Micro II - Monopoly (Solutions)

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Exercise 1

Consider a monopolist operating in a market with demand function $q = a - bq$. The marginal cost is c. Show that if the marginal cost increases by dc the monopoly price increases by $dc/2$.

Solution: the monopolist maximizes profits $\pi = (a - bp)(p - c)$. *The FOC is* $a - 2bp + bc = 0 \implies p^M = \frac{a+bc}{2b} \implies \frac{dp^M}{dc} = \frac{1}{2}$.

Exercise 2

Consider a monopolist with cost function $c(q) = cq$. The demand function is $q(p) = \alpha p^{-\epsilon}$. Assume $\epsilon > 1$, derive the optimal price and quantity chosen by the monopolist.

Solution: the monopolist maximizes profits $\pi = \alpha p^{-\epsilon}(p - c)$. *The FOC is:* $\frac{d\pi}{dp} = -\epsilon \alpha p^{-\epsilon-1}(p-c) + \alpha p^{-\epsilon} = 0$. *The SOC is satisfied given that* $\epsilon > 1$. *Multiplying the FOC by* $p^{\epsilon+1}$ *we get:* $-\epsilon\alpha(p-c) + \alpha p = 0 \iff p^M = \frac{\epsilon}{\epsilon-1} \implies q^M = \frac{\alpha\epsilon^{-\epsilon}}{(\epsilon-1)^{-\epsilon}}$.

Exercise 3

Suppose that the government wants to introduce a tax or a subsidy to a monopolist with cost function $c(q)$ and inverse demand function $p(q)$. Assume that both functions are differentiable and that the objective function of the monopolist is concave. What is the tax or subsidy that the government has to introduce in order to induce the monopolist to produce the efficient level of output?

Solution: The monopolist maximizes profits $\pi = p(q)q - c(q) - tq$. *The FOC is* $\frac{d\pi}{dq} = p'(q)q + p(q) - c'(q) - t = 0$. *In order to ensure that the optimal quantity chosen by the monopolist coincides with the efficient quantity (the quantity such that price equals marginal cost) we need* $t = p'(q^{eff})q^{eff} < 0$ *with* q^{eff} *such that* $p(q^{eff}) = c'(q^{eff})$. *Given that* $p'(q) < 0$, *also* $t < 0$ *is a subsidy rather than a tax.*

Exercise 4

Consider a market characterized by an inverse demand function $p(q) = 10/q$. The marginal cost is constant and equal to 1. Determine the level of output that maximizes profits.

Solution: the monopolist wants to maximize profits: $\pi = p(q)q - c(q) = 10 - q$. *Notice that in this case* ($\epsilon = 1$)*, the profits are strictly decreasing in* q*. In this case the monopolist wants to produce an arbitrarily small level of output and the maximization problem has no well-defined solution.*

Exercise 5

Consider a market characterized by the inverse demand function $p(q) = 10 - q$. The marginal cost for the monopolist is c. Determine the price that maximizes the monopolist's profits as a function of c and the level of c such that the optimal quantity is 6.

Solution: The monopolist maximizes profits given by $\pi = p(q)q - c(q) = (10 - q - c)q$. *The FOC is* $\frac{d\pi}{dq} = 10 - 2q - c = 0 \Longleftrightarrow q^M = \frac{10-c}{2}$. *It does not exist a positive level of* c *such that the monopolist produces 6.*

Exercise 6 (from March 2017 exam)

Consider a firm that produces one good demanded by two different types of consumers in equal number, type 1 and type 2 consumers with (individual) demand functions given by:

- Type 1: $D_1(p) = 1 - p$
- Type 2: $D_2(p) = 2(1 - p)$

The marginal cost is constant and equal to 0.

1. Show that if the monopolist is precluded from using non-linear pricing, then the optimal price is $p = \frac{1}{2}$ and profit (per consumer) $\pi = \frac{3}{8}$.

   Solution: the aggregate demand function is given by the horizontal sum of the demand functions of all consumers in the market: $D(p) = \frac{1}{2}D_1(p) + \frac{1}{2}D_2(p) = \frac{3}{2} - \frac{3}{2}p$ *for any* $p \leq 1$ *and* $D(p) = 0$ *for* $p > 1$*. The monopolist's problem is:*

$$\max_p \left(\frac{3}{2} - \frac{3}{2}p\right) \cdot p$$

   The FOC is $\frac{3}{2} - 3p = 0$ *and the SOC is always satisfied, thus the optimal price charged by the monopolist is* $p^M = \frac{1}{2}$ *and profit per consumer is* $\pi = \left(\frac{3}{2} - \frac{3}{2} \cdot \frac{1}{2}\right) \cdot \frac{1}{2} = \frac{3}{8}$.

2. Show that if the seller can set only a single two-part tariff (the same for both type of consumers) and he wants to serve both types of consumers, then the optimal values are $T = \frac{9}{32}$ and $p = \frac{1}{4}$ for a profit

of $\frac{9}{16}$.

Solution: if the seller can set only a single two-part tariff and he wants to sell to both types of consumers, he will have to choose the fixed part T so as to satisfy the Individual Rationality (IR) constraints of the low type (type-1). Indeed, the IR constraint of the high type is slack whenever the IR of the low type is satisfied. The IR of the low type is the difference between the consumer surplus associated with the two-part tariff and the fixed fee T:

$$\frac{1}{2}(1-p)^2 - T \geq 0 \quad (IR_1)$$

The IR constraint is binding at the optimum, else the monopolist could increase the fixed component T and increase profits while still satisfying the IR constraint. Hence, $T = \frac{1}{2}(1-p)^2$. *Thus the monopolist problem is*

$$\max_p \frac{1}{2}(1-p)^2 + \left(\frac{3}{2} - \frac{3}{2}p\right) \cdot p$$

The FOC is $-1 + p + \frac{3}{2} - 3p = 0$ *and the SOC is always satisfied, thus the optimal per-unit price charged by the monopolist is* $p^M = \frac{1}{4}$, $T = \frac{1}{2}(1-\frac{1}{4})^2 = \frac{9}{32}$ *and profit per consumer is* $\pi = \frac{9}{32} + \left(\frac{3}{2} - \frac{3}{2}\cdot\frac{1}{4}\right)\cdot\frac{1}{4} = \frac{9}{16}$.

3. Show that if the seller can set multiple two-part tariffs, but he cannot distinguish among the two types of consumers, then the optimal values are $T_1 = \frac{1}{8}$ and $p_1 = \frac{1}{2}$, $T_2 = \frac{7}{8}$ and $p_2 = 0$, for a profit of $\frac{5}{8}$.

   Solution: if the seller can set multiple two-part tariffs, but cannot distinguish across types of consumers, he will have to solve the following constrained maximization problem:

$$\max_{T_1,T_2,p_1,p_2} \frac{1}{2}T_1 + \frac{1}{2}T_2 + \frac{1}{2}(1-p_1)p_1 + (1-p_2)p_2$$

$$\textit{subject to}: \quad \frac{1}{2}(1-p_1)^2 - T_1 \geq 0 \quad (IR_1)$$

$$(1-p_2)^2 - T_2 \geq 0 \quad (IR_2)$$

$$\frac{1}{2}(1-p_1)^2 - T_1 \geq \frac{1}{2}(1-p_2)^2 - T_2 \quad (IC_1)$$

$$(1-p_2)^2 - T_2 \geq (1-p_1)^2 - T_1 \quad (IC_2)$$

   We will proceed in two steps: first, we will reduce the number of contraints by showing that some contraints are redundant, and only some contraints are binding. Second, we will solve the resulting (simplified) maximization problem.

   Notice that IR_2 *is implied by* IR_1 *and* IC_2 *and thus can be disregarded:*

$$\underbrace{(1-p_2)^2 - T_2 \geq (1-p_1)^2 - T_1}_{IC_2} > \underbrace{\frac{1}{2}(1-p_1)^2 - T_1 \geq 0}_{IR_1} \Longrightarrow \underbrace{(1-p_2)^2 - T_2 \geq 0}_{IR_2}$$

Further notice that IC_2must be binding at the optimum: suppose not, the monopolist can increase T_2 (and profits) while keeping both IR_1 unaffected and relaxing the IC_1 constraint. Thus we have: $(1-p_2)^2 - (1-p_1)^2 = T_2 - T_1$. Substituing this expression in IC_1, we get

$$(1-p_2)^2 - (1-p_1)^2 \geq \frac{1}{2}(1-p_2)^2 - \frac{1}{2}(1-p_1)^2 \Longleftrightarrow (1-p_2)^2 - (1-p_1)^2 \geq 0 \Longleftrightarrow p_2 \leq p_1$$

Finally, notice that IR_1 must be binding at the optimum: suppose not, the monopolist can increase both T_2 and T_1 so as to keep IC_2 satisfied with equality and increase profits. To summarize, we proved that IR_1 and IC_2 are binding, IR_2 is slack and can be disregarded and IC_1 can be replaced by the constraint $p_2 \leq p_1$. Hence the maximization problem of the monopolist becomes:

$$\max_{T_1, T_2, p_1, p_2} \frac{1}{2}T_1 + \frac{1}{2}T_2 + \frac{1}{2}(1-p_1)p_1 + (1-p_2)p_2$$

$$\textit{subject to}: \quad \frac{1}{2}(1-p_1)^2 - T_1 = 0$$

$$p_1 \geq p_2$$

$$(1-p_2)^2 - T_2 = (1-p_1)^2 - T_1$$

which can be further simplified by substituting the first and third constraints into the objective function:

$$\max_{p_1, p_2} \frac{1}{4}(1-p_1)^2 + \frac{1}{2}(1-p_2)^2 - \frac{1}{4}(1-p_1)^2 + \frac{1}{2}(1-p_1)p_1 + (1-p_2)p_2$$

$$\textit{subject to}: \quad p_1 \geq p_2$$

We will proceed by disregarding the constraint and checking ex post whether the solution that we find satisfies it. The FOCs are:

$$1 - 2p_1 = 0 \Longleftrightarrow p_1 = \frac{1}{2}$$

$$-1 + p_2 + 1 - 2p_2 = 0 \Longleftrightarrow p_2 = 0$$

The candidates p_1 and p_2 satisfy the constraint $p_1 \geq p_2$. We can plug these values back in the expressions we *derived above for $T_1$and T_2: $T_1 = \frac{1}{2}(1-p_1)^2 = \frac{1}{8}$ and $T_2 = (1-p_2)^2 - \frac{1}{2}(1-p_1)^2 = 1 - \frac{1}{8} = \frac{7}{8}$. Profits are $\frac{1}{2}T_1 + \frac{1}{2}T_2 + \frac{1}{2}(1-p_1)p_1 + (1-p_2)p_2 = \frac{1}{16} + \frac{7}{16} + \frac{1}{8} = \frac{5}{8}$.*