A NEW THEOREM IN ELECTROSTATICS WITH APPLICATIONS TO CALCULABLE STANDARDS OF CAPACITANCE

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SUMMARY

The paper presents what is believed to be a new theorem in electrostatics, showing that the direct capacitance of a very general type of cylindrical 3-terminal capacitor is a constant, namely \( \frac{\log_2}{4\pi^2} \).

The presentation and proof of this theorem is followed by a detailed analysis of a cylindrical 3-terminal capacitor of rectangular cross-section, in which the effects of various practical limitations are evaluated in quantitative terms. The results of similar analyses of cylindrical 3-terminal capacitors of other cross-sections are stated.

As a consequence of this work, it appears that such cylindrical 3-terminal capacitors are extremely attractive as precise calculable standards of capacitance, only one precision length measurement being necessary to enable their capacitance to be computed to a high order of accuracy.

(1) INTRODUCTION

The practical system of electrical units is based on electromagnetic laws, and the corresponding standards of measurement are established by reference to some form of inductor whose value in electromagnetic units can be determined from its mechanical dimensions. A calculable capacitor, on the other hand, yields an electrostatic unit which is related to the electromagnetic unit by \( c \), the velocity of light. The precision with which \( c \) is known has increased considerably in recent years and, more recently, has been the subject of a series of investigations by several workers. The present paper describes investigations in some detail, and in Section 2 the general theorem mentioned above is stated and proved. Section 3 is devoted to the discussion of a cylindrical capacitor of rectangular cross-section, and, in particular, the effects of both finite gaps and departures from symmetry are examined. Results for cylindrical capacitors of other cross-sections, corresponding to those given in Section 3, are listed in Section 4. General results, concerning the effects of departures from symmetry, of finite gap widths and of dielectric films on the electrode surfaces, are obtained as corollaries of some of these examples.

(2) A GENERAL THEOREM ON CYLINDRICAL CAPACITORS

Let the closed curve \( S \) (see Fig. 1) be the right cross-section of a conducting cylindrical shell, the cross-section having one axis of symmetry \( AC \), but being otherwise arbitrary. Further, let this shell be divided into four parts by two planes at right angles, the line of intersection of the planes being parallel to the gene-

![Cross-section of cylindrical capacitor](https://example.com/cross-section.png)
rators of the cylinder, and one of the planes containing the line AC.

Then, the direct capacitance per unit length of the cylinder, between opposing parts of the shell (e.g. $\alpha\delta$ to $\beta\gamma$), due to the field inside (or outside) the shell, is a constant:

$$C_0 = \frac{\log_2 2}{4\pi^2} \text{ e.s.u.} = 0.0175576 \text{ e.s.u.} \quad (1)$$

**Proof.**

An existence theorem (see Reference 7, p. 186) due to Riemann states:

If $\Sigma$ is the open domain bounded by a simple closed Jordan curve $S$, there exists a unique analytic function $f(z)$ regular in $\Sigma$, such that $\omega = f(z)$ maps $\Sigma$ conformally on $|\omega| < 1$, and also transforms a given point $z = z_0$ within $S$ into the origin, and a given direction at $z = z_0$ into the positive direction of the real axis.

Referring to Fig. 1, any point is now chosen which is inside $S$ and which lies on $AC$, as the point $z = z_0$. Applying Riemann's theorem, it is seen that the cross-section of the cylindrical shell of Fig. 1 may be transformed into a circle, as shown in Fig. 2,

![Fig. 2.—Cross-section of cylindrical capacitor after conformal transformation.](image)

in such a way that the line $AC$ maps into the line $A'C'$, and that symmetry is preserved.

The direct capacitance per unit length of cylinder between the opposing parts, say $\alpha\delta^*$ and $\beta\gamma^*$, due to the field inside the cylindrical shell of Fig. 2 can now be computed. The computation for the capacitance due to the field outside the cylindrical shell is similar and details are therefore omitted.

It is convenient first to redraw Fig. 2, at the same time adopting a system of polar co-ordinates with the line $OX$ as origin for the angular co-ordinate.

By definition, the direct capacitance per unit length of cylinder between the faces $\beta\gamma^*$ and $\alpha\delta^*$ is simply the total charge per unit length of the cylinder which appears on the face $\alpha\delta^*$, all faces of the cylinder being at zero potential with the exception of $\beta\gamma^*$, which is at unit potential. This direct capacitance is simply Maxwell's 'coefficient of induction' with sign reversed (see Reference 9, p. 38).

Thus, if $V(r, \phi)$ denotes the potential function inside the cylinder (whose cross-section is circular with radius $a$, say), it is clear that the following boundary conditions must be imposed:

$$V(a, \phi) = 1, \quad -\frac{\theta}{2} \leq \phi \leq \frac{\theta}{2} \quad \ldots \quad (2a)$$

$$= 0, \quad \frac{\theta}{2} < \phi < 2\pi - \frac{\theta}{2} \quad \ldots \quad (2b)$$

At all points inside the shell, $V(r, \phi)$ must satisfy Laplace's equation, namely

$$\frac{\partial^2 V}{\partial r^2} + \frac{1}{r} \frac{\partial V}{\partial r} + \frac{1}{r^2} \frac{\partial^2 V}{\partial \phi^2} = 0 \quad \ldots \quad (3)$$

It is further required that $V(r, \phi)$

(a) be an even, periodic function of $\phi$, because of the choice of $OX$ as origin for the angular co-ordinate,

(b) have no singularity at $r = 0$.

It is easy to show that the most general solution of eqn. (3) which satisfies these requirements is of the form

$$V(r, \phi) = \sum_{m=0}^{\infty} A(m) r^m \cos m\phi \quad \ldots \quad (4)$$

When we set $r = a$, we have

$$V(a, \phi) = \sum_{m=0}^{\infty} A(m) a^m \cos m\phi \quad \ldots \quad (5)$$

which is a Fourier cosine series in $\phi$, and, using the boundary conditions (2), we find

$$A(m) a^m = \frac{2^m}{\pi} \int_0^{\pi/2} \cos m\theta \sin \frac{m\theta}{2} \cos m\phi \quad \ldots \quad (6)$$

$$A(0) = \frac{\theta}{2\pi} \quad \ldots \quad (7)$$

so that, finally,

$$V(r, \phi) = \frac{\theta}{2\pi} + \sum_{m=1}^{\infty} \frac{2^m}{m\pi} \left( \frac{r}{a} \right)^m \sin \frac{m\theta}{2} \cos m\phi \quad (9)$$

The charge density per unit length at the surface, $\sigma(\phi)$, is obtained by application of Gauss's theorem (see Reference 9, p. 17) and is given by

$$\sigma(\phi) = \frac{1}{4\pi} \frac{\partial}{\partial r} \int_{r=a}^{\infty} \frac{\partial V}{\partial r} \quad \ldots \quad (10)$$

$$= -\frac{1}{2\pi a^2} \sum_{m=1}^{\infty} \frac{2^m}{m\pi} \sin \frac{m\theta}{2} \cos m\phi \quad (11)$$

The series on the right-hand side of eqn. (11) is easily trans-
formed into a pair of geometric series of complex exponentials, which can be summed to give
\[ \sigma(\phi) = \frac{1}{8\pi^2 a} \left[ \frac{\sin(\theta/2 + \phi)}{1 - \cos(\theta/2 + \phi)} + \frac{\sin(\theta/2 - \phi)}{1 - \cos(\theta/2 - \phi)} \right] \] (12)

The capacitance per unit length of the cylinder is now obtained by integrating \( \sigma(\phi) \) over the face \( \alpha \beta \).

Thus, from eqn. (12), we find
\[ \int_{-\theta/2}^{2\pi - 3\theta/2} \sigma(\phi) a d\phi = -\frac{1}{8\pi^2} \left[ \log e (1 - \cos \theta) - \log e (1 - \cos 2\theta) - 2 \log e (1 + \cos \theta) \right] \] (13)

The most general solution of Laplace's equation, which satisfies boundary conditions (15a), (15b) and (15c), is of the form
\[ V(x, y) = \sum_{n=1}^{\infty} A(n) \sin \frac{mnx}{a} \sinh \frac{mny}{a} \] . . . . (16)

Thus, from the theory of Fourier series, it follows that, after setting \( y = b \), and making use of condition (15d),
\[ A(n) \sin \frac{nnb}{a} = \frac{2}{\pi} \int_0^a \sin \frac{mnx}{a} dx \] . . . . (17)

so that the complete solution is
\[ V(x, y) = \frac{2}{\pi n} \sum_{n=1}^{\infty} \frac{1}{1 - (-1)^n} \frac{\sin \frac{mny}{a}}{\sinh \frac{mny}{a}} \] . . . . (19)

It follows from Gauss's theorem that \( \sigma(x) \), the surface charge density per unit length on the face \( y = 0 \), is given by
\[ \sigma(x) = \frac{1}{4\pi} \frac{\partial}{\partial y} V(x, y) \] . . . . (20)

\[ = \frac{1}{2\pi a} \sum_{n=1}^{\infty} \frac{1}{1 - (-1)^n} \frac{\sin \frac{mny}{a}}{\sinh \frac{mny}{a}} \] . . . . (21)

Integration of \( \sigma(x) \) over the face \( y = 0 \) gives, finally, the capacitance per unit length between the faces \( y = 0 \) and \( y = b \) as
\[ C(\beta) = \left[ \int_0^a \sigma(x) dx \right] \] . . . . (22)

\[ = \frac{2}{\pi^2} \sum_{n=0}^{\infty} \frac{1}{(2n + 1) \sinh (2n + 1)\pi \beta} \] . . . . (23)

where
\[ \beta = b/a \] . . . . (24)

In eqn. (23) may be summed to give
\[ C(\beta) = \frac{1}{\pi^2} \log e \frac{\theta_1(0; e^{-n\beta})}{\theta_2(0; e^{-n\beta})} \] . . . . (25)

where
\[ \theta_1(0; q) = 1 + 2 \sum_{n=1}^{\infty} q^n = \left[ \frac{2}{\pi} K(k) \right]^{1/2} \] . . . . (26)

\[ \theta_2(0; q) = 1 + 2 \sum_{n=1}^{\infty} (-1)^n q^n = \left[ \frac{2}{\pi} (1 - k^2)^{1/2} K(k) \right]^{1/2} \] (27)

are theta functions (see Reference 10, p. 464), and \( K(k) \) is the complete elliptic integral of the first kind* with modulus \( k \) chosen so that
\[ \frac{K[\sqrt{(1 - k^2)}]}{K(k)} = \frac{1}{\pi} \log e \left( \frac{1}{q} \right) \] . . . . (28)

From eqns. (26), (27) and (28), we have
\[ C(\beta) = \frac{1}{4\pi^2} \log e \left( \frac{1}{1 - k^2} \right) \] . . . . (29)

where \( k \) satisfies
\[ \frac{K[\sqrt{(1 - k^2)}]}{K(k)} = \beta \] . . . . (30)

When it is recalled that \( C(\beta) \) denotes the direct capacitance per unit length between the faces \( y = 0 \) and \( y = b \) with \( \beta = b/a \),

* The theta functions and elliptic integrals which occur here arise quite naturally as a result of applying the Schwarz-Christoffel transformation to a rectangle. Some very general results concerning the solution of Laplace's equation inside a rectangular boundary have been given in a recent paper.11
it can be seen that \( C(\frac{1}{\beta}) \) is the direct capacitance per unit length between the faces \( x = 0 \) and \( x = a \).

It follows from eqns. (29) and (30) that

\[
C(\frac{1}{\beta}) = \frac{1}{4\pi^2} \log e \frac{1}{\beta} \quad \ldots \ldots \quad (31)
\]

and hence, again making use of eqn. (30), \( C(\beta) \) and \( C(\frac{1}{\beta}) \) are connected by the identity

\[
e^{-4\pi^2 C(\beta)} + e^{-4\pi^2 C(\frac{1}{\beta})} = 1 \quad \ldots \ldots \quad (32)
\]

In particular, when \( \beta = 1 \) the rectangle becomes a square for which the diagonal is an axis of symmetry, and it follows immediately from eqn. (32) that

\[
C(1) = \frac{1}{4\pi^2} \log e \frac{1}{2} \quad \ldots \ldots \quad (33)
\]

which is in agreement with the theorem given in Section 1.

Explicit solutions of eqn. (30) are known (see Reference 10, p. 525) for some other values of \( \beta \). [The solution for \( \beta = 1 \) is closely related to the theory of lemniscate functions (see Reference 10, p. 524).] These solutions, together with corresponding values of \( C(\beta) \) and \( C(\frac{1}{\beta}) \) obtained by substituting these solutions in eqns. (29) and (31), are given in Table 1.

The behaviour of the functions \( C(\beta) \) and \( C(\frac{1}{\beta}) \) and their arithmetic mean is shown in Fig. 5 for \( \frac{1}{2} < \beta < 2 \). It will be seen that the mean capacitance per unit length has a minimum at \( \beta = 1 \), showing that, to the first order, small departures from squareness produce no change in the mean capacitance per unit length. Before taking up this point in more detail, the asymptotic (\( \beta \to \infty \)) behaviour of \( C(\beta) \) and \( C(\frac{1}{\beta}) \) is investigated.

Returning to expression (25) for \( C(\beta) \), and making use of the series definitions (26) and (27) for the theta functions, we obtain

\[
C(\beta) = \frac{1}{\pi^2} \left( 4e^{-\pi \beta} + \frac{16}{3} e^{-3\pi \beta} - 8e^{-5\pi \beta} + 20e^{-9\pi \beta} + \ldots \right) \quad (34)
\]

\[
\sim \frac{4}{\pi^2} e^{-\pi \beta} \quad \text{as} \quad \beta \to \infty \quad \ldots \ldots \quad (35)
\]

where the Taylor expansion has been used for the logarithm occurring in eqn. (25). The corresponding expression for \( C(\frac{1}{\beta}) \) is obtained by making use of the identity (32), thus:

\[
C(\frac{1}{\beta}) = \frac{1}{4\pi^2} \log e \left( 1 - e^{-4\pi^2 C(\beta)} \right)^{-1} \quad (36)
\]

\[
\sim \frac{1}{4\pi^2} \log e \left( \frac{e^{2\pi \beta}}{16} \right) \quad \text{as} \quad \beta \to \infty \quad \ldots \ldots \quad (37)
\]

\[
\sim \frac{\beta}{\pi^2} \quad \text{as} \quad \beta \to \infty \quad \ldots \ldots \quad (38)
\]

We note that the results in eqns. (35) and (38) may readily be obtained by working directly with expression (23) for \( C(\beta) \). To see the way in which small deviations from squareness

<table>
<thead>
<tr>
<th>( \beta )</th>
<th>( k )</th>
<th>( C(\beta) )</th>
<th>( C(\frac{1}{\beta}) )</th>
<th>( \frac{1}{2} [C(\beta) + C(\frac{1}{\beta})] )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( \frac{1}{\sqrt{2}} )</td>
<td>( \frac{1}{4\pi^2} \log e \frac{\sqrt{2}}{2} )</td>
<td>( \frac{1}{4\pi^2} \log e \frac{\sqrt{2}}{2} + \frac{1}{2} )</td>
<td>( \frac{1}{4\pi^2} \log e \frac{\sqrt{2}}{2} )</td>
</tr>
<tr>
<td>( \sqrt{2} )</td>
<td>( \sqrt{2} - 1 )</td>
<td>( \frac{1}{4\pi^2} \log e \frac{\sqrt{2}}{2} + \frac{1}{2} )</td>
<td>( \frac{1}{4\pi^2} \log e \frac{\sqrt{2}}{2} + \frac{1}{2} )</td>
<td>( \frac{1}{4\pi^2} \log e \frac{\sqrt{2}}{2} + \frac{1}{2} )</td>
</tr>
<tr>
<td>( \sqrt{3} )</td>
<td>( \frac{1}{2} (2 - \sqrt{3})^{1/2} )</td>
<td>( \frac{1}{4\pi^2} \log e \frac{\sqrt{2}}{2} + \frac{1}{2} )</td>
<td>( \frac{1}{4\pi^2} \log e \frac{\sqrt{2}}{2} + \frac{1}{2} )</td>
<td>( \frac{1}{4\pi^2} \log e \frac{\sqrt{2}}{2} + \frac{1}{2} )</td>
</tr>
<tr>
<td>2</td>
<td>( \frac{\sqrt{2} - 1}{\sqrt{2} + 1} )</td>
<td>( \frac{1}{4\pi^2} \log e \frac{\sqrt{2}}{2} + \frac{1}{2} )</td>
<td>( \frac{1}{4\pi^2} \log e \frac{\sqrt{2}}{2} + \frac{1}{2} )</td>
<td>( \frac{1}{4\pi^2} \log e \frac{\sqrt{2}}{2} + \frac{1}{2} )</td>
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</table>

Fig. 5.—Cylindrical capacitor: rectangular cross-section.
affect the mean capacitance per unit length, we use eqn. (34) and put
\[ \beta = 1 + \delta, \text{ where } \delta \ll 1 \]  
(39)

We then have
\[ C(1 + \delta) = \frac{1}{\pi^2} \left[ 4e^{-\pi(1 + \beta)} + \frac{16}{3} e^{-3\pi(1 + \beta)} - 8e^{-5\pi(1 + \beta)} + 20e^{-9\pi(1 + \beta)} + \ldots \right] \]
(40)
\[ = C(1) - \frac{\delta}{\pi} (4e^{-\pi} + 16e^{-3\pi} - 40e^{-5\pi} + 180e^{-9\pi} + \ldots) \]
\[ + \frac{\delta^2}{2} (4e^{-\pi} + 48e^{-3\pi} - 200e^{-5\pi} + 1600e^{-9\pi} + \ldots) + \text{terms in } \delta^3 \text{ and higher} \]
(41)

Similarly we find
\[ C\left( \frac{1}{1 + \delta} \right) = \frac{1}{\pi^2} \left[ 4e^\pi + 16e^{3\pi} - 40e^{5\pi} + 180e^{9\pi} + \ldots \right] \]
\[ + \frac{\delta^2}{2} \left[ (1 - \frac{2}{\pi}) e^{-\pi} + 16 \left( 3 - \frac{2}{\pi} \right) e^{-3\pi} - 40 \left( 5 - \frac{2}{\pi} \right) e^{-5\pi} + \ldots \right] + \text{terms in } \delta^3 \text{ and higher} \]
(42)

Adding eqns. (41) and (42) we obtain finally
\[ \frac{1}{\pi^2} \left[ C(1) + C\left( \frac{1}{1 + \delta} \right) \right] = \frac{1}{\pi^2} \left[ C(1) - \frac{\delta}{\pi} (4e^{-\pi} + 16e^{-3\pi} - 40e^{-5\pi} + 180e^{-9\pi} + \ldots) \right] \]
\[ + \frac{\delta^2}{2} \left[ (1 - \frac{2}{\pi}) e^{-\pi} + 16 \left( 3 - \frac{2}{\pi} \right) e^{-3\pi} - 40 \left( 5 - \frac{2}{\pi} \right) e^{-5\pi} + \ldots \right] + \text{terms in } \delta^3 \text{ and higher} \]
(43)

which shows clearly that small deviations from squareness produce only a second-order change in the mean capacitance per unit length.

To complete our investigation of the cylindrical capacitor of rectangular cross-section, it would be desirable to have an estimate of the error introduced by the presence of the necessarily finite insulating gaps. An exact determination of this error is complicated, but an upper bound is readily obtained by computing the fraction of the total charge (on the face \( y = 0 \)) which lies in a pair of narrow strips, each of fractional width \( \xi \), lying at each end \( x = 0 \) and \( x = a \) of the face \( y = 0 \). It is thought that this upper bound, while probably being considerably larger than the true error, is nevertheless a useful figure, by means of which gap effects in different types of capacitor may be roughly compared.

The surface charge density per unit length on the face \( y = 0 \) is given by eqn. (21), and the charge which lies in the strip \( 0 < x < \xi a \) is
\[ \int_0^\xi a \sigma(x) dx = \frac{1}{2\pi a} \sum_{n=0}^\infty \left[ 1 - (-1)^n \right] \frac{\sin \pi nx}{n \beta} a \]  
(45)
where, as before, \( \beta = \frac{b}{a} \).
and we see that the fractional change in capacitance produced by the small but finite insulating gaps is a second-order effect. This property and those properties expressed by eqns. (33) and (44), show that the cylindrical capacitor of square cross-section would be an attractive design for a calculable capacitance standard.

(4) RESULTS FOR CYLINDRICAL CAPACITORS OF OTHER CROSS-SECTIONS

In this Section we list, for some cylindrical capacitors of other cross-sections, results corresponding to those given in Section 3 for the capacitor of rectangular cross-section.

**Mean capacitance per unit length.**

\[ C_M(\theta_1, \theta_2, \theta_3) = \frac{1}{2} (C_1 + C_2) = \frac{1}{8\pi^2} \log_e \left[ \frac{1 - \cos(\theta_1 + \theta_2)\left(1 - \cos(\theta_1 + \theta_2 + \theta_3)\right)}{\left(1 - \cos(\theta_1\theta_2\theta_3)(1 - \cos(\theta_1 + \theta_2 + \theta_3))\right)} \right] \]

... (62)

**Fraction** \(D(\xi)\) of total charge which lies in the gaps \(\beta, \delta, \) the gaps each having an angular width \(\frac{\pi}{2}\).

**Surface charge density per unit length.**

\[ \sigma(\phi) = \frac{1}{8\pi^2} \log_e \left[ \frac{1 - \cos(\theta_1 + \phi)\left(1 - \cos(\theta_1 - \phi)\right)}{1 - \cos(\theta_1 + \phi)} \right] \]

... (59)

**Capacitance per unit length between hatched portions.**

\[ C_1(\theta_1, \theta_2, \theta_3) = \frac{1}{8\pi^2} \log_e \left[ \frac{1 - \cos(\theta_1 + \theta_2)\left(1 - \cos(\theta_1 + \theta_2 + \theta_3)\right)}{\left(1 - \cos(\theta_1\theta_2\theta_3)(1 - \cos(\theta_1 + \theta_2 + \theta_3))\right)} \right] \]

... (60)

**Capacitance per unit length between unhatched portions.**

\[ C_2(\theta_1, \theta_2, \theta_3) = \frac{1}{8\pi^2} \log_e \left[ \frac{1 - \cos(\theta_1 + \theta_2)\left(1 - \cos(\theta_1 + \theta_2 + \theta_3)\right)}{\left(1 - \cos(\theta_1\theta_2\theta_3)(1 - \cos(\theta_1 + \theta_2 + \theta_3))\right)} \right] \]

... (61)

**Some Special Cases of these Results.**

If we set

\[ \begin{align*}
\theta_1 &= \theta + \delta_1, \\
\theta_2 &= \theta - \delta, \\
\theta_3 &= \theta - \delta_3
\end{align*} \]

... (64)

we obtain the system of Fig. 3, analysed in Section 2. Substituting conditions (64) in eqn. (61) we obtain

\[ C_1(\theta) = \frac{1}{8\pi^2} \log_e \left[ \frac{2(1 - \cos 2\theta)}{(1 - \cos \theta)(1 + \cos \theta)} \right] \]

... (65)

\[ = \frac{1}{4\pi^2} \log_e 2 \] \begin{align*}
\theta_1 &= \theta + \delta_1, \\
\theta_2 &= \theta - \delta, \\
\theta_3 &= \theta - \delta_3
\end{align*} \]

... (66)

as before.

It is of particular interest in this case to investigate the effect, on the mean capacitance per unit length, of small deviations of all the angles from their nominal values [conditions (64)]. Substituting the conditions

\[ \begin{align*}
\theta_1 &= \theta + \delta_1, \\
\theta_2 &= \pi - \theta + \delta_2, \\
\theta_3 &= \pi - \theta - \delta_3
\end{align*} \]

... (67)

in expression (62) for the mean capacitance per unit length, and expanding in powers of \(\delta\), we find, after some reduction,

\[ C_M(\theta) = \frac{1}{4\pi^2} \log_e 2 + \frac{1}{16\pi^2} \left[ \frac{1}{1 - \cos \theta} \left[ \frac{\delta_1^2 + (\delta_1 + \delta_2 + \delta_3)^2}{2} \right] \right] \]

... (68)

which shows that the mean capacitance per unit length remains constant to the first order. It is clear that this result furnishes a corollary to the main theorem given in Section 2, which may be stated (referring to Fig. 1) as follows:

**Corollary.** The arithmetic mean of the two direct capacitances per unit length, i.e. \(\frac{1}{2} [C(\alpha\beta) + C(\gamma\delta)]\), remains constant and equal to \(C_o\), to the first order, for small but arbitrary departures from symmetry.
APPLICATIONS TO CALCULABLE STANDARDS OF CAPACITANCE

Surface change density per unit length.

\[ \sigma(x) = -\frac{1}{2\pi^2} \left[ \int_{0}^{\infty} \frac{ka}{2} \cos kxdk \right] \]

Capacitance per unit length between hatched portions.

\[ C(a/b) = \frac{1}{4\pi^2} \left[ \int_{0}^{\infty} \frac{ka}{2} \sin \frac{k(b + a/2)}{k} dk \right] \]

Note that \( C(1) = \frac{1}{4\pi^2} \log 2 \), which is in agreement with the results of the general theorem.

Mean capacitance per unit length.

\[ C_M(a/b) = \frac{1}{2} [C(a/b) + C(b/a)] = \frac{1}{4\pi^2} \log \left(1 + \frac{a}{b}\right) \]

Another case of interest, shown systematically in Fig. 7, is obtained by setting

\[ \Theta_1 = \theta \]
\[ \Theta_2 = \pi - \theta \]
\[ \Theta_3 = \theta \]

When we substitute conditions (69) into eqns. (60) and (61) we obtain

\[ C_1(\theta) = \frac{1}{4\pi^2} \log \left(1 + \cos \theta \right) \]
\[ C_2(\theta) = \frac{1}{4\pi^2} \log \left(1 - \cos \theta \right) \]

The capacitances \( C_1(\theta) \) and \( C_2(\theta) \) are connected by the identity

\[ e^{-4\pi^2 C_1(0)} + e^{-4\pi^2 C_2(0)} = 1 \]

which is analogous to eqn. (32) for the capacitor of rectangular cross-section.

Finally we note that, in general, because of the first-order gap effect [see eqn. (63)], cylindrical capacitors of circular cross-section are not particularly suitable for calculable standards of high precision.

(4.2) Infinite-Plane Cylinder

The system of the infinite-plane cylinder, shown in Fig. 8, arises from application of the Schwarz–Christoffel transformation

\[ \frac{y}{x} = \sin \frac{k}{2} e^{-\xi} \cos kxdk, \quad y > 0 \]

Potential at point \( (x, y) \).

\[ V(x, y) = \frac{2}{\pi} \left[ \int_{0}^{\infty} \frac{ka}{2} \cos kxdk, \quad y > 0 \right] \]

which is of the first order in \( \xi \).

\( \omega = \frac{\partial}{\partial \theta} \)

Fig. 9.—The neighbourhood of an insulating gap in a cylindrical capacitor.

(a) Cross-section.
(b) Conformal transformation.

* The integral of eqn. (92) is readily put in the form of a Frullani integral (see Reference 12, p. 479), or may be evaluated as a Fourier integral.
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It should be noted that the results of this Section on the infinite-plane capacitor provide an alternative proof for the general theorem of Section I. Thus, if a polygon with 4n sides is inscribed in the cross-section of Fig. 1, the result of applying the Schwarz–Christoffel transformation is, in the limit \( n \to \infty \), the system of Fig. 8 with \( a = b \). The proofs of the theorem and corollary then follow from eqns. (75) and (77).

With this procedure in mind, a result concerning the order of magnitude of the error produced by small but finite gaps can be established for the general case.

Thus, considering the neighbourhood of the insulating gap shown in Fig. 1, it can be supposed that any discontinuity in direction of the curve \( S \) lies within the gap itself as shown in Fig. 9(a).

Application of the Schwarz–Christoffel transformation results in the system shown in Fig. 9(b).

In this neighbourhood, this transformation has the functional form (see Reference 7, p. 197)

\[
z = \int F(\omega)(\omega - \omega_0)^{-1}d\omega + C \quad \ldots \quad (79)
\]

where \( C \) is a constant and \( F(\omega) \) is a relatively slowly varying function of \( \omega \).

In particular we have

\[
\Delta z = \int_{-\infty}^{+\infty} F(\omega) \left( \omega - \omega_0 \right)^{-1}d\omega \quad \ldots \quad (80)
\]

\[
= F(\omega_0) \left( \frac{\Delta \omega}{2} \right)^{1/2} \left( 1 - (-1)^{\alpha+2} \right) + \text{higher terms} \quad \ldots \quad (81)
\]

\[
= \Delta \omega \left[ a_0 + a_1 \Delta \omega + a_2 (\Delta \omega)^2 + \ldots \right] \quad \ldots \quad (82)
\]

Reversion of this power series (see Reference 10, p. 129) then gives

\[
\Delta \omega = b_1 \left[ \frac{\Delta z}{F(\omega_0)} \right]^{1/2} + b_2 \left[ \frac{\Delta z}{F(\omega_0)} \right]^{2/2} + \ldots \quad (83)
\]

where the \( b_i \)'s are functions of the \( a_i \)'s of eqn. (83) and hence expressible in terms of \( F(\omega_0) \), its derivatives and \( \alpha \). Taking the modulus of both sides of eqn. (84) we obtain

\[
\Delta \omega \leq C |\Delta z|^{1/2} + \text{terms in higher powers of } |\Delta z| \quad \ldots \quad (85)
\]

Eqn. (78) shows that the fraction of charge in a narrow gap in the infinite plane system is linear to the first order in the gap width. Hence, from eqn. (85), it can be seen that a gap of width \( |\Delta z| \) in the system shown in Fig. 1 will produce an error in the direct capacitance whose magnitude is proportional, to the first order, to \( |\Delta z|^{1/2} \), where \( \alpha \) is the angle between tangents to the electrode surfaces at the gap.

For example, for the rectangular system treated in Section 3, we have \( \alpha = \frac{\pi}{2} \) at every gap, and hence the fractional error in capacitance is proportional to \( (\text{gap width})^2 \), which is in agreement with eqn. (57).

It is clear from eqn. (85) that a configuration highly insensitive to the width of the gaps used would result from placing the gap in a re-entrant position such that \( \alpha \) is as small as possible. A configuration which achieves this objective, and at the same time has considerable advantages in practice, is shown in Fig. 10.

Calculable capacitors based on this design have been constructed, and will be described in detail in a forthcoming publication, by the author's colleague A. M. Thompson, to whom this configuration is due.

Of some practical interest is the effect, on the direct capacitance of a cylindrical capacitor, of filling part of the interior of the cylinder with material whose dielectric constant differs from unity. To obtain some idea of the magnitude of this effect, particularly in the limiting case of thin dielectric films on the electrode surfaces (representing a possible surface contamination), an analysis of the symmetrical infinite-plane system shown in Fig. 11 has been made, with the following results.

Potential at point \((x, y)\) in Region 1, \( y \leq c \).

\[
V_1(x, y) = \frac{2}{\pi} \int_0^\infty \frac{ka}{\cos kx} \left\{ \left( \frac{1 + \frac{1}{\epsilon}}{\epsilon} \right)^{kay} + \left( \frac{1 - \frac{1}{\epsilon}}{\epsilon} \right)^{kay} - \frac{e^{-2k\epsilon\epsilon}}{e^{2k\epsilon}} \right\} dk \quad \ldots \quad (86)
\]

Potential at point \((x, y)\) in Region 2, \( y \geq c \).

\[
V_2(x, y) = \frac{4}{\pi} \int_0^\infty \frac{ka}{\cos kx} \left\{ \left( \frac{1 + \frac{1}{\epsilon}}{\epsilon} \right)^{kay} + \left( \frac{1 - \frac{1}{\epsilon}}{\epsilon} \right)^{kay} - \frac{e^{-2k\epsilon\epsilon}}{e^{2k\epsilon}} \right\} dk \quad \ldots \quad (87)
\]
Applications to Calculable Standards of Capacitance

Surface charge density per unit length.

\[ \sigma(x) = -\frac{\varepsilon}{2\pi^2} \left[ \frac{\sin (ka)}{2} \cos kx \cdot \frac{1 + \frac{1}{\varepsilon} e^{\frac{-k\ell}{\varepsilon}} - (1 - \frac{1}{\varepsilon} e^{2k\ell})}{1 + \frac{1}{\varepsilon} e^{\frac{-k\ell}{\varepsilon}}} \right] \, dk \]  

(88)

Direct capacitance per unit length between hatched portions.

\[ C(\varepsilon; \frac{c}{a}) = \frac{1}{4\pi^2} \log_2 \frac{1 + 4\left(\frac{\varepsilon}{a}\right)}{1 + \left(\frac{\varepsilon}{a}\right)^2} \]  

(90)

In particular, for \( \varepsilon < 1 \), corresponding to thin films, we find:

\[ C(\varepsilon; \frac{c}{a}) = \frac{1}{4\pi^2} \log_2 2 + \frac{3}{16\pi^2} \left(\frac{c}{a}\right)^2 + \text{terms in } \left(\frac{c}{a}\right)^4 \]  

and higher (90)

which shows that thin dielectric films produce only a second-order effect on the capacitance.

We note the following special cases of eqn. (89):

\[ C(\varepsilon; 0) = C(\varepsilon; \infty) = \frac{\varepsilon}{4\pi^2} \log_2 \frac{1 + 4\left(\frac{\varepsilon}{a}\right)}{1 + \left(\frac{\varepsilon}{a}\right)^2} \]  

(91)

and

\[ C(\varepsilon; \infty) = \frac{1}{4\pi^2} \log_2 2 \]  

(92)

We note the following special cases of eqn. (89):

\[ C(\varepsilon; 0) = C(\varepsilon; \infty) = \frac{\varepsilon}{4\pi^2} \log_2 \frac{1 + 4\left(\frac{\varepsilon}{a}\right)}{1 + \left(\frac{\varepsilon}{a}\right)^2} \]  

(91)

and

\[ C(\varepsilon; \infty) = \frac{1}{4\pi^2} \log_2 2 \]  

(92)

Finally, the effect of the necessarily finite insulating gaps has been examined, in an approximate way, and it has been shown that, providing that the gaps are placed in suitably re-entrant corners, their effect may be made negligible.

These investigations show that cylindrical 3-terminal capacitors are extremely attractive as calculable capacitance standards, one precision length measurement only being required to compute the capacitance.

Conclusions

A theorem has been established which states that the direct capacitance per unit length of a certain class of cylindrical 3-terminal capacitors is a constant.

A first essential requirement in this class of capacitors is that the cylinder cross-section shall have at least one axis of symmetry, but it has been shown that when the symmetry is slightly disturbed in an arbitrary way, the residual error in the mean of the two cross-capacitances (per unit length) is of the second order.

Finally, the effect of the necessarily finite insulating gaps has been examined, in an approximate way, and it has been shown that, providing that the gaps are placed in suitably re-entrant corners, their effect may be made negligible.

These investigations show that cylindrical 3-terminal capacitors are extremely attractive as calculable capacitance standards, one precision length measurement only being required to compute the capacitance.

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(7) REFERENCES


(3) SNOW, C: 'A Standard of Small Capacitance', ibid., 1947, 42, p. 848.


(8) APPENDIX

Evaluation of \( \sum_{n=0}^{\infty} \frac{1}{\sinh \theta} \) \( (2n + 1) \sinh [(2n + 1)\pi\beta] \)

By using the result

\[ \frac{1}{\sinh \theta} = 2 \sum_{n=0}^{\infty} e^{-2(2n+1)\theta} \]  

we may write

\[ \sum_{n=0}^{\infty} \frac{1}{(2n + 1) \sinh [(2n + 1)\pi\beta]} = 2 \sum_{s=0}^{\infty} \sum_{n=0}^{\infty} \frac{1}{(2n + 1)} e^{-(2n+1)(2s+1)\pi\beta} \]  

\[ = 2 \sum_{z=0}^{\infty} \frac{\pi(2s + 1)}{\sinh [(2s + 1)\pi\beta]} \sum_{n=0}^{\infty} e^{-(2n+1)(2s+1)\pi\beta} \]  

\[ = \sum_{z=0}^{\infty} \frac{\pi(2s + 1)}{\sinh [(2s + 1)\pi\beta]} \sum_{n=0}^{\infty} d\beta \]  

where, in passing from eqn. (106) to eqn. (107), we have used the result

\[ \int \frac{dx}{\sinh ax} = \frac{1}{a} \log \left( \frac{\tanh \frac{ax}{2}}{2} \right) \]  

Now the theta functions \( \theta_3(z; q) \) have the well-known infinite product representation (see Reference 10, p. 469):

\[ \theta_3(z; q) = G \prod_{n=0}^{\infty} (1 + 2q^{2n+1} \cos 2z + q^{2n+2}) \]  

\[ \theta_4(z; q) = G \prod_{n=0}^{\infty} (1 - 2q^{2n+1} \cos 2z + q^{2n+2}) \]

and thus it follows that

\[ \theta_3(0; q) = G \prod_{n=0}^{\infty} (1 + q^{2n+1}) \]  

\[ \theta_4(0; q) = G \prod_{n=0}^{\infty} (1 - q^{2n+1}) \]

When we set \( q = e^{-\pi\beta} \) and make use of eqns. (112) and (109), we obtain

\[ \sum_{n=0}^{\infty} \frac{1}{(2n + 1) \sinh [(2n + 1)\pi\beta]} = \frac{1}{2} \log \frac{\theta_3(0; e^{-\pi\beta})}{\theta_4(0; e^{-\pi\beta})} \]  

which is the required result.