\xi) = \mathcal{A}$$

(clearly $\Xi := \iota_A \circ \Phi : \mathbb{R}^n \to A \hookrightarrow E$ is the injective linear part of ξ and satisfies $A = \mathrm{Im}(\Xi)$).

Conversely, if $\xi : \mathbb{R}^n \to \mathcal{E}$ is an injective affine map, with linear part $\Xi : \mathbb{R}^n \to E$, then $\mathrm{Im}(\xi)$ is an n-dimensional affine subspace of $\mathcal{E}$, modelled on the vector subspace $\mathrm{Im}(\Xi)$ of E.

So the affine subspaces are all and only the subsets of $\mathcal{E}$ that can be parametrized by means of injective, affine maps.

Such a kind of *affine parametrization* –which characterizes the affine subspaces– exhibits the following differential-topological properties: [17]

Proposition 20 *If ξ is an affine parametrization of $\mathcal{A}$, then*
(1) *$\xi : \mathbb{R}^n \to \mathcal{A}$ is a homeomorphism;*
(2) *$\xi : \mathbb{R}^n \to \mathcal{E}$ is C^∞, with injective differential $d_q\xi : \mathbb{R}^n \to E$ at each $q \in \mathbb{R}^n$.*

Proof Recall that
(1) $\xi : \mathbb{R}^n \to \mathcal{A}$ is an affine isomorphism and therefore a homeomorphism; [18]
(2) $\xi : \mathbb{R}^n \to \mathcal{E}$ is affine and therefore C^∞, with injective differential $d_q\xi = \Xi$ at each $q \in \mathbb{R}^n$. □

---

[17] The map induced by $\xi : \mathbb{R}^n \to \mathcal{E}$ onto its own image $\mathcal{A}$ will be denoted by $\xi : \mathbb{R}^n \to \mathcal{A}$. Moreover, $\mathcal{A}$ will be meant to be equipped with its subspace topology.

[18] See section 5.1 *Continuous maps* and the above property (iv).

Smooth manifolds

We shall now pass on to the study of 'smooth manifolds' in a Euclidean environment, conceived as 'loci' which admit (local) parametrizations exhibiting the same differential-topological properties as those of the (global) parametrizations of the affine subspaces.

Going into detail, a non-empty subset $Q \subset \mathcal{E}$ will be said to be an n-dimensional *smooth manifold* embedded in $\mathcal{E}$ (n being a positive integer), if each point $\mathrm{p}_o \in Q$ admits an open neighbourhood $\mathcal{U} \in \tau_{Q}$ [19] which is the image

$$\mathcal{U} = \mathrm{Im}(\xi)$$

of a parametrization

$$\xi : W \subset \mathbb{R}^n \rightarrow \mathcal{E}$$

defined on an open subset W of $\mathbb{R}^n$ and satisfying the following differential-topological properties: [20]

1. $\xi : W \rightarrow \mathcal{U}$ is a homeomorphism;
2. $\xi : W \rightarrow \mathcal{E}$ is C^∞, with injective differential $d_q\xi : \mathbb{R}^n \rightarrow E$ at each $q \in W$.

Each point $\mathrm{p} \in \mathcal{U}$ is then given a unique n-tuple of *coordinates* $q = (q^1, \ldots, q^n) = \xi^{-1}(\mathrm{p}) \in W$ by the n-dimensional *chart* $\xi^{-1} : \mathcal{U} \rightarrow W$, [21] which is said to be *local* or *global* according to whether its *coordinate domain* $\mathcal{U}$ is strictly contained in Q or coincides with Q, respectively. [22]

A collection of charts whose coordinate domains cover the whole manifold, is said to be an *atlas*.

Special names are adopted for Q in the following cases: Q is said to be a smooth *curve*, *surface* or (in the case $\dim(\mathcal{E}) > 3$) *hypersurface*, according to whether $\dim(Q) = 1$, $\dim(Q) = 2$ or $\dim(Q) = \dim(\mathcal{E}) - 1$, respectively.

---

[19] τ_Q denotes the subspace topology of Q.

[20] The map induced by $\xi : W \rightarrow \mathcal{E}$ onto its own image $\mathcal{U}$, will be denoted by $\xi : W \rightarrow \mathcal{U}$.

[21] In the sequel, the name 'chart' will often be referred to the parametrization ξ as well.

[22] For instance, a flat manifold admits global charts, determined by its affine parametrizations. Moreover, if $\xi : W \rightarrow \mathcal{U}$ is a (local) chart on a smooth manifold Q, its coordinate domain $\mathcal{U}$ is a smooth manifold as well, admitting ξ as a global chart.

Proposition 21 $\dim(Q) \leq \dim(\mathcal{E})$

Proof From the well known dimensional property

$$\dim(\mathrm{Im}(d_q\xi)) = \dim(\mathbb{R}^n) - \dim(\mathrm{Ker}(d_q\xi))$$

and the injectivity

$$\mathrm{Ker}(d_q\xi) = \{0\}$$

of the linear map $d_q\xi : \mathbb{R}^n \to E$, we infer that $\mathrm{Im}(d_q\xi)$ is an n-dimensional vector subspace of E and hence $n \leq \dim(E)$.

As $n = \dim(Q)$ and $\dim(E) = \dim(\mathcal{E})$, that proves our claim. □

Locally Euclidean topology

From a topological point of view, Q is a *locally Euclidean* topological subspace of $\mathcal{E}$, in the sense that it is covered – owing to *topological property* 1 – by open subsets (namely, the coordinate domains of its charts) homeomorphic to open subsets of Euclidean space $\mathbb{R}^n$.

Smoothness

From a differential point of view, Q is a *smooth* subspace of $\mathcal{E}$, in the sense that it admits – owing to *smoothness property* 2 – an n-dimensional 'tangent vector space' at each one of its points, as will now be shown.

Smooth curves and tangent vector spaces

Remark that 'tangency' is an 'infinitesimal' concept, linked to derivation as follows.

First consider a *smooth curve* of $\mathcal{E}$, i.e. a C^∞ differentiable curve

$$\gamma : I \subset \mathbb{R} \to \mathcal{E} : t \mapsto \mathrm{p} = \mathrm{p}(t)$$

The vector $\mathbf{v} = \dot{\mathrm{p}}(t) \in E$, obtained from γ by derivation at any $t \in I$, is said to be *tangent* at $\mathrm{p} = \mathrm{p}(t)$ to γ. [23]

---

[23] Think of the representation of $\dot{\mathrm{p}}(t) := \lim_{a\to 0} \frac{1}{a}\big(\mathrm{p}(t+a) - \mathrm{p}(t)\big)$ – when it does not vanish – as an oriented segment attached at $\mathrm{p}(t)$, obtained via limit from a secant segment attached at the same point (and pointing towards the 'future' determined by the increasing 'time' t).

Now consider a *smooth curve* of Q, i.e. a parametrized curve

$$\gamma : I \subset \mathbb{R} \to Q : t \mapsto \mathrm{p} = \mathrm{p}(t)$$

which, for each point

$$\mathrm{p} \in \mathrm{Im}(\gamma)$$

of its orbit, admits a C^∞ *coordinate expression* $\gamma_\xi = \xi^{-1} \circ \gamma|_{I_*}$ in one (and then every) chart $\xi : W \to \mathcal{U}$ of Q near p (i.e. with $\mathcal{U} \ni \mathrm{p}$). [24]

Clearly a smooth curve of Q is a smooth curve of $\mathcal{E}$ whose orbit entirely lies on Q, i.e.

$$\mathrm{Im}(\gamma) \subset Q$$

(the converse could be proved as well).

A vector of E tangent to a smooth curve γ of Q at a point p of its orbit, will then be said to be *tangent* at p to Q.

For any $\mathrm{p} \in Q$, the set $T_{\mathrm{p}}Q$ of all the vectors of E tangent at p to Q –i.e. tangent at p to some smooth curve of Q passing through p– will be called the *tangent vector space* of Q at p, owing to the following

Proposition 22 *$T_{\mathrm{p}}Q$ is a vector subspace of E and* $\dim(T_{\mathrm{p}}Q) = \dim(Q)$.

Proof Let $\xi : W \to \mathcal{U}$ be a chart of Q near $\mathrm{p} = \xi(q) \in \mathcal{U}$.

Recall that, owing to property 2, $\mathrm{Im}(d_q\xi) \subset E$ is a vector subspace of dimension $n = \dim(Q)$.

So our claim will be established by proving the following [25]

Lemma ($\star$) $\mathrm{Im}(d_q\xi) = T_{\mathrm{p}}Q$

Let $\mathbf{v} \in T_{\mathrm{p}}Q$, i.e.

$$\mathbf{v} = \dot{\mathrm{p}}(t_*)$$

for some smooth curve $\gamma : t \in I \mapsto \mathrm{p}(t) \in Q$ passing, at $t_* \in I$, through $\mathrm{p}(t_*) = \mathrm{p}$.

As $\mathcal{U}$ is an open neighbourhood of $\mathrm{p}(t_*) = \mathrm{p}$ in Q, there exists (by continuity) an open interval $I_* \subset I$ containing t_* s.t. $\gamma(I_*) \subset \mathcal{U}$

Then γ can be given the C^∞ coordinate expression

$$\gamma_\xi := \xi^{-1} \circ \gamma|_{I_*} : I_* \to \mathbb{R}^n : t \mapsto q(t) := \xi^{-1}(\mathrm{p}(t))$$

satisfying $q(t_*) = \xi^{-1}(\mathrm{p}(t_*)) = \xi^{-1}(\mathrm{p}) = q$.

---

[24] See section 5.1, C^∞ *differentiable curves*, *Coordinate expression*.

[25] Remark that the following Lemma ($\star$) implies $T_{\mathrm{p}}\mathcal{U} = T_{\mathrm{p}}Q$ (see footnote [22]).

By derivation at t_* of γ or, equivalently, of

$$\gamma|_{I_*} = \xi \circ \gamma_{\xi} : I_* \to Q : t \mapsto \mathrm{p}(t) = \xi(q(t))$$

we obtain [26]

$$\mathbf{v} = \dot{\mathrm{p}}(t_*) = d_{q(t_*)}\xi\,(\dot{q}(t_*)) = d_q\xi\,(\dot{q}(t_*))$$

Hence $\mathbf{v} \in \mathrm{Im}(d_q\xi)$.

Conversely, let $\mathbf{v} \in \mathrm{Im}(d_q\xi)$, i.e.

$$\mathbf{v} = d_q\xi\,(v)$$

for some $v \in \mathbb{R}^n$.

Choose $t_* \in \mathbb{R}$ and consider the affine mapping

$$\lambda : \mathbb{R} \to \mathbb{R}^n : t \mapsto q(t) := q + v\,(t - t_*)$$

with linear part $\Lambda : dt \in \mathbb{R} \mapsto v\,dt \in \mathbb{R}^n$, satisfying $q(t_*) = q \in W$ and $\dot{q}(t_*) = d_{t_*}\lambda\,(1) = \Lambda(1) = v$.

As W is an open neighbourhood of $q(t_*) = q$ in $\mathbb{R}^n$, there exists (by continuity) an open interval I_* containig t_* s.t. $\lambda(I_*) \subset W$.

Then $\gamma_\xi := \lambda|_{I_*}$ is the coordinate expression of

$$\gamma := \xi \circ \gamma_\xi : I_* \to \mathcal{U} : t \mapsto \mathrm{p}(t) = \xi(q(t))$$

which is a smooth curve of Q passing through

$$\mathrm{p}(t_*) = \xi(q(t_*)) = \xi(q) = \mathrm{p}$$

with derivative

$$\dot{\mathrm{p}}(t_*) = d_{q(t_*)}\xi\,(\dot{q}(t_*)) = d_q\xi\,(v) = \mathbf{v}$$

Hence $\mathbf{v} \in T_{\mathrm{p}}Q$. □

Some remarks follow from Lemma ($\star$).

Notice that

$$\mathbf{v} \in T_{\mathrm{p}}Q = \mathrm{Im}(d_q\xi) \longmapsto v = (d_q\xi)^{-1}(\mathbf{v}) \in \mathbb{R}^n$$

is the linear isomorphism which takes any vector $\mathbf{v} \in T_{\mathrm{p}}Q$ to its *components* $v \in \mathbb{R}^n$ *in* ξ , that is, to its components in the basis of $T_{\mathrm{p}}Q$ given by

$$\left.\frac{\partial}{\partial q^h}\right|_{\mathrm{p}} := d_q\xi(\delta_h) = \left.\frac{\partial \mathrm{p}}{\partial q^h}\right|_q , \quad h = 1, \ldots, n$$

Moreover, if $\mathrm{p} = \xi(q) \in \mathcal{U} \subset Q$ and $\mathbf{v} = d_q\xi(v) \in T_{\mathrm{p}}Q$ (with 'small' $|\mathbf{v}|$), then – up to higher-order infinitesimals – $\mathrm{p} + \mathbf{v} \in \mathcal{U} \subset Q$, since

$$\mathrm{p} + \mathbf{v} = \xi(q) + d_q\xi(v) \approx \xi(q + v)$$

---

[26] See section 5.1, C^∞ *differentiable curves*, *Coordinate expression.*

Open submanifolds

Let U be an open subset of $\mathcal{E}$.

For instance, if

$$g = (g_1, \ldots, g_\mu) : \mathcal{E} \to \mathbb{R}^\mu : \mathrm{p} \mapsto g(\mathrm{p}) = (g_1(\mathrm{p}), \ldots, g_\mu(\mathrm{p}))$$

is a continuous mapping, then

$$\begin{aligned} U &:= g^{-1}(\mathbb{R}^+)^\mu \\ &= \{\mathrm{p} \in \mathcal{E} \mid g(\mathrm{p}) \in (\mathbb{R}^+)^\mu\} \\ &= \{\mathrm{p} \in \mathcal{E} \mid g_1(\mathrm{p}) > 0, \ldots, g_\mu(\mathrm{p}) > 0\} \end{aligned}$$

is an open subset of $\mathcal{E}$. [27]

If $\varphi : \mathbb{R}^m \to \mathcal{E}$ is a Cartesian system in $\mathcal{E}$, its restriction $\xi := \varphi|_W$ to the open subset $W := \varphi^{-1}(U) \subset \mathbb{R}^m$ determines a global m-dimensional chart on U (since $\mathrm{Im}(\xi) = U$), which is therefore a smooth manifold of maximal dimension

$$\dim(U) = \dim(\mathcal{E})$$

Owing to obvious dimensional reasons,

$$T_\mathrm{p} U = E$$

for all $\mathrm{p} \in U$.

Implicit function theorem

Non-trivial examples of smooth manifolds embedded in $\mathcal{E}$ arise from the well known Implicit Function Theorem, concerning ‘loci’ described by means of algebraic equations.

Proposition 23 *Let*

$$f = (f_1, \ldots, f_\kappa) : U \subset \mathcal{E} \to \mathbb{R}^\kappa : \mathrm{p} \longmapsto f(\mathrm{p}) = (f_1(\mathrm{p}), \ldots, f_\kappa(\mathrm{p}))$$

be a differentiable map, defined on an open subset $U \subset \mathcal{E}$ *(say* $U = g^{-1}(\mathbb{R}^+)^\mu$ *) and taking values in* $\mathbb{R}^\kappa$ *(* $\kappa < m := \dim \mathcal{E}$ *), with surjective differential* $d_\mathrm{p} f : E \to \mathbb{R}^\kappa$ *at each point* p *of*

$$\begin{aligned} Q &:= f^{-1}(0) \\ &= \{\mathrm{p} \in U \mid f(\mathrm{p}) = 0\} \\ &= \{\mathrm{p} \in U \mid f_1(\mathrm{p}) = 0, \ldots, f_\kappa(\mathrm{p}) = 0\} \\ &= \{\mathrm{p} \in \mathcal{E} \mid g_1(\mathrm{p}) > 0, \ldots, g_\mu(\mathrm{p}) > 0 \,,\; f_1(\mathrm{p}) = 0, \ldots, f_\kappa(\mathrm{p}) = 0\} \end{aligned}$$

---

[27] Recall that $\mathbb{R}^+ := (0, +\infty)$ is an open subset of $\mathbb{R}$ and then $(\mathbb{R}^+)^\mu$ is an open subset of $\mathbb{R}^\mu$.

Q is then an n-dimensional smooth manifold with

$$n := m - \kappa$$

and, for all $\mathrm{p} \in Q$,

$$T_{\mathrm{p}}Q = \mathrm{Ker}(d_{\mathrm{p}}f)$$

Proof The above mentioned theorem of Analysis states that, for any $\mathrm{p} \in Q$, the fact of $d_{\mathrm{p}}f$ being surjective implies the existence of an open neighbourhood $\mathcal{U}$ of p in Q , an open subset W of $\mathbb{R}^n$ and a κ-tuple of real-valued functions $\varphi^1 : W \to \mathbb{R}, \ldots, \varphi^\kappa : W \to \mathbb{R}$ such that

$$\mathcal{U} = \mathrm{Im}(\xi)$$

where $\xi : W \to \mathcal{E}$ is defined (with a suitable choice of a Cartesian systen $\varphi : \mathbb{R}^m \to \mathcal{E}$) by

$$(q^1, \ldots, q^n) \overset{\xi}{\mapsto} \varphi(\, q^1, \ldots, q^n,\, \varphi^1(q^1, \ldots, q^n), \ldots, \varphi^\kappa(q^1, \ldots, q^n)\,)$$

and fulfils the properties characterizing an n-dimensional chart on Q . [28]

That proves our first claim.

As the above chart ξ is obviously composable with f and $f \circ \xi = 0$, we have $d_{\mathrm{p}}f \circ d_q\xi = d_q(f \circ \xi) = 0$ with $q := \xi^{-1}(\mathrm{p})$, whence

$$\mathrm{Im}(d_q\xi) \subset \mathrm{Ker}(d_{\mathrm{p}}f)$$

Moreover, the surjectivity of $d_{\mathrm{p}}f$ and the injectivity of $d_q\xi$ imply the dimensional result

$$\dim(\mathrm{Ker}(d_{\mathrm{p}}f)) = \dim(E) - \dim(\mathrm{Im}(d_{\mathrm{p}}f)) = m - \kappa = n = \dim(\mathrm{Im}(d_q\xi))$$

As a consequence,

$$\mathrm{Im}(d_q\xi) = \mathrm{Ker}(d_{\mathrm{p}}f)$$

Owing to Lemma ($\star$), that proves our second claim. □

---

[28] So, on $\varphi^{-1}(\mathcal{U})$, the algebraic equations

$$f_1\big(\varphi(x^1, \ldots, x^n, x^{n+1}, \ldots, x^{n+\kappa})\big) = 0, \ldots, f_\kappa\big(\varphi(x^1, \ldots, x^n, x^{n+1}, \ldots, x^{n+\kappa})\big) = 0$$

implicitly define $x^{n+1}, \ldots, x^{n+\kappa}$ as functions of $(x^1, \ldots, x^n) \in W$, i.e.

$$x^{n+1} = \varphi^1(x^1, \ldots, x^n), \ldots, x^{n+\kappa} = \varphi^\kappa(x^1, \ldots, x^n)$$

Smooth maps and differential calculus

We shall now extend the differential calculus to maps whose domain of definition is a smooth manifold.

We start with a real-valued function

$$f : Q \to \mathbb{R}$$

f will be said to be a *differentiable* (resp. *smooth*) *function*, if, for each point $\mathrm{p} \in Q$, it admits a differentiable (resp. C^∞ differentiable) *coordinate expression*

$$f_\xi := f \circ \xi$$

in one (and then every) chart ξ of Q near p.

Now define the *directional derivative* of a differentiable function f along a tangent vector $\mathbf{v} \in T_{\mathrm{p}}Q$ by putting

$$\mathbf{v}(f) := \frac{d}{dt}\Big|_{t_*} (f \circ \gamma)$$

where $\gamma : t \in I \subset \mathbb{R} \mapsto \mathrm{p}(t) \in Q$ is a a smooth parametrized curve of Q such that $\mathrm{p} = \mathrm{p}(t_*)$ and $\mathbf{v} = \dot{\mathrm{p}}(t_*)$ for some $t_* \in I$.

The above derivative does not depend on the choice of γ, as is shown by the following coordinate expression:

Proposition 24 *If* $(\mathrm{p}, \mathbf{v}) = (\xi(q), d_q\xi(v))$ *in a chart* ξ, *then*

$$\mathbf{v}(f) = d_q f_\xi(v) = v^h \frac{\partial f_\xi}{\partial q^h}\Big|_q$$

In particular,

$$\frac{\partial}{\partial q^k}\Big|_{\mathrm{p}}(f) = \frac{\partial f}{\partial q^k}\Big|_{\mathrm{p}} := \frac{\partial f_\xi}{\partial q^k}\Big|_q$$

Proof By considering the coordinate expressions of γ and f in ξ, we obtain [29]

$$\begin{aligned} \frac{d}{dt}\Big|_{t_*}(f \circ \gamma) &= \frac{d}{dt}\Big|_{t_*}(f \circ \xi \circ \xi^{-1} \circ \gamma|_{I_\star}) \\ &= \frac{d}{dt}\Big|_{t_*}(f_\xi \circ \gamma_\xi) \\ &= (\, d_{t_*}(f_\xi \circ \gamma_\xi)\,)(1) \\ &= (\, d_{q(t_*)}f_\xi \circ d_{t_*}\gamma_\xi)\,)(1) \\ &= d_{q(t_*)}f_\xi(\dot{q}(t_*)) \\ &= d_q f_\xi(v) \end{aligned}$$

---

[29] See section 5.1, C^∞ differentiable curves, *Coordinate expression.*

Moreover

$$\begin{aligned} d_q f_\xi(v) &= d_q f_\xi(v^h \delta_h) \\ &= v^h d_q f_\xi(\delta_h) \\ &= v^h \frac{\partial f_\xi}{\partial q^h}\Big|_q \end{aligned}$$

That proves our claim. □

From the above result, it follows that

$$\begin{aligned} \mathbf{v}(f) &= d_q f_\xi(v) \\ &\approx f_\xi(q+v) - f_\xi(q) \\ &= f(\xi(q+v)) - f(\xi(q)) \\ &\approx f(\mathrm{p}+\mathbf{v}) - f(\mathrm{p}) \end{aligned}$$

and, for any $a, b \in \mathbb{R}$ and $\mathbf{u}, \mathbf{v} \in T_{\mathrm{p}}Q$,

$$(a\,\mathbf{u} + b\,\mathbf{v})(f) = a\,\mathbf{u}(f) + b\,\mathbf{v}(f)$$

which motivates the definition of *differential* of f at p as the linear map given by

$$d_{\mathrm{p}} f : T_{\mathrm{p}}Q \to \mathrm{R} : \mathbf{v} \mapsto \mathbf{v}(f)$$

The above definitions of differentiability (resp. smoothness) and differential at a point, can be extended without any change to a map

$$f : Q \subset \mathcal{E} \to \mathcal{A}$$

whose codomain is an arbitrary Euclidean affine space $\mathcal{A}$, modelled on a vector space A (clearly, if f is a differentiable map, then its differential at a point will take values in A). [30]

5.3 Tangent bundle and differential equations

The tangent bundle of a manifold is the first geometric arena of differential equations.

---

[30] If f is defined on an open manifold $Q \subset \mathcal{E}$, the above definitions of differentiability and differential are equivalent to those given in section 5.1.

Tangent bundle and canonical projection

The *tangent bundle* of an n-dimensional smooth manifold Q is the disjoint union of its tangent vector spaces, i.e.

$$TQ = \bigcup_{\mathrm{p} \in Q} \{\mathrm{p}\} \times T_{\mathrm{p}}Q$$

(so, for any $\mathrm{p} \in Q$, $(\mathrm{p}, \mathbf{v}) \in TQ$ means $\mathbf{v} \in T_{\mathrm{p}}Q$). [31]

The *canonical projection* of TQ onto its *base manifold* Q is the map

$$\pi_{Q} : TQ \to Q : (\mathrm{p}, \mathbf{v}) \mapsto \mathrm{p}$$

Vector fibre bundle structure

TQ is a *vector fibre bundle* over Q, in the sense that its *fibre* $T_{\mathrm{p}}Q$ over any $\mathrm{p} \in Q$ is a vector space. Such fibres are 'glued' together by *natural charts*

$$(q, v) \in TW \longmapsto (\mathrm{p}, \mathbf{v}) = (\xi(q), d_q\xi\,(v)) \in T\mathcal{U}$$

(arising from the charts $\xi : W \to \mathcal{U}$ of Q [32]), which could be shown to give TQ the structure of a $2n$-dimensional smooth manifold embedded in $T\mathcal{E} = \mathcal{E} \times E$. [33]

Tangent lift

If

$$\gamma : I \subset \mathbb{R} \to Q : t \mapsto \gamma(t) = \mathrm{p}(t)$$

is a smooth curve of Q, its *tangent lift*

$$\dot{\gamma} : I \subset \mathbb{R} \to TQ : t \mapsto \dot{\gamma}(t) = (\mathrm{p}(t), \dot{\mathrm{p}}(t))$$

is a smooth curve of TQ that π_Q projects down onto γ, i.e.

$$\pi_{Q} \circ \dot{\gamma} = \gamma$$

Owing to the very definition of tangent vector spaces, TQ is the region of $T\mathcal{E}$ 'swept' by the tangent lifts of all the smooth curves of Q (i.e. covered by the orbits of such lifts).

---

[31] Remark that the tangent bundle of an open manifold $U \subset \mathcal{E}$ is trivial, i.e. $TU = U \times E$. In particular, $T\mathcal{E} = \mathcal{E} \times E$ is a Euclidean affine space modelled on $E \times E$.

[32] Recall that $TW = W \times \mathbb{R}^n$ and $T\mathcal{U} = \pi_{Q}^{-1}(\mathcal{U})$.

[33] See [11], Proposition 5.5.5.

Implicit differential equations

As a consequence, if a subset

$$D \subset TQ$$

is assigned (by prescribing some geometric or algebraic properties), the problem may arise of determining the smooth curves $\gamma : I \subset \mathbb{R} \to Q$ whose tangent lifts live in D, i.e.

$$\mathrm{Im}(\dot{\gamma}) \subset D \tag{$\diamond$}$$

that is to say,

$$\dot{\gamma}(t) = (\mathrm{p}(t), \dot{\mathrm{p}}(t)) \in D\,, \quad \forall t \in I \tag{$\diamond$}$$

Such a D will then be called a *first-order, implicit differential equation* on Q. [34]

Integral curves

Any solution to the above problem, i.e. any smooth curve γ of Q satisfying condition ($\diamond$), is called an *integral curve* of D (or *maximal* integral curve, if it is *not* restriction of any other integral curve).

Integrable part

The problem of establishing the existence of integral curves is the same as that of determining the *integrable part* of D, which is the region

$$D^{(i)} \subset D$$

swept by the tangent lifts of all its integral curves.

D is said to be *integrable*, if

$$\emptyset \neq D = D^{(i)}$$

---

[34] We shall not consider the case of a *time-dependent* equation $D \subset J^1Q$, subset of the *jet bundle* $J^1Q := \mathbb{R} \times TQ$, whose solutions are the smooth curves γ of Q with *jet extension* $j^1\gamma : t \in I \mapsto j^1\gamma\,(t) := (t, \dot{\gamma}(t)) \in J^1Q$ living in D, i.e. $\mathrm{Im}(j^1\gamma) \subset D$.

Vector fields and normal form

The main case of integrability is that of a differential equation D reducible to *normal form*, that is, one for which there exists a smooth *section* of TQ or *vector field* on Q [35]

$$\begin{aligned} X : Q \to TQ \quad &: \quad \mathrm{p} \mapsto (\mathrm{p}, X(\mathrm{p})) \\ &: \quad \mathrm{p} \mapsto X(\mathrm{p}) \in T_{\mathrm{p}}Q \end{aligned}$$

such that

$$D = \mathrm{Im}(X) = \{(\mathrm{p}, \mathbf{v}) \in TQ \mid \mathbf{v} = X(\mathrm{p})\}$$

For such an equation, condition ($\diamond$) reads

$$\dot{\gamma} = X \circ \gamma$$

that is to say,

$$\dot{\mathrm{p}}(t) = X(\mathrm{p}(t)) \ , \quad \forall t \in I$$

Cauchy problems and determinism

The integrability of

$$D = \mathrm{Im}(X) \neq \emptyset$$

is a consequence of the celebrated *determinism* of normal equations (here stated without proof):

Determinism theorem *For any* $t_o \in \mathbb{R}$ *and* $\mathrm{p}_o \in Q$ *, there exists a unique maximal solution of* Cauchy problem (D, t_o, p_o) *, i.e. a unique maximal integral curve* $\gamma : t \in I \mapsto \mathrm{p}(t) \in Q$ *of* D *, with* $I \ni t_o$ *, satisfying* initial condition $\mathrm{p}(t_o) = \mathrm{p}_o$ *(all of the other solutions of the problem being restrictions of the maximal one).*

In fact, if $(\mathrm{p}_o, \mathbf{v}_o) \in D$ –i.e. $\mathrm{p}_o \in Q$, $\mathbf{v}_o = X(\mathrm{p}_o)$– and γ is the maximal solution of (D, t_o, p_o), we obtain

$$(\mathrm{p}_o, \mathbf{v}_o) = (\mathrm{p}(t_o), X(\mathrm{p}(t_o))) = (\mathrm{p}(t_o), \dot{\mathrm{p}}(t_o)) \in \mathrm{Im}(\dot{\gamma}) \subset D^{(i)}$$

Hence

$$D = D^{(i)}$$

---

[35] X is said to be *smooth*, if it admits smooth component functions

$$X^h : \mathcal{U} \to \mathbb{R} : \mathrm{p} \mapsto X^h(\mathrm{p}) \ , \quad h = 1, \ldots, n$$

in one (and then every) atlas of charts $\xi^{-1} : \mathcal{U} \to \mathbb{R}^n$ of Q.

5.4 Second tangent bundle and second-order differential equations

The second tangent bundle of a manifold is the geometric arena of second-order differential equations.

Second tangent lift and second tangent bundle

Consider the *iterated tangent bundle* of a smooth manifold $Q \subset \mathcal{E}$, that is, the tangent bundle

$$TTQ \subset TT\mathcal{E}$$

of $TQ \subset T\mathcal{E}$.

Recall that TTQ is the region of $TT\mathcal{E} = (\mathcal{E} \times E) \times (E \times E)$ swept by the tangent lifts of all the smooth curves of TQ, the tangent lift of any such curve

$$c : I \to TQ : t \mapsto c(t) = (\mathrm{p}(t), \mathbf{v}(t))$$

being given by

$$\dot{c} : I \to TTQ : t \mapsto \dot{c}(t) = (\mathrm{p}(t), \mathbf{v}(t); \dot{\mathrm{p}}(t), \dot{\mathbf{v}}(t))$$

In particular, the *second tangent lift* of a smooth curve

$$\gamma : I \to Q : t \mapsto \gamma(t) = \mathrm{p}(t)$$

that is, the tangent lift of

$$\dot{\gamma} : I \to TQ : t \mapsto \dot{\gamma}(t) = (\mathrm{p}(t), \dot{\mathrm{p}}(t))$$

is a smooth curve

$$\ddot{\gamma} : I \to TTQ : t \mapsto \ddot{\gamma}(t) = (\mathrm{p}(t), \dot{\mathrm{p}}(t); \dot{\mathrm{p}}(t), \ddot{\mathrm{p}}(t))$$

which lives in the region

$$T^2Q \subset TTQ$$

defined by

$$T^2Q := \{(\mathrm{p}, \mathbf{v}; \mathbf{u}, \mathbf{w}) \in TTQ \mid \mathbf{u} = \mathbf{v}\}$$

Such a region, called *second tangent bundle* of Q, is the disjoint union

$$T^2Q = \bigcup_{(\mathrm{p},\mathbf{v}) \in TQ} \{(\mathrm{p}, \mathbf{v})\} \times \{\mathbf{v}\} \times T^2_{(\mathrm{p},\mathbf{v})}Q$$

of the fibres

$$T^2_{(\mathrm{p},\mathbf{v})}Q := \{\mathbf{w} \in E \mid (\mathrm{p}, \mathbf{v}; \mathbf{v}, \mathbf{w}) \in TTQ\}$$

(so, for any $(\mathrm{p}, \mathbf{v}) \in TQ$, $(\mathrm{p}, \mathbf{v}; \mathbf{v}, \mathbf{w}) \in T^2Q$ means $\mathbf{w} \in T^2_{(\mathrm{p},\mathbf{v})}Q$).

Affine fibre bundle structure

T^2Q is an *affine fibre bundle* over TQ, in the sense that its *fibre* $T^2_{(\mathrm{p},\mathbf{v})}Q$ over any $(\mathrm{p},\mathbf{v}) \in TQ$ is an affine space, as will now be shown.

Proposition 25 *For any* $(\mathrm{p},\mathbf{v}) \in TQ$ *,* $T^2_{(\mathrm{p},\mathbf{v})}Q$ *is an* n*-dimensional affine subspace of* E *, modelled on* $T_{\mathrm{p}}Q$ *.*

Proof Let $\xi : W \to \mathcal{U}$ be a chart of Q near $\mathrm{p} = \xi(q) \in \mathcal{U}$, where $\mathbf{v} = d_q\xi(v)$.

Owing to Lemma ($\star$), it will suffice to prove the following

Lemma ($\star\star$) $\quad T^2_{(\mathrm{p},\mathbf{v})}Q = \mathbf{z}(q,v) + \mathrm{Im}(d_q\xi)\,, \quad \mathbf{z}(q,v) := v^h v^k \frac{\partial^2 \mathrm{p}}{\partial q^h \partial q^k}\Big|_q \,.$

Let $\mathbf{w} \in T^2_{(\mathrm{p},\mathbf{v})}Q$, that is, $(\mathrm{p},\mathbf{v};\mathbf{v},\mathbf{w}) \in TTQ$ or

$$(\mathrm{p},\mathbf{v};\mathbf{v},\mathbf{w}) = \dot{c}(t_*) = (\mathrm{p}(t_*),\mathbf{v}(t_*);\dot{\mathrm{p}}(t_*),\dot{\mathbf{v}}(t_*))$$

for some smooth curve $c : t \in I \mapsto c(t) = (\mathrm{p}(t),\mathbf{v}(t)) \in TQ$ and $t_* \in I$.

As $\mathcal{U}$ is an open neighbourhood of $\mathrm{p} = \mathrm{p}(t_*)$ in Q, there exists (by continuity) an open interval $I_* \subset I$ containing t_* s.t. $\mathrm{p}(t) \in \mathcal{U}$ for all $t \in I_*$.

Then $c|_{I_*}$ can be given the C^∞ coordinate expression

$$\mathrm{p}(t) = \xi(q(t))\,,\ \mathbf{v}(t) = d_{q(t)}\xi\,(v(t)) = v^h(t)\frac{\partial \mathrm{p}}{\partial q^h}\Big|_{q(t)}$$

with

$$q(t_*) = q\ ,\ \ \dot{q}(t_*) = v(t_*) = v$$

By derivation at t_* of c (or, equivalently, $c|_{I_*}$), we obtain

$$\begin{aligned}
\mathbf{w} &= \dot{\mathbf{v}}(t_*) \\
&= v^h(t_*)\dot{q}^k(t_*)\frac{\partial^2 \mathrm{p}}{\partial q^h \partial q^k}\Big|_{q(t_*)} + \dot{v}^h(t_*)\frac{\partial \mathrm{p}}{\partial q^h}\Big|_{q(t_*)} \\
&= v^h v^k \frac{\partial^2 \mathrm{p}}{\partial q^h \partial q^k}\Big|_q + w^h \frac{\partial \mathrm{p}}{\partial q^h}\Big|_q\,, \quad w := \dot{v}(t_*) \\
&= \mathbf{z}(q,v) + d_q\xi\,(w)
\end{aligned}$$

Hence $\mathbf{w} \in \mathbf{z}(q,v) + \mathrm{Im}(d_q\xi)$.

Conversly, let $\mathbf{w} \in \mathbf{z}(q,v) + \mathrm{Im}(d_q\xi)$, that is,

$$\mathbf{w} = \mathbf{z}(q,v) + d_q\xi\,(w)$$

for some $w \in \mathbb{R}^n$.

Choose $t_* \in \mathbb{R}$ and consider the map

$$\alpha : \mathbb{R} \to \mathbb{R}^n : t \mapsto q(t) := q + v(t - t_*) + \frac{1}{2}w(t - t_*)^2$$

satisfying

$$q(t_*) = q \ , \ \ \dot{q}(t_*) = v \ , \ \ \ddot{q}(t_*) = w$$

As W is an open neighbourhood of $q(t_*) = q$ in $\mathbb{R}^n$, there exists (by continuity) an open interval I_* containing t_* s.t. $\alpha(I_*) \subset W$. Then $\gamma_\xi := \alpha|_{I_{t_*}}$ is the coordinate expression of

$$\gamma = \xi \circ \gamma_\xi : I_* \to Q : t \mapsto \mathrm{p}(t) = \xi(q(t))$$

which is a smooth curve of Q passing through

$$\mathrm{p}(t_*) = \xi(q(t_*)) = \xi(q) = \mathrm{p}$$

with derivatives

$$\begin{aligned}
\dot{\mathrm{p}}(t_*) &= \dot{q}^h(t_*)\frac{\partial \mathrm{p}}{\partial q^h}\Big|_{q(t_*)} \\
&= d_{q(t_*)}\xi\,(\dot{q}(t_*)) \\
&= d_q\xi\,(v) \\
&= \mathbf{v}
\end{aligned}$$

and

$$\begin{aligned}
\ddot{\mathrm{p}}(t_*) &= \dot{q}^h(t_*)\dot{q}^k(t_*)\frac{\partial^2 \mathrm{p}}{\partial q^h \partial q^k}\Big|_{q(t_*)} + \ddot{q}^h(t_*)\frac{\partial \mathrm{p}}{\partial q^h}\Big|_{q(t_*)} \\
&= v^h v^k \frac{\partial^2 \mathrm{p}}{\partial q^h \partial q^k}\Big|_q + w^h \frac{\partial \mathrm{p}}{\partial q^h}\Big|_q \\
&= \mathbf{z}(q, v) + d_q\xi\,(w) \\
&= \mathbf{w}
\end{aligned}$$

So

$$(\mathrm{p}, \mathbf{v}; \mathbf{v}, \mathbf{w}) = (\mathrm{p}(t_*), \dot{\mathrm{p}}(t_*); \dot{\mathrm{p}}(t_*), \ddot{\mathrm{p}}(t_*)) = \ddot{\gamma}(t_*) \in \mathrm{Im}(\ddot{\gamma})$$

Hence $\mathbf{w} \in T^2_{(\mathrm{p},\mathbf{v})}Q$. □

Remark that

$$\mathbf{w} \in T^2_{(\mathrm{p},\mathbf{v})}Q \longmapsto w = (d_q\xi)^{-1}\Big(\mathbf{w} - \mathbf{z}(q, v)\Big) \in \mathbb{R}^n$$

is the affine isomorphism which takes any vector $\mathbf{w} \in T^2_{(\mathrm{p},\mathbf{v})}Q$ to its *components* $w \in \mathbb{R}^n$ *in* ξ, that is, to its ‘coordinates’ in the Cartesian system of $T^2_{(\mathrm{p},\mathbf{v})}Q$ determined by the origin $\mathbf{z}(q, v) \in T^2_{(\mathrm{p},\mathbf{v})}Q$ and the basis $\left(\frac{\partial}{\partial q^h}\Big|_{\mathrm{p}}\right)_{h=1,\ldots,n} \subset T_{\mathrm{p}}Q$.

Second-order implicit differential equations

The proof of Lemma ($\star\star$) has also shown that T^2Q is the region of TTQ 'swept' by the second tangent lifts of all the smooth curves of Q.

As a consequence, if a subset

$$\mathcal{D} \subset T^2Q$$

is assigned (by prescribing some geometric or algebraic properties), the problem may arise of determining the smooth curves $\gamma : I \subset \mathbb{R} \to Q$ whose second tangent lifts live in $\mathcal{D}$, i.e.

$$\operatorname{Im}(\ddot{\gamma}) \subset \mathcal{D} \tag{$\diamond\diamond$}$$

that is to say,

$$\ddot{\gamma}(t) = (\mathrm{p}(t), \dot{\mathrm{p}}(t); \dot{\mathrm{p}}(t), \ddot{\mathrm{p}}(t)) \in \mathcal{D}\ , \quad \forall t \in I \tag{$\diamond\diamond$}$$

Such a $\mathcal{D}$ will then be called a *second-order, implicit differential equation* on Q.

Integral curves and base integral curves

A second-order differential equation on Q, say $\mathcal{D} \subset T^2Q \subset TTQ$, is a first-order differential equation on TQ as well, whose integral curves exhibit the following peculiar property.

Proposition 26 *Each integral curve c of $\mathcal{D} \subset T^2Q$ is the tangent lift of its own projection $\pi_Q \circ c$, i.e.*

$$(\pi_Q \circ c)^{\boldsymbol{\cdot}} = c$$

Proof Let

$$c : t \in I \mapsto c(t) = (\mathrm{p}(t), \mathbf{v}(t)) \in TQ$$

be a smooth curve of TQ, and

$$\pi_Q \circ c : t \in I \mapsto \pi_Q(c(t)) = \mathrm{p}(t) \in Q$$

its projection onto Q.

If c is an integral curve of $\mathcal{D}$, it satisfies, $\forall\, t \in I$,

$$\dot{c}(t) = (\mathrm{p}(t), \mathbf{v}(t); \dot{\mathrm{p}}(t), \dot{\mathbf{v}}(t)) \in \mathcal{D} \subset T^2Q$$

whence

$$\dot{\mathrm{p}}(t) = \mathbf{v}(t)$$

that is,

$$(\pi_Q \circ c)^{\boldsymbol{\cdot}}(t) = (\mathrm{p}(t), \dot{\mathrm{p}}(t)) = (\mathrm{p}(t), \mathbf{v}(t)) = c(t)$$

which is our claim. □

The *base integral curves* $\gamma = \pi_Q \circ c$ of $\mathcal{D}$ –projections of the integral curves c of $\mathcal{D}$– are the solution curves of problem ($\diamond\diamond$), as will now be shown.

Proposition 27 *A smooth curve* γ *of* Q *is a base integral curve of* $\mathcal{D}$*, iff it is a solution curve of problem* $(\diamond\diamond)$.

Proof Let $\operatorname{Im}(\ddot{\gamma}) \subset \mathcal{D}$. Putting $c := \dot{\gamma}$, whence $\pi_Q \circ c = \pi_Q \circ \dot{\gamma}$ and $\dot{c} = \ddot{\gamma}$, we obtain $\pi_Q \circ c = \gamma$ and $\operatorname{Im}(\dot{c}) \subset \mathcal{D}$.

Conversely, let $\gamma = \pi_Q \circ c$ and $\operatorname{Im}(\dot{c}) \subset \mathcal{D}$. From Proposition 26, we obtain $\dot{\gamma} = (\pi_Q \circ c)^{\boldsymbol{\cdot}} = c$, whence $\ddot{\gamma} = \dot{c}$, and then $\operatorname{Im}(\ddot{\gamma}) \subset \mathcal{D}$. □

The integral curves and the base integral curves of $\mathcal{D}$ are then bijectively related to one another (Proposition 26) and, owing to the above bijection, the first-order problem of determining the integral curves of $\mathcal{D}$ proves to be naturally equivalent to the second-order problem of determining its base integral curves, i.e. the solution curves of $(\diamond\diamond)$ (Proposition 27).

Semi-sprays and normal form

A second-order equation $\mathcal{D}$ on Q (first-order on TQ) is reducible to normal form, if there exists a smooth *semi-spray* on TQ (i.e. a smooth vector field on TQ with values in T^2Q) [36]

$$\begin{aligned} \Gamma : TQ \to T^2Q \subset TTQ \quad &: \quad (\mathrm{p}, \mathbf{v}) \mapsto (\mathrm{p}, \mathbf{v}; \mathbf{v}, \Gamma(\mathrm{p}, \mathbf{v})) \\ &: \quad \mathbf{v} \in T_{\mathrm{p}}Q \mapsto \Gamma(\mathrm{p}, \mathbf{v}) \in T^2_{(\mathrm{p},\mathbf{v})}Q \end{aligned}$$

such that

$$\mathcal{D} = \operatorname{Im}(\Gamma) = \{(\mathrm{p}, \mathbf{v}; \mathbf{u}, \mathbf{w}) \in TTQ \mid \mathbf{u} = \mathbf{v}\,,\ \mathbf{w} = \Gamma(\mathrm{p}, \mathbf{v})\}$$

As is already known, condition $\operatorname{Im}(\dot{c}) \subset \mathcal{D}$, characterizing the integral curves c of $\mathcal{D} = \operatorname{Im}(\Gamma)$, reads

$$\dot{c} = \Gamma \circ c$$

As a consequence, condition $\operatorname{Im}(\ddot{\gamma}) \subset \mathcal{D}$, characterizing the base integral curves γ of $\mathcal{D} = \operatorname{Im}(\Gamma)$, reads

$$\ddot{\gamma} = \Gamma \circ \dot{\gamma}$$

Cauchy problems and determinism

If $\mathcal{D} = \operatorname{Im}(\Gamma)$, then – for any $t_o \in \mathbb{R}$ and $(\mathrm{p}_o, \mathbf{v}_o) \in TQ$ – Cauchy problem $\Big(\mathcal{D}, t_o, (\mathrm{p}_o, \mathbf{v}_o)\Big)$ admits a unique maximal solution, which amounts to saying that there exists a unique maximal base integral curve $\gamma : t \in I \mapsto \mathrm{p}(t) \in Q$ of $\mathcal{D}$, with $t_o \in I$, satisfying *initial conditions* $(\mathrm{p}(t_o), \dot{\mathrm{p}}(t_o)) = (\mathrm{p}_o, \mathbf{v}_o)$ (all of the other solutions of the problem being restrictions of the maximal one).

---

[36] Γ is said to be *smooth*, if it admits smooth component functions

$$\Gamma^h : T\mathcal{U} \to \mathbb{R} : (\mathrm{p}, \mathbf{v}) \mapsto \Gamma^h(\mathrm{p}, \mathbf{v})\,, \quad h = 1, \ldots, n$$

in one (and then every) atlas of charts $\xi^{-1} : \mathcal{U} \to \mathbb{R}^n$ of Q.

5.5 Cotangent bundle and differential forms

The cotangent bundle of a manifold is the geometric arena of differential forms.

Cotangent vector spaces

Let Q be a smooth manifold.

For any $\mathrm{p} \in Q$, to the tangent vector space $T_{\mathrm{p}}Q$ there corresponds, by duality, the *cotangent vector space* $T^*_{\mathrm{p}}Q$ made up of all the *covectors*

$$\pi : T_{\mathrm{p}}Q \to \mathbb{R} : \mathbf{v} \mapsto \langle \pi \mid \mathbf{v} \rangle$$

(linear maps of $T_{\mathrm{p}}Q$ in $\mathbb{R}$, summed to one another and multiplied by real numbers according to the rule $\langle a_1\pi_1 + a_2\pi_2 \mid \mathbf{v}\rangle = a_1\langle \pi_1 \mid \mathbf{v}\rangle + a_2\langle \pi_2 \mid \mathbf{v}\rangle$).

Typical covector is the *differential at* p of a smooth real-valued function $f : Q \to \mathbb{R}$, defined by [37]

$$d_{\mathrm{p}}f : T_{\mathrm{p}}Q \to \mathbb{R} : \mathbf{v} \mapsto \langle d_{\mathrm{p}}f \mid \mathbf{v}\rangle := \mathbf{v}(f)$$

Coordinate formalism

Let $\xi : W \to \mathcal{U}$ be a chart of Q near $\mathrm{p} = \xi(q) \in \mathcal{U}$.

For any $\pi \in T^*_{\mathrm{p}}Q$ and $\mathbf{v} = v^h \frac{\partial}{\partial q^h}\Big|_{\mathrm{p}} \in T_{\mathrm{p}}Q$, we have

$$\begin{aligned} \langle \pi \mid \mathbf{v}\rangle &= \left\langle \pi \mid v^h \frac{\partial}{\partial q^h}\Big|_{\mathrm{p}} \right\rangle \\ &= \left\langle \pi \mid \frac{\partial}{\partial q^h}\Big|_{\mathrm{p}} \right\rangle v^h \\ &= p_h v^h \end{aligned}$$

where the coefficients

$$p_h := \left\langle \pi \mid \frac{\partial}{\partial q^h}\Big|_{\mathrm{p}} \right\rangle$$

are called the *components* of π in the chart.

The name is motivated by the followin considerations.

Let

$$\begin{aligned} q^h &:= pr^h \circ \xi^{-1} \\ &: \quad \mathrm{p} = \xi(q) \in \mathcal{U} \overset{\xi^{-1}}{\longmapsto} q = (q^1, \ldots, q^n) \in W \subset \mathbb{R}^n \overset{pr^h}{\longmapsto} q^h \in \mathbb{R} \end{aligned}$$

be the *h-th coordinate function* determined by ξ^{-1}.

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[37] See section 5.2, *Smooth maps and differential calculus.*

For all $\mathbf{v} = d_q\xi(v) = v^h \frac{\partial}{\partial q^h}\Big|_{\mathrm{p}} \in T_{\mathrm{p}}Q$, we have

$$\langle d_{\mathrm{p}}q^h \mid \mathbf{v}\rangle = \mathbf{v}(q^h) = d_q q^h_{\xi}(v) = d_q pr^h(v) = pr^h(v) = v^h$$

(i.e. the n-tuple $(d_{\mathrm{p}}q^h)$ is the dual basis in $T^*_{\mathrm{p}}Q$ of the basis $\left(\frac{\partial}{\partial q^h}\Big|_{\mathrm{p}}\right)$ of $T_{\mathrm{p}}Q$) and then

$$\langle \pi \mid \mathbf{v}\rangle = p_h v^h = p_h \langle d_{\mathrm{p}}q^h \mid \mathbf{v}\rangle = \langle p_h d_{\mathrm{p}}q^h \mid \mathbf{v}\rangle$$

Hence

$$\pi = p_h\, d_{\mathrm{p}}q^h$$

In particular,

$$d_{\mathrm{p}}f = \frac{\partial f}{\partial q^h}\Big|_{\mathrm{p}} d_{\mathrm{p}}q^h$$

since [38]

$$\langle d_{\mathrm{p}}f \mid \frac{\partial}{\partial q^h}\Big|_{\mathrm{p}}\rangle = \frac{\partial}{\partial q^h}\Big|_{\mathrm{p}}(f) = \frac{\partial f}{\partial q^h}\Big|_{\mathrm{p}}$$

Cotangent bundle and canonical projection

The *cotangent bundle* of an n-dimensional smooth manifold Q is the disjoint union of its cotangent vector spaces, i.e.

$$T^*Q := \bigcup_{\mathrm{p}\in Q} \{\mathrm{p}\} \times T^*_{\mathrm{p}}Q$$

(so, for any $\mathrm{p} \in Q$, $(\mathrm{p}, \pi) \in T^*Q$ means $\pi \in T^*_{\mathrm{p}}Q$). [39]

The *canonical projection* of T^*Q onto its *base* manifold Q is the map

$$\pi^*_Q : T^*Q \to Q : (\mathrm{p}, \pi) \mapsto \mathrm{p}$$

Vector bundle structure

T^*Q is a *vector fibre bundle* over Q , in the sense that its *fibre* $T^*_{\mathrm{p}}Q$ over each $\mathrm{p} \in Q$ is a vector space. Such fibres are 'glued' together by *natural charts*

$$(q, p) \in T^*W \longmapsto (\mathrm{p}, \pi) = (\xi(q), p_h\, d_{\mathrm{p}}q^h) \in T^*\mathcal{U}$$

(arising from the charts $\xi : W \to \mathcal{U}$ of Q [40]), which could be shown to give T^*Q the structure of a $2n$-dimensional smooth manifold. [41]

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[38] See section 5.2, Proposition 24.

[39] Remark that the cotangent bundle of an open manifold $U \subset \mathcal{E}$ is trivial, i.e. $T^*U = U \times E^*$ or, if E is identified with E^* through its Euclidean metric structure, $T^*U = U \times E$.

[40] Recall that $T^*W = W \times \mathbb{R}^n$ and $T^*\mathcal{U} = (\pi^*_Q)^{-1}(\mathcal{U})$.

[41] Unlike $TQ \subset T\mathcal{E} = \mathcal{E} \times E$, cotangent bundle T^*Q is *not* naturally embedded in some Euclidean space related to $\mathcal{E}$, and therefore its structure of smooth manifold is to be meant

Differential 1-forms

A *differential 1-form* on Q is a smooth *section* of T^*Q or *covector field* on Q, i.e.

$$\begin{aligned} \mathsf{f} : Q \to T^*Q \quad &: \quad \mathrm{p} \mapsto (\mathrm{p}, \mathsf{f}(\mathrm{p})) \\ &: \quad \mathrm{p} \mapsto \mathsf{f}(\mathrm{p}) \in T^*_{\mathrm{p}}Q \end{aligned}$$

At each point p of a coordinate domain, the value

$$\mathsf{f}(\mathrm{p}) = \mathsf{f}_h(\mathrm{p})\, d_{\mathrm{p}} q^h$$

can be expressed in terms of its components [42]

$$\mathsf{f}_h(\mathrm{p}) := \left\langle \mathsf{f}(\mathrm{p}) \mid \frac{\partial}{\partial q^h}\Big|_{\mathrm{p}} \right\rangle$$

In particular, a smooth real-valued function $f : Q \to \mathbb{R}$ gives rise to an *exact 1-form* on Q through its *differential*

$$df : Q \to T^*Q : \mathrm{p} \mapsto (\mathrm{p}, d_{\mathrm{p}} f)$$

whose smooth component functions, in any chart of of Q, are the derivatives

$$\frac{\partial f}{\partial q^h} : \mathcal{U} \to \mathbb{R} : \mathrm{p} \mapsto \frac{\partial f}{\partial q^h}\Big|_{\mathrm{p}}$$

Semi-basic 1-forms

The concept of 1-form will now be generalized.

A *semi-basic 1-form* on TQ is a smooth *bundle morphism* of TQ in T^*Q, i.e.

$$\begin{aligned} F : TQ \to T^*Q \quad &: \quad (\mathrm{p}, \mathbf{v}) \mapsto (\mathrm{p}, F(\mathrm{p}, \mathbf{v})) \\ &: \quad \mathbf{v} \in T_{\mathrm{p}}Q \mapsto F(\mathrm{p}, \mathbf{v}) \in T^*_{\mathrm{p}}Q \end{aligned}$$

---

in the generalized sense illustrated in section 5.6, *Intrinsic geometry of smooth manifolds* (see, in particular, Proposition 39).

[42] f has been required to be *smooth*, that is, to admit smooth component functions

$$\mathsf{f}_h : \mathcal{U} \to \mathbb{R} : \mathrm{p} \mapsto \mathsf{f}_h(\mathrm{p}), \quad h = 1, \dots, n$$

in one (and then every) atlas of charts $\xi^{-1} : \mathcal{U} \to \mathbb{R}^n$ of Q.

At each point p of a coordinate domain, the value

$$F(\mathrm{p},\mathbf{v}) = F_h(\mathrm{p},\mathbf{v})\, d_{\mathrm{p}}q^h$$

can be expressed in terms of its components [43]

$$F_h(\mathrm{p},\mathbf{v}) := \left\langle\, F(\mathrm{p},\mathbf{v}) \mid \frac{\partial}{\partial q^h}\Big|_{\mathrm{p}} \right\rangle$$

A *semi-basic 1-form* on T^2Q is a smooth *bundle morphism* of T^2Q in T^*Q over π_Q, i.e.

$$\begin{aligned} \Phi : T^2Q \to T^*Q \quad &: \quad (\mathrm{p},\mathbf{v};\mathbf{v},\mathbf{w}) \mapsto (\mathrm{p},\Phi(\mathrm{p},\mathbf{v},\mathbf{w})) \\ &: \quad \mathbf{w} \in T^2_{(\mathrm{p},\mathbf{v})}Q \mapsto \Phi(\mathrm{p},\mathbf{v},\mathbf{w}) \in T^*_{\mathrm{p}}Q\,, \quad \mathrm{p} = \pi_Q(\mathrm{p},\mathbf{v}) \end{aligned}$$

At each point p of a coordinate domain, the value

$$\Phi(\mathrm{p},\mathbf{v},\mathbf{w}) = \Phi_h(\mathrm{p},\mathbf{v},\mathbf{w})\, d_{\mathrm{p}}q^h$$

can expressed in terms of its components [44]

$$\Phi_h(\mathrm{p},\mathbf{v},\mathbf{w}) = \left\langle\, \Phi(\mathrm{p},\mathbf{v},\mathbf{w}) \mid \frac{\partial}{\partial q^h}\Big|_{\mathrm{p}} \right\rangle$$

Special kinds of semi-basic 1-forms will now be described.

Semi-Riemannian metric

A *semi-Riemannian metric* on an n-dimensional manifold Q is a *symmetric vector bundle isomorphism* of TQ onto T^*Q, that is, a semi-basic 1-form

$$\begin{aligned} g : TQ \to T^*Q \quad &: \quad (\mathrm{p},\mathbf{v}) \mapsto (\mathrm{p}, g_{\mathrm{p}}(\mathbf{v})) \\ &: \quad \mathbf{v} \in T_{\mathrm{p}}Q \mapsto g_{\mathrm{p}}(\mathbf{v}) \in T^*_{\mathrm{p}}Q \end{aligned}$$

---

[43] F has been required to be *smooth*, that is, to admit smooth component functions

$$F_h : T\mathcal{U} \to \mathbb{R} : (\mathrm{p},\mathbf{v}) \mapsto F_h(\mathrm{p},\mathbf{v})\,, \quad h = 1,\dots,n$$

in one (and then every) atlas of charts $\xi^{-1} : \mathcal{U} \to \mathbb{R}^n$

[44] Φ has been required to be *smooth*, that is, to admit smooth component functions

$$\Phi_h : T^2\mathcal{U} \to \mathbb{R} : (\mathrm{p},\mathbf{v};\mathbf{v},\mathbf{w}) \mapsto \Phi_h(\mathrm{p},\mathbf{v},\mathbf{w})\,, \quad h = 1,\dots,n$$

in one (and then every) atlas of charts $\xi^{-1} : \mathcal{U} \to \mathbb{R}^n$ of Q.

such that, for each $\mathrm{p} \in Q$, its restriction

$$g_{\mathrm{p}} : T_{\mathrm{p}}Q \to T^*_{\mathrm{p}}Q : \mathbf{v} \mapsto g_{\mathrm{p}}(\mathbf{v})$$

is a linear isomorphism (*non-degenerateness*) and the bilinear *inner product*

$$(\mathbf{u}, \mathbf{v}) \in T_{\mathrm{p}}Q \times T_{\mathrm{p}}Q \mapsto \langle g_{\mathrm{p}}(\mathbf{u}) \mid \mathbf{v} \rangle \in \mathbb{R}$$

satisfies the symmetry condition (*commutativity*)

$$\langle g_{\mathrm{p}}(\mathbf{u}) \mid \mathbf{v} \rangle = \langle g_{\mathrm{p}}(\mathbf{v}) \mid \mathbf{u} \rangle$$

If non-degenerateness is strengthened by requiring *positive definiteness*, i.e.

$$\langle g_{\mathrm{p}}(\mathbf{v}) \mid \mathbf{v} \rangle > 0$$

for all $(\mathrm{p}, \mathbf{v}) \in TQ$ with $\mathbf{v} \neq \mathbf{0}$, then g is said to be a *Riemannian metric.*

So a (semi-)Riemannian metric g smoothly determines a (pseudo-)Euclidean metric structure g_{p} in each fibre $T_{\mathrm{p}}Q$ of TQ and it is characterized by its *quadratic form*

$$K : TQ \to \mathbb{R} : (\mathrm{p}, \mathbf{v}) \mapsto K(\mathrm{p}, \mathbf{v}) := \frac{1}{2} \langle g_{\mathrm{p}}(\mathbf{v}) \mid \mathbf{v} \rangle$$

owing to *polarization identity*

$$\langle g_{\mathrm{p}}(\mathbf{u}) \mid \mathbf{v} \rangle = K(\mathrm{p}, \mathbf{u} + \mathbf{v}) - K(\mathrm{p}, \mathbf{u}) - K(\mathrm{p}, \mathbf{v})$$

A manifold equipped with a (semi-)Riemannian metric will be called (*semi-*)*Riemannian manifold* and denoted by (Q, K).

Remark Any semi-spray

$$\Gamma_K : TQ \to T^2Q : \mathbf{v} \in T_{\mathrm{p}}Q \mapsto \Gamma_K(\mathrm{p}, \mathbf{v}) \in T^2_{(\mathrm{p},\mathbf{v})}Q$$

associated with a (semi-)Riemannian manifold (Q, K), naturally transforms g (semi-basic 1-form on TQ) into

$$[K] : T^2Q \to T^*Q : \mathbf{w} \in T^2_{(\mathrm{p},\mathbf{v})}Q \mapsto [K](\mathrm{p}, \mathbf{v}, \mathbf{w})) \in T^*_{\mathrm{p}}Q$$

(semi-basic 1-form on T^2Q) by putting

$$[K](\mathrm{p}, \mathbf{v}, \mathbf{w}) := g_{\mathrm{p}}(\mathbf{w} - \Gamma_K(\mathrm{p}, \mathbf{v}))$$

Coordinate formalism

In a chart $\xi : W \rightarrow \mathcal{U}$ of Q, g is characterized by the symmetric, non-singular, $n \times n$ matrix of C^∞ real-valued functions on W given by

$$g_{hk}(q) := \left\langle\, g_{\xi(q)}\Big(\frac{\partial}{\partial q^h}\Big|_{\xi(q)}\Big) \;\mid\; \frac{\partial}{\partial q^k}\Big|_{\xi(q)} \right\rangle$$

since the component functions of g, i.e. the components of $\pi = g_{\mathrm{p}}(\mathbf{v})$ for any $(\mathrm{p}, \mathbf{v}) = (\xi(q), d_q\xi(v)) \in T\mathcal{U}$, have coordinate expression

$$\begin{aligned} p_h &= \left\langle\, \pi \;\mid\; \frac{\partial}{\partial q^h}\Big|_{\mathrm{p}} \right\rangle \\ &= \left\langle\, g_{\mathrm{p}}(\mathbf{v}) \;\mid\; \frac{\partial}{\partial q^h}\Big|_{\mathrm{p}} \right\rangle \\ &= \left\langle\, g_{\xi(q)}\left(v^k \frac{\partial}{\partial q^k}\Big|_{\mathrm{p}}\right) \;\mid\; \frac{\partial}{\partial q^h}\Big|_{\mathrm{p}} \right\rangle \\ &= v^k \left\langle\, g_{\xi(q)}\left(\frac{\partial}{\partial q^k}\Big|_{\xi(q)}\right) \;\mid\; \frac{\partial}{\partial q^h}\Big|_{\xi(q)} \right\rangle \\ &= v^k g_{kh}(q) \\ &= g_{hk}(q)\, v^k \end{aligned}$$

(the last passage being due to symmetry).[45]

The inner product is then expressed (owing to bilinearity) by

$$\begin{aligned} \langle\, g_{\mathrm{p}}(\mathbf{u}) \mid \mathbf{v} \,\rangle &= \left\langle\, g_{\xi(q)}\Big(u^h \frac{\partial}{\partial q^h}\Big|_{\xi(q)}\Big) \;\mid\; v^k \frac{\partial}{\partial q^k}\Big|_{\xi(q)} \right\rangle \\ &= u^h \left\langle\, g_{\xi(q)}\Big(\frac{\partial}{\partial q^h}\Big|_{\xi(q)}\Big) \;\mid\; \frac{\partial}{\partial q^k}\Big|_{\xi(q)} \right\rangle v^k \\ &= g_{hk}(q)\, u^h v^k \end{aligned}$$

and therefore K has coordinate expression

$$K(q, v) = \frac{1}{2}\, g_{hk}(q)\, v^h v^k$$

---

[45] The final result should be read, in matrix notation, as a row by column multiplication. Check that the elements of the (inverse) matrix characterizing g^{-1}, are given by

$$g^{hk}(q) := \left\langle\, d_{\xi(q)} q^h \;\mid\; g^{-1}_{\xi(q)}(d_{\xi(q)} q^k) \right\rangle$$

Almost-symplectic structure

An *almost-symplectic structure* on a $2n$-dimensional manifold S is a *skew-symmetric vector bundle isomorphism* of TS onto T^*S, that is, a semi-basic 1-form

$$\begin{aligned} \omega : TS \to T^*S \quad &: \quad (\pi, X) \mapsto (\pi, \omega_\pi(X)) \\ &: \quad X \in T_\pi S \mapsto \omega_\pi(X) \in T^*_\pi S \end{aligned}$$

such that, for each $\pi \in S$, its restriction

$$\omega_\pi : T_\pi S \to T^*_\pi S : X \mapsto \omega_\pi(X)$$

is a linear isomorphism (*non-degenerateness*) and the bilinear product

$$(X, Y) \in T_\pi S \times T_\pi S \mapsto \langle \omega_\pi(X) \mid Y \rangle \in \mathbb{R}$$

satisfies the skew-symmetry condition (*anticommutativity*)

$$\langle \omega_\pi(X) \mid Y \rangle = -\langle \omega_\pi(Y) \mid X \rangle$$

Coordinate formalism

In a chart of S, ω is characterized by a skew-symmetric, non-singular, $2n \times 2n$ matrix of C^∞ real-valued functions

$$\begin{bmatrix} \omega_{(1)(1)} & \omega_{(1)(2)} \\ \omega_{(2)(1)} & \omega_{(2)(2)} \end{bmatrix}$$

(where each block $\omega_{(\alpha)(\beta)}$ is a $n \times n$ matrix). If, for any $\pi \in S$,

$$\begin{bmatrix} \Theta_{(1)} & \Theta_{(2)} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} X^{(1)} \\ X^{(2)} \end{bmatrix}$$

are the $2n$-tuples of components of $\Theta \in T^*_\pi S$ and $X \in T_\pi S$, respectively, then the components of $\Theta = \omega_\pi(X)$ are expressed (owing to skew-symmetry) by [46]

$$\Theta_{(\alpha)} = -\,\omega_{(\alpha)(\beta)}\, X^{(\beta)}$$

In particular, ω is said to be a *symplectic structure*, if it is characterized $-$in a suitable atlas of charts of $S-$ by the constant *symplectic matrix*

$$\begin{bmatrix} 0 & \delta \\ -\delta & 0 \end{bmatrix}$$

where 0 is the zero $n \times n$ matrix and δ is the identity $n \times n$ matrix.

On a cotangent bundle $S = T^*Q$ there exists a *canonical* symplectic structure, characterized by the symplectic matrix in the atlas of the natural charts.

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[46] The procedure is the same as that shown for a semi-Riemannian metric. Summation over $(\beta) = (1), (2)$ is understood. Multiplication of matrixes is meant to be row by column.

5.6 Intrinsic approach to smooth manifods

A deeper insight into the structure of a smooth manifold embedded in a Euclidean affine space, will provide useful suggestions for a generalized definition of 'smooth manifold' and an 'intrinsic' approach to its geometry (without reference to any embedding into a Euclidean environment).

Intrinsic geometry of embedded manifolds

We shall consider an n-dimensional smooth manifold Q embedded in a Euclidean affine space $\mathcal{E}$ (modelled on a Euclidean vector space E), with the aim of investigating the possible properties of its manifold structure that are independent of the Euclidean embedding.

Manifold structure

As is known, a (local) parametrization of Q, say $\xi : W \to \mathcal{U}$, gives rise to an n-dimensional coordinate system or *chart*

$$\varphi := \xi^{-1} : \mathcal{U} \to W = \varphi(\mathcal{U})$$

which, not considering the embedding of Q into $\mathcal{E}$, can only be seen as a bijection of a non-empty subset $\mathcal{U} \subset Q$ (*coordinate domain*) onto a subset $\varphi(\mathcal{U}) \subset \mathbb{R}^n$.

It is also known that the collection $\mathcal{A}_Q$ of all the above charts is an *atlas* of Q, in the sense that the collection of their coordinate domains is a covering of Q.

Moreover $\mathcal{A}_Q$ could be shown to be C^∞, i.e. any two charts φ and

$$\varphi' = \xi'^{-1} : \mathcal{U}' \to W' = \varphi'(\mathcal{U}')$$

belonging to $\mathcal{A}_Q$, with $\mathcal{U} \cap \mathcal{U}' \neq \emptyset$, are C^∞ related to each other, in the sense that the *transition function*

$$\tau_{(\varphi,\varphi')} := \varphi' \circ \varphi^{-1}|_{\varphi(\mathcal{U}\cap\mathcal{U}')} : q \in \varphi(\mathcal{U}\cap\mathcal{U}') \overset{\varphi^{-1}}{\longmapsto} \mathrm{p} \in \mathcal{U}\cap\mathcal{U}' \overset{\varphi'}{\longmapsto} q' \in \varphi'(\mathcal{U}\cap\mathcal{U}')$$

from φ to φ', as well as its inverse, is C^∞ differentiable (if $\mathcal{U} \cap \mathcal{U}' = \emptyset$, the charts φ and φ' are still said to be C^∞ related to each other). [47]

Finally $\mathcal{A}_Q$ could be shown to be *complete*, i.e. any chart C^∞ related to all of the charts of $\mathcal{A}_Q$ belongs to $\mathcal{A}_Q$. [48]

$\mathcal{A}_Q$ is called the *manifold structure* of Q.

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[47] Remark that, owing to the differentiability of the transition functions, any subset of type $\varphi(\mathcal{U}\cap\mathcal{U}') \subset \mathbb{R}^n$ is open and then (considering the case $\varphi' = \varphi$) any subset of type $\varphi(\mathcal{U}) \subset \mathbb{R}^n$ is open as well.

[48] As to the C^∞ differentiability and completeness of $\mathcal{A}_Q$, see [11], Lemma 6.1.2 (i).

Topology

From the topological point of view, Q has been thought of as a topological subspace of $\mathcal{E}$. However, its subspace topology (whose non-empty open subsets are the intersections of Q with the subsets belonging to $\tau_{\mathcal{E}}$) can as well be described in terms of the manifold structure $\mathcal{A}_Q$, as follows.

First we remark that the coordinate domains of the charts belonging to $\mathcal{A}_Q$ turn out to be the base of a *manifold topology* (whose non-empty open subsets are then unions of coordinate domains appearing in $\mathcal{A}_Q$). [49]

Then we prove the following

Proposition 28 *The subspace topology of Q coincides with its manifold topology.*

Proof The subspace topology and the manifold topology of Q will be denoted by τ_Q and τ'_Q, respectively.

We have $\tau'_Q \subset \tau_Q$, since the coordinate domains belong to τ_Q and then so does any union of coordinate domains.

We also have $\tau_Q \subset \tau'_Q$, since any non-empty $\mathrm{U} \in \tau_Q$ is union of all the coordinate domains of the charts of type $\varphi|_{\mathrm{U}\cap\mathcal{U}} \in \mathcal{A}_Q$ (where $\varphi \in \mathcal{A}_Q$ denotes a chart whose coordinate domain $\mathcal{U}$ contains a point of U). $\square$

Tangent bundle

From the differential point of view, Q has been shown to carry a bundle of n-dimensional tangent vector spaces, meant as vector subspaces of E. However, each tangent vector space can as well be described in terms of the manifold structure $\mathcal{A}_Q$, as follows.

The idea arises from the fact that, for any $\mathrm{p} \in Q$, a tangent vector $\mathbf{v} \in T_{\mathrm{p}}Q$ is completely determined by any one of the ordered couples of type (φ, v) with $\varphi := \xi^{-1} \in \mathcal{A}_{\mathrm{p}}$ ($\mathcal{A}_{\mathrm{p}} \subset \mathcal{A}_Q$ being the set of the charts whose coordinate domains contain p) and $v = (d_q\xi)^{-1}(\mathbf{v}) \in \mathbb{R}^n$ (with $q = \xi^{-1}(\mathrm{p})$).

Moreover, for any two charts $\varphi = \xi^{-1}$ and $\varphi' = \xi'^{-1}$ in $\mathcal{A}_{\mathrm{p}}$, where $\mathrm{p} = \xi(q) = \xi'(q')$ and $\mathbf{v} = d_q\xi\,(v) = d_{q'}\xi'\,(v')$, we have

$$\begin{aligned}
d_{q'}\xi'\,(v') &= d_q\xi\,(v) \\
&= d_q\varphi^{-1}\,(v) \\
&= d_q\varphi^{-1}|_{\varphi(\mathcal{U}\cap\mathcal{U}')}\,(v) \\
&= d_q\left(\xi' \circ \varphi' \circ \varphi^{-1}|_{\varphi(\mathcal{U}\cap\mathcal{U}')}\right)\,(v) \\
&= d_q\left(\xi' \circ \tau_{(\varphi,\varphi')}\right)\,(v) \\
&= \left(d_{q'}\xi' \circ d_q\tau_{(\varphi,\varphi')}\right)\,(v) \\
&= d_{q'}\xi'\left(d_q\tau_{(\varphi,\varphi')}\,(v)\right)
\end{aligned}$$

---

[49] See Proposition 31 in the next subsection *Intrinsic geometry of smooth manifolds.*

whence $-d_{q'}\xi'$ being injective$-$ the transformation law

$$v' = d_q\tau_{(\varphi,\varphi')}\,(v)$$

Now we remark that the binary relation

$$(\varphi, v) \sim_{\mathrm{p}} (\varphi', v') \quad \text{iff} \quad v' = d_{\varphi(\mathrm{p})}\tau_{(\varphi,\varphi')}\,(v)$$

defined in $\mathcal{A}_{\mathrm{p}} \times \mathbb{R}^n$ by the above transformation law, is an equivalence relation and there exists a unique vector space structure on the quotient $\mathcal{A}_{\mathrm{p}} \times \mathbb{R}^n / \sim_{\mathrm{p}}$ [50] such that, for all $\varphi \in \mathcal{A}_{\mathrm{p}}$, the bijection

$$\varphi_{\mathrm{p}} : \mathbb{R}^n \to \mathcal{A}_{\mathrm{p}} \times \mathbb{R}^n / \sim_{\mathrm{p}} : \; v \mapsto \varphi_{\mathrm{p}}(v) := [(\varphi, v)]_{\sim_{\mathrm{p}}}$$

is an isomorphism. [51]

Then we prove the following

Proposition 29 *The map*

$$\iota_{\mathrm{p}} : T_{\mathrm{p}}Q \to \mathcal{A}_{\mathrm{p}} \times \mathbb{R}^n / \sim_{\mathrm{p}} : \; \mathbf{v} = d_q\xi\,(v) \mapsto \iota_{\mathrm{p}}(\mathbf{v}) := \varphi_{\mathrm{p}}(v)$$

is an invariant isomorphism. [52]

Proof $\iota_{\mathrm{p}} = \varphi_{\mathrm{p}} \circ (d_q\xi)^{-1}$ is composition of isomorphisms and its invariance is due to the fact that, once chosen any two charts $\varphi = \xi^{-1}$ and $\varphi' = \xi'^{-1}$ in $\mathcal{A}_{\mathrm{p}}$ where $\mathrm{p} = \xi(q) = \xi'(q')$, then $-$for all $\mathbf{v} = d_q\xi\,(v) = d_{q'}\xi'\,(v') \in T_{\mathrm{p}}Q-$ we have $(\varphi, v) \sim_{\mathrm{p}} (\varphi', v')$, whence $\varphi_{\mathrm{p}}(v) = \varphi'_{\mathrm{p}}(v')$. □

The above canonical isomorphism, owing to which the tangent vector space $T_{\mathrm{p}}Q$ can be identified with the quotient vector space $\mathcal{A}_{\mathrm{p}} \times \mathbb{R}^n / \sim_{\mathrm{p}}$, will be denoted by

$$T_{\mathrm{p}}Q \;\equiv\; \mathcal{A}_{\mathrm{p}} \times \mathbb{R}^n / \sim_{\mathrm{p}}$$

meaning that any tangent vector $\mathbf{v} = d_q\xi\,(v)$ is identified with its image $\varphi_{\mathrm{p}}(v)$, i.e.

$$\mathbf{v} \;\equiv\; \varphi_{\mathrm{p}}(v)$$

In particular, for the vectors $\frac{\partial}{\partial q^h}\Big|_{\mathrm{p}} = d_q\xi\,(\delta_h)$ [53] of the basis determined in $T_{\mathrm{p}}Q$ by a chart $\varphi = \xi^{-1} \in \mathcal{A}_{\mathrm{p}}$, we have

$$\frac{\partial}{\partial q^h}\Big|_{\mathrm{p}} \;\equiv\; \varphi_{\mathrm{p}}(\delta_h)$$

---

[50] $[(\varphi, v)]_{\sim_{\mathrm{p}}} \in \mathcal{A}_{\mathrm{p}} \times \mathbb{R}^n / \sim_{\mathrm{p}}$ will denote the equivalence class of (φ, v).

[51] See Propositions 33 and 34 in the next subsection *Intrinsic geometry of smooth manifolds.*

[52] 'Invariant' means 'not depending on the choice of the chart'.

[53] Recall that $(\delta_h)_{h=1,\dots,n}$ denotes the canonical basis of $\mathbb{R}^n$.

Finally, if $\gamma : t \in I \subset \mathbb{R} \to \mathrm{p}(t) \in Q$ is a smooth curve of Q and $\varphi = \xi^{-1}$ is a chart near $\mathrm{p}(t) = \xi(q(t))$), for the tangent vector $\dot{\mathrm{p}}(t) = d_{q(t)}\xi\,(\dot{q}(t))$ we have

$$\dot{\mathrm{p}}(t) \;\equiv\; \varphi_{\mathrm{p}(t)}(\dot{q}(t))$$

Second tangent bundle

Also the second tangent bundle of Q can be described in intrinsic way, as follows.

First we recall the canonical projection of the iterated tangent bundle TTQ onto its base manifold TQ, i.e. the map

$$\pi_{_{TQ}} : TTQ \to TQ$$

defined by

$$\zeta = (\mathrm{p}, \mathbf{v}; \mathbf{u}, \mathbf{w}) \longmapsto \pi_{_{TQ}}(\zeta) := (\mathrm{p}, \mathbf{v})$$

whose coordinate expression is [54]

$$\zeta \overset{\varphi}{\equiv} (q, v; u, w) \longmapsto \pi_{_{TQ}}(\zeta) \overset{\varphi}{\equiv} (q, v)$$

Then we introduce the *tangent map* of $\pi_{_Q}$, i.e. the map

$$T\pi_{_Q} : TTQ \to TQ$$

defined by [55]

$$\zeta = (\mathrm{p}, \mathbf{v}; \mathbf{u}, \mathbf{w}) = (\mathrm{p}(t_*), \mathbf{v}(t_*); \dot{\mathrm{p}}(t_*), \dot{\mathbf{v}}(t_*)) \longmapsto T\pi_{_Q}(\zeta) := (\mathrm{p}(t_*), \dot{\mathrm{p}}(t_*)) = (\mathrm{p}, \mathbf{u})$$

whose coordinate expression is

$$\zeta \overset{\varphi}{\equiv} (q, v; u, w) = (q(t_*), v(t_*); \dot{q}(t_*), \dot{v}(t_*)) \longmapsto T\pi_{_Q}(\zeta) \overset{\varphi}{\equiv} (q(t_*), \dot{q}(t_*)) = (q, u)$$

Hence

$$\begin{aligned} T^2Q &= \{(\mathrm{p}, \mathbf{v}; \mathbf{u}, \mathbf{w}) \in TTQ \mid \mathbf{u} = \mathbf{v}\} \\ &= \{\zeta \in TTQ \mid T\pi_{_Q}(\zeta) = \pi_{_{TQ}}(\zeta)\} \end{aligned}$$

---

[54] The symbol $\overset{\varphi}{\equiv}$ will denote the coordinates determined in Q, TQ and TTQ by a chart $\varphi \in \mathcal{A}_Q$.

[55] Recall that a vector tangent to TQ is the derivative of a smooth curve $t \mapsto \big(\mathrm{p}(t), \mathbf{v}(t)\big)$ of TQ at some ‘time’ t_*.

Intrinsic geometry of smooth manifolds

We are now ready for a generalized definition of smooth manifold and an intrinsic approach to its geometry.

Manifold structure

Let M be a non-empty set.

If m denotes a positive integer, a (local) m-dimensional coordinate system or *chart* on M is a bijection

$$\varphi : \mathcal{U} \to \varphi(\mathcal{U})$$

of a non-empty subset $\mathcal{U} \subset M$ (*coordinate domain*) onto a subset $\varphi(\mathcal{U}) \subset \mathbb{R}^m$, owing to which each point $\mathrm{p} \in \mathcal{U}$ is characterized by an m-tuple of coordinates $x = \varphi(\mathrm{p}) \in \varphi(\mathcal{U})$.

A collection $\mathcal{A}$ of such charts is said to be an m-dimensional *atlas* of M, if the collection of their coordinate domains is a covering of M.

Moreover an m-dimensional atlas $\mathcal{A}$ is said to be to be C^∞, if any two charts φ and

$$\varphi' : \mathcal{U}' \to \varphi'(\mathcal{U}')$$

belonging to $\mathcal{A}$, with $\mathcal{U} \cap \mathcal{U}' \neq \emptyset$, are C^∞ related to each other, in the sense that the *transition function*

$$\tau_{(\varphi,\varphi')} := \varphi' \circ \varphi^{-1}|_{\varphi(\mathcal{U}\cap\mathcal{U}')} : x \in \varphi(\mathcal{U}\cap\mathcal{U}') \xmapsto{\varphi^{-1}} \mathrm{p} \in \mathcal{U}\cap\mathcal{U}' \xmapsto{\varphi'} x' \in \varphi'(\mathcal{U}\cap\mathcal{U}')$$

from φ to φ', as well as its inverse, is C^∞ differentiable (if $\mathcal{U} \cap \mathcal{U}' = \emptyset$, the charts φ and φ' are still said to be C^∞ related to each other). [56]

Finally a C^∞, m-dimensional atlas $\mathcal{A}$ is said to be *complete*, if any m-dimensional chart C^∞ related to all of the charts of $\mathcal{A}$ belongs to $\mathcal{A}$.

M is said to be an m-dimensional *smooth manifold*, if it is endowed with a *manifold structure* given by of a complete, C^∞, m-dimensional atlas $\mathcal{A}_M$.

Notice that any C^∞, m-dimensional atlas $\mathcal{A}$ uniquely determines a manifold structure on M, owing to the following

Proposition 30 *A C^∞, m-dimensional atlas $\mathcal{A}$ is contained in a unique complete, C^∞, m-dimensional atlas $\mathcal{A}_M$, given by the collection of the m-dimensional charts C^∞ related to all of the charts belonging to $\mathcal{A}$.*

Proof As $\mathcal{A}$ is C^∞, we have $\mathcal{A}_M \supset \mathcal{A}$. So $\mathcal{A}_M$ is an m-dimensional atlas.

Now let $\varphi_1, \varphi_2 \in \mathcal{A}_M$. For any point x in the domain of $\tau_{(\varphi_1,\varphi_2)}$, choose a chart $\varphi \in \mathcal{A}$ whose domain contains $\varphi_1^{-1}(x)$. As a composition of C^∞ maps, $\tau_{(\varphi,\varphi_2)} \circ \tau_{(\varphi_1,\varphi)}$ is C^∞. Since such a composite map equals $\tau_{(\varphi_1,\varphi_2)}$ in an open

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[56] Remark that, owing to the differentiability of the transition functions, any subset of type $\varphi(\mathcal{U}\cap\mathcal{U}') \subset \mathbb{R}^m$ is open and then (considering the case $\varphi' = \varphi$) any subset of type $\varphi(\mathcal{U}) \subset \mathbb{R}^m$ is open as well.

neighbourhood of x, the latter is C^∞ on that neighbourhood. As a consequence, owing to the arbitrarinness of x and the local character of C^∞ differentiability, $\tau_{(\varphi_1,\varphi_2)}$ turns out to be C^∞. So $\mathcal{A}_M$ is C^∞.

Moreover, by the very definition of $\mathcal{A}_M$, any m-dimensional chart C^∞ related to all of the charts of $\mathcal{A}_M$ belongs to $\mathcal{A}_M$. So $\mathcal{A}_M$ is complete.

Finally, any complete, C^∞, m-dimensional atlas $\mathcal{A}'_M$ containing $\mathcal{A}$ satisfies $\mathcal{A}'_M \subset \mathcal{A}_M$ (owing to its being C^∞) and then $\mathcal{A}'_M \supset \mathcal{A}_M$ (owing to its completeness). So $\mathcal{A}'_M = \mathcal{A}_M$. □

Topology

The manifold structure $\mathcal{A}_M$, through the collection $\beta_{_M}$ of the coordinate domains of its charts, determines a *manifold topology* $\tau_{_M}$ on M (whose elements are said to be the *open subsets* of M).

That is shown by the following

Proposition 31 *$\beta_{_M}$ is the basis of a topology $\tau_{_M}$ on M.*

Proof First we recall that, if $\mathcal{W} \in \beta_{_M}$, then $\varphi(\mathcal{U} \cap \mathcal{W})$ is an open subset of $\mathbb{R}^m$ for all $\varphi : \mathcal{U} \to \varphi(\mathcal{U})$ in $\mathcal{A}_M$.

Then we consider the whole collection $\tau_{_M}$ (containing $\beta_{_M}$) of the subsets $W \subset M$ such that $\varphi(\mathcal{U} \cap W)$ is an open subset of $\mathbb{R}^m$ for all $\varphi : \mathcal{U} \to \varphi(\mathcal{U})$ in $\mathcal{A}_M$.

Clearly, $\tau_{_M}$ is a topology, that is , it satisfies the properties

$$\emptyset \in \tau_M\,, \quad M \in \tau_M$$

$$W_1, W_2 \in \tau_M \Longrightarrow W_1 \cap W_2 \in \tau_M$$

$$\{W_\alpha\} \subset \tau_M \Longrightarrow \bigcup_\alpha W_\alpha \in \tau_M$$

since, for all $\varphi : \mathcal{U} \to \varphi(\mathcal{U})$,

$$\varphi(\mathcal{U} \cap \emptyset) = \emptyset\,, \quad \varphi(\mathcal{U} \cap M) = \varphi(\mathcal{U})$$

$$\varphi(\mathcal{U} \cap W_1 \cap W_2) = \varphi(\mathcal{U} \cap W_1) \cap \varphi(\mathcal{U} \cap W_2)$$

$$\varphi\Big(\mathcal{U} \cap (\bigcup_\alpha W_\alpha)\Big) = \bigcup_\alpha \varphi(\mathcal{U} \cap W_\alpha)$$

are all open subsets of $\mathbb{R}^m$.

Finally, $\beta_{_M}$ is a basis of $\tau_{_M}$, that is, any $W \in \tau_{_M}$ is union of elements of $\beta_{_M}$, since ($\beta_{_M}$ being a covering of M)

$$W = \bigcup_{\mathcal{U} \in \beta_{_M}} (\mathcal{U} \cap W)$$

and each $\mathcal{W} := \mathcal{U} \cap W$ (if non-empty) belongs to $\beta_{_M}$.

The above property $\mathcal{W} \in \beta_M$ is due to the fact that $\mathcal{W}$ belongs to $\tau_{_M}$ and is contained in the coordinate domain $\mathcal{U}$ of a chart $\varphi \in \mathcal{A}_M$, whence $\varphi|_{\mathcal{W}} \in \mathcal{A}_M$ (in fact $\varphi|_{\mathcal{W}}$ is C^∞ related to every chart $\psi : \mathcal{V} \to \psi(\mathcal{V})$ of $\mathcal{A}_M$, since the transition functions $\tau_{(\varphi|_{\mathcal{W}},\psi)}$ and $\tau_{(\psi,\varphi|_{\mathcal{W}})}$ are restrictions of the C^∞ functions $\tau_{(\varphi,\psi)}$ and $\tau_{(\psi,\varphi)}$ to the open subsets $\varphi|_{\mathcal{W}}(\mathcal{W} \cap \mathcal{V}) = \varphi(\mathcal{U} \cap \mathcal{W}) \cap \varphi(\mathcal{U} \cap \mathcal{V}) \subset \varphi(\mathcal{U} \cap \mathcal{V})$ and $\psi(\mathcal{V} \cap \mathcal{W}) \subset \psi(\mathcal{V} \cap \mathcal{U})$ of their domains). □

M, equipped with its manifold topology, is a locally Euclidean topological space, that is, each point of M admits an open neighbourhood homeomorphic to an open subset of $\mathbb{R}^m$, owing to the following

Proposition 32 *Each chart $\varphi : \mathcal{U} \to \varphi(\mathcal{U})$ belonging to $\mathcal{A}_M$, is a homeomorphism.*

Proof φ is an open map, since it takes any open subset W of M contained in $\mathcal{U}$ onto

$$\varphi(W) = \varphi(\mathcal{U} \cap W)$$

which is an open subset of $\mathbb{R}^m$ contained in $\varphi(\mathcal{U})$.

φ^{-1} is open too, since it takes any open subset A of $\mathbb{R}^m$ contained in $\varphi(\mathcal{U})$ onto

$$W := \varphi^{-1}(A)$$

which is an open subset of M contained in $\mathcal{U}$.

The above property $W \in \tau_{_M}$ is due to the fact that, for all $\psi : \mathcal{V} \to \psi(\mathcal{V})$ belonging to $\mathcal{A}_M$, the image

$$\begin{aligned} \psi(\mathcal{V} \cap W) &= \psi(\mathcal{V} \cap \mathcal{U} \cap W) \\ &= \psi\Big(\varphi^{-1}(\varphi(\mathcal{V} \cap \mathcal{U})) \cap \varphi^{-1}(A)\Big) \\ &= \tau_{(\varphi,\psi)}\left(\varphi(\mathcal{U} \cap \mathcal{V}) \cap A\right) \end{aligned}$$

is an open subset of $\mathbb{R}^m$. [57] □

Tangent bundle

Let $\mathrm{p} \in M$.

Consider the set $\mathcal{A}_{\mathrm{p}} \subset \mathcal{A}_M$ of all the charts near p, (i.e. the charts of $\mathcal{A}_M$ whose coordinate domains contain p).

---

[57] Recall that a C^∞ transition function $\tau_{(\varphi,\psi)}$ is a homeomorphism between open subsets of $\mathbb{R}^m$.

Then define a binary relation $\sim_{\mathrm{p}}$ in $\mathcal{A}_{\mathrm{p}} \times \mathbb{R}^m$ through the following transformation law: [58]

$$(\varphi, v) \sim_{\mathrm{p}} (\varphi', v') \quad \text{iff} \quad v' = d_{\varphi(\mathrm{p})}\tau_{(\varphi,\varphi')}\,(v)$$

Proposition 33 $\sim_{\mathrm{p}}$ *is an equivalence relation.*

Proof The binary relation $\sim_{\mathrm{p}}$ is reflexive

$$(\varphi, v) \sim_{\mathrm{p}} (\varphi, v)$$

since

$$v = \mathrm{id}_{\mathbb{R}^m}(v) = d_{\varphi(\mathrm{p})}\tau_{(\varphi,\varphi)}\,(v)$$

Moreover it is symmetric

$$(\varphi, v) \sim_{\mathrm{p}} (\varphi', v') \Longrightarrow (\varphi', v') \sim_{\mathrm{p}} (\varphi, v)$$

since

$$v = \left(d_{\varphi(\mathrm{p})}\tau_{(\varphi,\varphi')}\right)^{-1}(v') = d_{\varphi'(\mathrm{p})}\tau_{(\varphi',\varphi)}\,(v')$$

Finally it is transitive

$$(\varphi, v) \sim_{\mathrm{p}} (\varphi', v')\,,\ (\varphi', v') \sim_{\mathrm{p}} (\varphi'', v'') \Longrightarrow (\varphi, v) \sim_{\mathrm{p}} (\varphi'', v'')$$

since

$$v'' = d_{\varphi'(\mathrm{p})}\tau_{(\varphi',\varphi'')}\,(v') = d_{\varphi'(\mathrm{p})}\tau_{(\varphi',\varphi'')}\left(\, d_{\varphi(\mathrm{p})}\tau_{(\varphi,\varphi')}\,(v)\,\right) = d_{\varphi(\mathrm{p})}\tau_{(\varphi,\varphi'')}\,(v)$$

That proves our claim. □

As to the quotient $\mathcal{A}_{\mathrm{p}} \times \mathbb{R}^n/\sim_{\mathrm{p}}$, [59] we have the following

Proposition 34 *There exists a unique (n-dimensional) vector space structure on* $\mathcal{A}_{\mathrm{p}} \times \mathbb{R}^n/\sim_{\mathrm{p}}$ *such that, for all* $\varphi \in \mathcal{A}_{\mathrm{p}}$, *the bijection*

$$\varphi_{\mathrm{p}} : \mathbb{R}^n \to \mathcal{A}_{\mathrm{p}} \times \mathbb{R}^n/\sim_{\mathrm{p}} :\ v \longmapsto \varphi_{\mathrm{p}}(v) := [(\varphi, v)]_{\sim_{\mathrm{p}}}$$

is an isomorphism.

---

[58] If $x' = \tau_{(\varphi,\varphi')}(x)$ is a transition function, then the transformation law $v' = d_x\tau_{(\varphi,\varphi')}\,(v)$ (where $x := \varphi(\mathrm{p})$) can be expressed in terms of the Jacobian matrix of $\tau_{(\varphi,\varphi')}$, i.e.

$$v'^{\,i} = \frac{\partial x'^{\,i}}{\partial x^j}\bigg|_x v^j$$

(see section 5.1, C^∞ *differentiable maps*, *Remark*).

[59] $[(\varphi, v)]_{\sim_{\mathrm{p}}} \in \mathcal{A}_{\mathrm{p}} \times \mathbb{R}^n/\sim_{\mathrm{p}}$ will denote the equivalence class of (φ, v).

Proof Choose a chart $\varphi \in \mathcal{A}_{\mathrm{p}}$.

Clearly φ_{p} is surjective (since any equivalence class in the quotient is represented by a couple whose first element is φ) and injective (since $\varphi_{\mathrm{p}}(v) = \varphi_{\mathrm{p}}(w)$ means $(\varphi, v) \sim_{\mathrm{p}} (\varphi, w)$, whence $w = d_{\varphi(\mathrm{p})}\tau_{(\varphi,\varphi)}\,(v) = v$.)

It is also clear that φ_{p} turns into a linear isomorphism, if and only if the quotient is equipped with the vector space structure defined by putting, for all $\varphi_{\mathrm{p}}(v), \varphi_{\mathrm{p}}(w) \in \mathcal{A}_{\mathrm{p}} \times \mathbb{R}^n / \sim_{\mathrm{p}}$ and $a, b \in \mathbb{R}$,

$$a\,\varphi_{\mathrm{p}}(v) + b\,\varphi_{\mathrm{p}}(w) := \varphi_{\mathrm{p}}(a\,v + b\,w)$$

If the quotient is equipped with the above vector space structure, then, for any other chart $\varphi' \in \mathcal{A}_{\mathrm{p}}$, bijection φ'_{p} will be shown to be an isomorphism.

To that end, let

$$\varphi'_{\mathrm{p}}(v') = \varphi_{\mathrm{p}}(v)\,, \quad \varphi'_{\mathrm{p}}(w') = \varphi_{\mathrm{p}}(w)$$

that is,

$$v' = d_{\varphi(\mathrm{p})}\tau_{(\varphi,\varphi')}\,(v)\,, \quad w' = d_{\varphi(\mathrm{p})}\tau_{(\varphi,\varphi')}\,(w)$$

Hence (owing to the linearity of the differential)

$$a\,v' + b\,w' = d_{\varphi(\mathrm{p})}\tau_{(\varphi,\varphi')}\,(a\,v + b\,w)$$

which means

$$\begin{array}{rcl} \varphi'_{\mathrm{p}}(a\,v' + b\,w') & = & \varphi_{\mathrm{p}}(a\,v + b\,w) \\ & = & a\,\varphi_{\mathrm{p}}(v) + b\,\varphi_{\mathrm{p}}(w) \\ & = & a\,\varphi'_{\mathrm{p}}(v') + b\,\varphi'_{\mathrm{p}}(w') \end{array}$$

That proves our claim. □

The *tangent vector space* of M at p is then defined by

$$T_{\mathrm{p}}M := \mathcal{A}_{\mathrm{p}} \times \mathbb{R}^n / \sim_{\mathrm{p}}$$

A *tangent vector* $\mathbf{v} \in T_{\mathrm{p}}M$ is expressed, in a chart $\varphi \in \mathcal{A}_{\mathrm{p}}$, in the form

$$\mathbf{v} := \varphi_{\mathrm{p}}(v) = v^i \frac{\partial}{\partial x^i}\Big|_{\mathrm{p}}$$

where [60]

$$\frac{\partial}{\partial x^i}\Big|_{\mathrm{p}} := \varphi_{\mathrm{p}}(\delta_i)\,, \quad i = 1, \ldots, m$$

---

[60] $(\delta_i)_{i=1,\ldots,m}$ denotes the canonical basis of $\mathbb{R}^m$.

is a basis of $T_{\mathrm{p}}M$. [61]

The *tangent bundle* of M is the disjoint union of its tangent vector spaces, i.e.

$$TM = \bigcup_{\mathrm{p}\in M} \{\mathrm{p}\} \times T_{\mathrm{p}}M$$

(so, for any $\mathrm{p} \in M$, $(\mathrm{p}, \mathbf{v}) \in TM$ means $\mathbf{v} \in T_{\mathrm{p}}M$) and the *canonical projection* of TM onto its *base manifold* M is the map

$$\pi_{_M} : TM \to M : (\mathrm{p}, \mathbf{v}) \mapsto \mathrm{p}$$

TM is a *vector fibre bundle* over M , in the sense that its *fibre* $T_{\mathrm{p}}M$ over each $\mathrm{p} \in M$ is a vector space. Such fibres are 'glued' together by $2m$-dimensional *natural charts*

$$(\mathrm{p}, \mathbf{v}) \in \pi_M^{-1}(\mathcal{U}) \longmapsto (x, v) \in \varphi(\mathcal{U}) \times \mathbb{R}^m$$
$$x := \varphi(\mathrm{p}) \,, \quad v := \varphi_{\mathrm{p}}^{-1}(\mathbf{v})$$

arising from the charts $\varphi : \mathcal{U} \to \varphi(\mathcal{U})$ belonging to $\mathcal{A}_M$.

Proposition 35 *TM is given a $2m$-dimensional manifold structure by its natural charts.*

Proof The collection of the $2m$-dimensional natural charts is obviously an atlas of TM , since

$$M = \bigcup_{\mathcal{U}\in\beta_M} \mathcal{U} \Longrightarrow TM = \bigcup_{\mathcal{U}\in\beta_M} \pi_M^{-1}(\mathcal{U})$$

Moreover the above atlas is C^∞, since it exhibits C^∞ transition functions

$$(x, v) \in \varphi(\mathcal{U} \cap \mathcal{U}') \times \mathbb{R}^m \longmapsto (x', v') \in \varphi'(\mathcal{U}' \cap \mathcal{U}) \times \mathbb{R}^m$$

given by

$$x' = \tau_{(\varphi,\varphi')}(x) \,, \quad v' = d_x\tau_{(\varphi,\varphi')}(v)$$

that is,

$$x'^{\,i} = x'^{\,i}(x) \,, \quad v'^{\,i} = \left.\frac{\partial x'^{\,i}}{\partial x^j}\right|_x v^j \,, \quad i = 1, \ldots, n$$

Such an atlas uniquely determines a manifold structure on TM . [62] □

---

[61] Remark that $\left.\frac{\partial}{\partial x'^{\,j}}\right|_{\mathrm{p}} := \varphi'_{\mathrm{p}}(\delta_j) = \varphi_{\mathrm{p}}(\upsilon_j) = \upsilon^i_j \left.\frac{\partial}{\partial x^i}\right|_{\mathrm{p}}$, iff $\upsilon_j = d_{x'}\tau_{(\varphi',\varphi)}\,(\delta_j)$ (with $x' := \varphi'(\mathrm{p})$, i.e. $\upsilon^i_j = \left.\frac{\partial x^i}{\partial x'^{\,h}}\right|_{x'} \delta^h_j = \left.\frac{\partial x^i}{\partial x'^{\,j}}\right|_{x'}$. Hence the transformation law of bases

$$\left.\frac{\partial}{\partial x'^{\,j}}\right|_{\mathrm{p}} = \left.\frac{\partial x^i}{\partial x'^{\,j}}\right|_{x'} \left.\frac{\partial}{\partial x^i}\right|_{\mathrm{p}}$$

exhibiting the Jacobian matrix of $\tau_{(\varphi',\varphi)}$, whereas the *contravariant* transformation law of components in $T_{\mathrm{p}}M$ exhibits the Jacobian matrix of $\tau_{(\varphi,\varphi')}$ (see footnote [58]).

[62] See Proposition 30.

Now let

$$\gamma : I \subset \mathbb{R} \to M : t \mapsto \mathrm{p}(t)$$

be a *smooth curve* of M, i.e. a parametrized curve admitting – in every chart $\varphi \in \mathcal{A}_M$ whose coordinate domain encounters its orbit – a C^∞ coordinate expression $x(t) = \varphi(\mathrm{p}(t))$.

The *tangent vector* to γ at $\mathrm{p}(t)$ is defined by

$$\begin{aligned} \dot{\mathrm{p}}(t) &:= \varphi_{\mathrm{p}(t)}(\dot{x}(t)) \\ &= \dot{x}^i(t) \frac{\partial}{\partial x^i}\Big|_{\mathrm{p}(t)} \end{aligned}$$

Remark that the above vector does not depend on the choice of the chart, since transition to another chart

$$x'(t) = \tau_{(\varphi,\varphi')}(x(t))$$

implies

$$\dot{x}'(t) = d_{x(t)}\tau_{(\varphi,\varphi')}(\dot{x}(t))$$

that is,

$$\varphi'_{\mathrm{p}(t)}(\dot{x}'(t)) = \varphi_{\mathrm{p}(t)}(\dot{x}(t))$$

The smooth curve of TM given by

$$\dot{\gamma} : I \subset \mathbb{R} \to TM : t \mapsto \dot{\gamma}(t) := (\mathrm{p}(t), \dot{\mathrm{p}}(t))$$

is the *tangent lift* of γ, which π_M projects down onto γ, i.e. $\pi_M \circ \dot{\gamma} = \gamma$.

Proposition 36 *TM is entirely swept by the tangent lifts of the smooth curves of M.*

Proof Let $(\mathrm{p}, \mathbf{v}) \in TM$.

Choose a chart $\varphi : \mathcal{U} \to \varphi(\mathcal{U})$ in $\mathcal{A}_\mathrm{p}$, where

$$\mathrm{p} = \varphi^{-1}(x)\,, \quad \mathbf{v} = \varphi_\mathrm{p}(v)$$

with $x \in \varphi(\mathcal{U})$ and $v \in \mathbb{R}^n$.

Now choose a $t_* \in \mathbb{R}$ and consider the smooth curve

$$\gamma : I \subset \mathbb{R} \to M : t \mapsto \mathrm{p}(t) := \varphi^{-1}(x(t))$$

defined on a suitably small open interval $I \ni t_*$ such that, for all $t \in I$,

$$x(t) := x + v(t - t_*) \in \varphi(\mathcal{U})$$

Clearly we have

$$\mathrm{p}(t_*) = \varphi^{-1}(x(t_*)) = \varphi^{-1}(x) = \mathrm{p}$$

and

$$\dot{\mathrm{p}}(t_*) = \varphi_{\mathrm{p}(t_*)}(\dot{x}(t_*)) = \varphi_{\mathrm{p}}(v) = \mathbf{v}$$

So

$$(\mathrm{p}, \mathbf{v}) = (\mathrm{p}(t_*), \dot{\mathrm{p}}(t_*)) \in \mathrm{Im}(\dot{\gamma})$$

That proves our claim. □

On the base of the above definitions and results, the theory of first-order differential equations on a smooth manifold can be formulated in the same way as it was on an embedded manifold.

Second tangent bundle

Recall that the canonical projection of the iterated tangent bundle TTM onto its base manifold TM is the map

$$\pi_{TM} : TTM \to TM$$

whose restriction to each fibre $T_{(\mathrm{p},\mathbf{v})}TM$ is the constant map

$$\pi_{TM}\Big|_{T_{(\mathrm{p},\mathbf{v})}TM} : \zeta \in T_{(\mathrm{p},\mathbf{v})}TM \longmapsto \pi_{TM}(\zeta) = \mathbf{v} \in T_{\mathrm{p}}M$$

or, in natural coordinates, [63]

$$\zeta \overset{\varphi}{\equiv} (x, v; u, w) \longmapsto \pi_{TM}(\zeta) \overset{\varphi}{\equiv} (x, v)$$

Also consider the *tangent map* of π_M, i.e. the map

$$T\pi_M : TTM \to TM$$

whose restriction to each fibre $T_{(\mathrm{p},\mathbf{v})}TM$ is the linear map

$$T_{(\mathrm{p},\mathbf{v})}\pi_M : \zeta \in T_{(\mathrm{p},\mathbf{v})}TM \longmapsto T_{(\mathrm{p},\mathbf{v})}\pi_M(\zeta) = T\pi_M(\zeta) \in T_{\mathrm{p}}M$$

defined, in natural coordinates, by

$$\zeta \overset{\varphi}{\equiv} (x, v; u, w) \longmapsto T\pi_M(\zeta) \overset{\varphi}{\equiv} (x, u)$$

---

[63] The symbol $\overset{\varphi}{\equiv}$ will denote the coordinates determined in M, TM and TTM by a chart $\varphi \in \mathcal{A}_M$.

Remark that the above vector

$$T\pi_{M}(\zeta) := \varphi_{\mathrm{p}}(u)$$

does not depend on the choice of the chart $\varphi \in \mathcal{A}_{\mathrm{p}}$, since, after a transition to another chart $\varphi' \in \mathcal{A}_{\mathrm{p}}$, we have a transformation law of natural coordinates in TM given by

$$x'^{\,i} = x'^{\,i}(x)\,, \quad v'^{\,i} = \frac{\partial x'^{\,i}}{\partial x^j}\Big|_x v^j$$

and then a transformation law for the components of a tangent vector $\zeta \in T_{(\mathrm{p},\mathbf{v})}TM$ given by

$$u'^{\,i} = \frac{\partial x'^{\,i}}{\partial x^j}\Big|_{(x,v)} u^j + \frac{\partial x'^{\,i}}{\partial v^j}\Big|_{(x,v)} w^j\,, \quad w'^{\,i} = \frac{\partial v'^{\,i}}{\partial x^j}\Big|_{(x,v)} u^j + \frac{\partial v'^{\,i}}{\partial v^j}\Big|_{(x,v)} w^j$$

whence, in particular,

$$u'^{\,i} = \frac{\partial x'^{\,i}}{\partial x^j}\Big|_x u^j$$

that is,

$$\varphi'_{\mathrm{p}}(u') = \varphi_{\mathrm{p}}(u)$$

The *second tangent bundle* of M is then defined by putting

$$T^2M = \{\zeta \in TTM \mid T\pi_{M}(\zeta) = \pi_{TM}(\zeta)\}$$

So, for any $\zeta \overset{\varphi}{\equiv} (x,v;u,w)$ in TTM, we have $\zeta \in T^2M$ iff $u = v$.

T^2M is an *affine fibre bundle* over TM, in the sense that its *fibre* $T^2_{(\mathrm{p},\mathbf{v})}M$ over each $(\mathrm{p},\mathbf{v}) \in TM$ is an affine space, as will now be shown.

Proposition 37 *For any* $(\mathrm{p},\mathbf{v}) \in TM$, $T^2_{(\mathrm{p},\mathbf{v})}M$ *is an* m*-dimensional affine subspace of* $T_{(\mathrm{p},\mathbf{v})}TM$ *modelled on the vector subspace* $\mathrm{Ker}(T_{(\mathrm{p},\mathbf{v})}\pi_{M})$.

Proof Just notice that

$$T^2_{(\mathrm{p},\mathbf{v})}M = \{\zeta \in T_{(\mathrm{p},\mathbf{v})}TM \mid T_{(\mathrm{p},\mathbf{v})}\pi_{M}(\zeta) = \mathbf{v}\}$$

Hence, for any $\zeta \in T^2_{(\mathrm{p},\mathbf{v})}M$,

$$T^2_{(\mathrm{p},\mathbf{v})}M = \zeta + \mathrm{Ker}(T_{(\mathrm{p},\mathbf{v})}\pi_{M})$$

That proves our claim. □

We inform that $\mathrm{Ker}(T_{(\mathrm{p},\mathbf{v})}\pi_{M})$ coincides (owing to the Implicit function Theorem) with the *vertical* vector subspace $T_{\mathbf{v}}(T_{\mathrm{p}}M)$ and the latter can be identified (throug a canonical isomorphism) with $T_{\mathrm{p}}M$.

Now let

$$\gamma : I \subset \mathbb{R} \to M$$

be a smooth curve of M, expressed in coordinates (near any point of its orbit) by

$$\gamma(t) \overset{\varphi}{\equiv} x(t)$$

Recall that its tangent lift

$$\dot{\gamma} : I \subset \mathbb{R} \to TM$$

is expressed in natural coordinates by

$$\dot{\gamma}(t) \overset{\varphi}{\equiv} (x(t), \dot{x}(t))$$

The *second tangent lift*

$$\ddot{\gamma} : I \subset \mathbb{R} \to TTM$$

i.e. the tangent lift of $\dot{\gamma}$, is then expressed in natural coordinates by

$$\ddot{\gamma}(t) \overset{\varphi}{\equiv} (x(t), \dot{x}(t)\,; \dot{x}(t), \ddot{x}(t))$$

As a consequence,

$$\mathrm{Im}(\ddot{\gamma}) \subset T^2M$$

Proposition 38 *T^2M is entirely swept by the second tangent lifts of the smooth curves of M.*

Proof Let $\zeta \in T^2_{(\mathrm{p},\mathbf{v})}M$

Choose a chart $\varphi : \mathcal{U} \to \varphi(\mathcal{U})$ in $\mathcal{A}_{\mathrm{p}}$, where

$$\zeta \overset{\varphi}{\equiv} (x, v; v, w)$$

Now choose a $t_* \in \mathbb{R}$ and consider the smooth curve

$$\gamma : I \subset \mathbb{R} \to M : t \mapsto \gamma(t) \overset{\varphi}{\equiv} x(t)$$

defined on a suitably small open interval $I \ni t_*$ such that, for all $t \in I$,

$$x(t) := x + v(t - t_*) + \frac{1}{2}w(t - t_*)^2 \in \varphi(\mathcal{U})$$

Clearly we have

$$\ddot{\gamma}(t_*) = (x(t_*), \dot{x}(t_*); \dot{x}(t_*), \ddot{x}(t_*)) = (x, v; v, w)$$

So

$$\zeta = \ddot{\gamma}(t_*) \in \mathrm{Im}(\ddot{\gamma})$$

That proves our claim. □

On the base of the above definitions and results, the theory of second-order differential equations on a smooth manifold can be formulated in the same way as it was on an embedded manifold.

Cotangent bundle

For any $\mathrm{p} \in M$, consider the *cotangent vector space* $T^*_{\mathrm{p}}M$, that is, the dual space of $T_{\mathrm{p}}M$.

Clearly any chart $\varphi \in \mathcal{A}_{\mathrm{p}}$ determines a linear isomorphism

$$\varphi^*_{\mathrm{p}} : \mathbb{R}^m \to T^*_{\mathrm{p}}M$$

which takes any m-tuple $p = (p_i)_{i=1,\ldots,m}$ to the *covector*

$$\pi = \varphi^*_{\mathrm{p}}(p)$$

whose *components* in φ are given by the above m-tuple, that is,

$$p_i = \left\langle \pi \mid \frac{\partial}{\partial x^i}\Big|_{\mathrm{p}} \right\rangle , \quad i = 1, \ldots, m$$

In another chart $\varphi' \in \mathcal{A}_{\mathrm{p}}$ we shall have

$$\pi = \varphi'^*_{\mathrm{p}}(p')$$

iff the m-tuples of components p and p' are related to each other by the transformation law [64]

$$p' = d_{x'}\tau_{(\varphi',\varphi)}(p)$$

(with $x' = \varphi'(\mathrm{p})$) since

$$\begin{aligned} p'_j &= \left\langle \pi \mid \frac{\partial}{\partial x'^{\,j}}\Big|_{\mathrm{p}} \right\rangle \\ &= \left\langle \pi \mid \frac{\partial x^i}{\partial x'^{\,j}}\Big|_{x'} \frac{\partial}{\partial x^i}\Big|_{\mathrm{p}} \right\rangle \\ &= \frac{\partial x^i}{\partial x'^{\,j}}\Big|_{x'} \left\langle \pi \mid \frac{\partial}{\partial x^i}\Big|_{\mathrm{p}} \right\rangle \\ &= \frac{\partial x^i}{\partial x'^{\,j}}\Big|_{x'} p_i \end{aligned}$$

---

[64] Here we encounter a *covariant* transformation law, exhibiting – like the transformation law of bases in $T_{\mathrm{p}}M$ – the Jacobian matrix of $\tau_{(\varphi',\varphi)}$.

The *cotangent bundle* of M is the disjoint union of its cotangent vector spaces, i.e.

$$T^*M = \bigcup_{\mathrm{p}\in M} \{\mathrm{p}\} \times T^*_{\mathrm{p}}M$$

(so, for any $\mathrm{p} \in M$, $(\mathrm{p}, \pi) \in T^*M$ means $\pi \in T^*_{\mathrm{p}}M$) and the *canonical projection* of T^*M onto its *base manifold* M is the map

$$\pi^*_{{}_M} : T^*M \to M : (\mathrm{p}, \pi) \mapsto \mathrm{p}$$

T^*M is a *vector fibre bundle* over M, in the sense that its *fibre* $T^*_{\mathrm{p}}M$ over each $\mathrm{p} \in M$ is a vector space. Such fibres are 'glued' together by $2m$-dimensional *natural charts*

$$(\mathrm{p}, \pi) \in \pi^{*-1}_M(\mathcal{U}) \longmapsto (x, p) \in \varphi(\mathcal{U}) \times \mathbb{R}^m$$
$$x := \varphi(\mathrm{p}), \quad p := \varphi^{*-1}_{\mathrm{p}}(\pi)$$

arising from the charts $\varphi : \mathcal{U} \to \varphi(\mathcal{U})$ belonging to $\mathcal{A}_M$.

Proposition 39 *T^*M is given a $2m$-dimensional manifold structure by its natural charts.*

Proof The collection of the $2m$-dimensional natural charts is obviously an atlas of TM, since

$$M = \bigcup_{\mathcal{U}\in\beta_M} \mathcal{U} \Longrightarrow T^*M = \bigcup_{\mathcal{U}\in\beta_M} \pi^{*-1}_M(\mathcal{U})$$

Moreover the above atlas is C^∞, since it exhibits C^∞ transition functions

$$(x, p) \in \varphi(\mathcal{U} \cap \mathcal{U}') \times \mathbb{R}^m \longmapsto (x', p') \in \varphi'(\mathcal{U}' \cap \mathcal{U}) \times \mathbb{R}^m$$

given by

$$x' = \tau_{(\varphi,\varphi')}(x), \quad p' = d_{x'}\tau_{(\varphi',\varphi)}(p)$$

that is,

$$x'^{\,j} = x'^{\,j}(x), \quad p'_j = \frac{\partial x^i}{\partial x'^{\,j}}\Big|_{x'} p_i, \quad j = 1, \ldots, n$$

Such an atlas uniquely determines a manifold structure on T^*M. [65] □

On the base of the above definitions and results, the theory of differential forms on a smooth manifold can be formulated in the same way as it was on an embedded manifold.

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[65] See Proposition 30.

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