Chapter 1

The structure of space-time and its symmetries

In this chapter we shall address the issue of continuous space-time symmetries as continuous groups of transformations. Their mathematical structure as Lie groups will be deepened in Chapter 27.

We shall begin with describing the geometrical nature of space-time as a manifold and introduce space-time splitting. The structure of space-time is analyzed by introducing a coordinate-free formulation of reference frame. Relativity groups emerge through the notion of equivalent frames. By means of the notion of objective existence, we show that transformations connecting equivalent frames are either Lorentz or generalized Galilei. Then, starting from the description of the Lorentz and Galilei group we shall introduce some of their most common subgroups of transformations of the configuration manifold, such as the **translation group** $T(n)$ in $\mathbb{R}^n$, the three-dimensional **special orthogonal** group $SO(3)$, the **special unitary** group $SU(2)$ and discuss their relationships.

From a geometric point of view, what all these groups have in common is that their elements can be labeled in a one-to-one way by some set of parameters, whose span can be given the structure of a differentiable manifold. This is what characterizes **Lie groups**, which will be considered in Chapter 27. Once given a manifold structure, the parameters span will be called the **group manifold**. It will turn out to be a compact manifold for $SO(3)$ and $SU(2)$, noncompact in the other cases. The inclusion of discrete operations (such as time-reversal and space-inversion), which are related with non-connected groups will also briefly be discussed.

1.1 The splitting of space-time

The general theory of relativity was formulated by Einstein and presented as a theory of gravitation based on completely new concepts of space and time. These concepts had remained essentially the same from Newton's time. Einstein arrived at this new formulation first through a deep epistemological analysis of electromagnetic phenomena and then generalized from flat manifolds to general ones. The major merit of Einstein was the realization that the bearing of the Lorentz transformations transcended its connection with Maxwell equations and was concerned with the nature of space and time in general. From his writings it emerges clearly that each *individual* (we shall identify this with a reference frame) should have a notion of space (space of the body) and a notion of time (earlier and later).

We should regard as real those perceptions which are common to different individuals, it will give rise to our concept of objective (inter-individual) existence and to the subsequent definition of compatible reference frames. The group of transformations connecting pairwise compatible reference frames will constitute the selected relativity group. Space and time will be perceived by different individuals as an impersonal whole entity. Each one of them will perform a division into time and space. The comparison of these various splittings into equivalence classes will identify the corresponding relativity group.

We assume the set of events (the space-time) to carry the structure of a smooth four dimensional manifold M. An individual, from here onward, will be called a reference frame and identified with a (1,1) tensor-field $R \in T^1_1(M)$, of rank one. We shall also require that it satisfies the relation $R \cdot R \propto R$. That is,

Definition. 1.1 *A reference frame is a (1,1) tensor field R of rank one and satisfying the condition* $R^2 = R$.

Each reference frame introduces a splitting of $T_m M$ in terms of the eigenspaces of $R(m)$ belonging to the zero eigenvalue and to the eigenvalue one respectively. We have

$$T_m M = \ker(R(m)) \oplus \mathrm{Im(R(m))}. \tag{1.1}$$

Equivalently we shall write

$$TM = R^s + R^t \tag{1.2}$$

where R^s and R^t stay for the corresponding vector bundles over the spacetime manifold M, they define the space and the time distributions respectively, associated with the reference frame. Given a mapping from a manifold N into the spacetime M, one can pull-back the vector bundles R^s and R^t to be vector bundles over the source manifold. In particular, the image of our mapping may be an integral manifold of R^s and will be called a space-axis or an integral manifold of R^t and will be called a time-axis. It is clear that a time-axis is a one-dimensional sub-manifold of the spacetime and will also be called an observer of the given reference frame. Thus, the family of integral curves of a vector field spanning the distribution R^t will constitute a family of observers of the given reference frame R. It should be noted that while R^t, being one-dimensional, is always an integrable distribution (cfr. def. (20.6)) and therefore each spacetime point lies on a time-axis, it is not necessarily true that each spacetime point lies on a space-axis: the spatial distribution need not be integrable. The family of reference frames for which each event lies on a space-axis coincides with those reference frames for which R^s is an integrable distribution. A reference frame whose spatial distribution is integrable will be called spatially integrable. The leaves of this spatial distribution associated with the reference frame R are a generalization of the instantaneous three space or rest frame of the family of observers attached to the reference frame [184]. For general (almost) product structures we refer the reader to [185].

To state more clearly the properties of the splitting of the spacetime associated with a reference frame R, it is convenient to decompose R into the tensor product of a vector field Γ and a one-form α. We shall write:

$$R = \alpha \otimes \Gamma. \tag{1.3}$$

The requirement $R^2 = R$ is equivalent to the condition $\alpha(\Gamma) = 1$. It should be noticed however that the family of one-forms $e^{-f}\alpha$ and the corresponding family of vector fields $e^f\Gamma$ provide an equivalent splitting of R. In particular this observation shows that integral leaves of R^t, time-axes or observers do not carry an intrinsic parametrization; this will come with the (gauge) choice of the particular decomposition of R. This freedom in the choice of Γ shows that by a proper choice of f we may select a vector field which is a complete vector field, i.e. it integrates to a one-parameter group of diffeomorphisms. Moreover the global existence of Γ implies that our spacetime is time-orientable.

In terms of the stated decomposition, R^s will be spatially integrable if and only if $\alpha \wedge d\alpha = 0$. This follows from Frobenius integrability theorem [176].

An equivalent way to state the integrability condition is the requirement

$$d\alpha = \alpha \wedge \eta \tag{1.4}$$

and η is called the recurrence one-form [184].

Remark. It is possible to show that $\Omega^s = \eta \wedge d\eta$ is closed and defines a cohomological class in the third de Rham cohomology group of M. The cohomology class of Ω^s is independent of the choice of the recurrence one-form η. From the geometrical view-point, the cohomology class of Ω^s measures the helical wobble of the leaves. It is to be remarked that the choice of α in the decomposition of R does not affect the integrability condition, i.e. if $\alpha \wedge d\alpha = 0$ then also $e^{-f}\alpha \wedge d(e^{-f}\alpha) = 0$.

Definition. A reference frame is said to be synchronizable or to satisfy the synchronizability condition if it admits a decomposition with $\alpha = d\mathscr{T}$. The function $\mathscr{T}$ will define a projection $\mathscr{T} : M \to \mathbb{R}$ with level sets defining simultaneity leaves.

If, moreover, the corresponding vector field Γ is a complete vector field, with $\mathscr{T}(\Gamma) = 1$, we will say that $R = d\mathscr{T} \otimes \Gamma$ represents a Galilean frame.

1.1.1 Compatible frames and objective existence

On the four-dimensional space-time $\mathbb{M}$, assumed to be diffeomorphic to $\mathbb{R}^4$, we select a linear structure represented by a dilation vector filed Δ. We assume that the family of reference frames we are considering satisfies the condition $L_\Delta R = 0$ and admits a decomposition $R = \alpha \otimes \Gamma$ with $L_\Delta\Gamma = -\Gamma$, $\alpha(\Gamma) = 1$. Any observer will be an integral curve of Γ. After the choice of a single observer has been made, the translation group 1.2.3 can be used to move it and thereby to construct a congruence of observers which are a family of solutions for the vector field Γ (we observe that the translation group is associated with smooth homogeneous vector fields of degree minus one with respect to Δ). The closed one form α defines a family of three-planes, transversal to the congruence defined by Γ. For any reference frame we can define time-like vectors as those $v_m \in T_m\mathbb{R}^4$ which satisfy $R(m)(v_m) = \lambda_m\Gamma(m)$, $\lambda_m \neq 0$. They will be future pointing if $\lambda_m > 0$.

We say that a point p is in the past of q ($p < q$) if p can be connected to q by a curve whose tangent vectors are future oriented. We can associate a clock with any observer by considering the curve

$$\gamma : \mathbb{R} \to \mathbb{M}$$

and setting

$$c(\gamma(s)) = \int_{s_0}^{s} \gamma^*(\alpha).$$

The mapping $c : \gamma(\mathbb{R}) \subset \mathbb{M} \to \mathbb{R}$ associates a "time parameter" with any observer with the property $c(\gamma(s_0)) = 0$.

Two frames in the family we are considering will be called a compatible pair if $\mathrm{Tr}\,(R_1 \cdot R_2) > 0$. This condition means that two compatible systems will perceive the observers of each other as representing some existing physical object; for this reason the requirement $\alpha_a(\Gamma_b) > 0, a, b = 1, 2$, will be called the *mutual objective existence condition*.

Remark. We could also accept $\alpha_a(\Gamma_b) < 0$. However, in this case we would be dealing with antiparticles instead of particles of the same species [64].

The restrictions on the stated condition of compatibility is meant to exclude the possibility of a time-axis of one reference frame to be contained in the space-axes of the other. This means that a particle at rest in one reference frame cannot be perceived by another one as a particle existing all over a real line only at a given instant of time, without past and without future.

We notice that the mutual objective existence condition is a symmetric and reflexive relation but it is not transitive. To build a partition of compatible systems into equivalence classes, the simplest way is to start with a fiducial reference frame, say R_0, and consider the subgroup of the inhomogeneous linear group (the semi-direct product of the general linear group and translations, which shall be defined in Sec.1.2.7) with the requirement that out of R_0 all transformed reference frames are pairwise compatible. The subgroup selected by implementing the mutual objective existence condition will be the selected relativity group.

Thus, from the point of view of the transformation group (i.e. transformations connecting physically equivalent systems) we have to exclude the possibility of transforming the time-axis of one frame into a real line lying in some space-axis of the other frame. It is therefore immediately clear that we should exclude rotations in the time-space planes.

Starting from a decomposition of R_0 into (α_0, Γ_0), we have to solve for all invertible linear transformations which may be composed among themselves while satisfying the conditions

$$\phi^*(\alpha_0)(\Gamma_0) > 0; \quad \alpha_0(\phi_*\Gamma_0) > 0.$$

To solve for transformations ϕ satisfying previous inequalities, we may restrict first to two-dimensional space-times.

On $M = \mathbb{R}^2$ we introduce coordinates x_0, x_1 and start with

$$R_0 = dx_0 \otimes (\frac{\partial}{\partial x_0} + b\frac{\partial}{\partial x_1}).$$

We notice that $L_\Delta R = 0$ means that we may restrict our considerations to the special linear group $SL(2\mathbb{R})$. This group coincides with the symplectic group $Sp(2, \mathbb{R})$. Thus, orbits of the one-parameter subgroups, that is subgroups connected to the identity transformation, will be given by level sets of quadratic functions in the variables (x_0, x_1). We have to exclude positive definite quadratic functions because they would generate rotations and therefore the transformed frame would not satisfy the mutual objective existence condition. Therefore we have to restrict to factorable quadratic functions, say

$$H = (ax_0 - bx_1)(mx_0 + nx_1).$$

The real lines $ax_0 - bx_1 = 0$ and $mx_0 + nx_1 = 0$ will be fixed lines of our allowed transformations. Therefore, space-time will be cut into four open regions and each region contains a branch of the level set $ax_0 - bx_1)(mx_0 + nx_1) = 1$.

The two lines represent the asymptotes of an hyperbola. By redefining the initial coordinate system by means of a linear transformation it is possible to bring our hyperbola to an equilateral one. The replacement of x_1 with x_2 and x_3 would provide us with similar hyperbolas in the other space-time planes. By assuming isotropy in x_1, x_2, x_3 and using an infinitesimal version of our quadratic form we would end up with a form $(dx_0)^2 - [(dx_1)^2 + (dx_2)^2 + (dx_3)^2]$ instead of H. This quadratic form may be obtained by interpreting it as the map that takes Γ into α for the whole family of compatible frames, i.e. generated by a relativity group. Before undertaking this construction in the coming section, we shall first comment on some limiting cases. It is quite easy to figure out that the transformed frames will satisfy the mutual objective existence condition iff the time-axis and the space-axis of our fiducial starting reference frame R_0 intersect the orbit of the selected transformations in disjoint branches [48]. It is therefore clear that in the generic situation the mutual objective existence condition selects a Lorentz-type subgroup.

When $a = m$ and $b = -n$ we get a degenerate situation. In particular when the quadratic forms degenerate into pairs of lines, $b = n = 0$, we get the Galilei group as an allowed transformation group provided that $dx_0 \wedge \alpha = 0$, i.e. the level setes of $\mathbb{M}$ should coincide with the space-axes of R_0.

It is also intuitively clear the quadratic form we have identified to select the relativity group allowed by the mutual existence condition defines a symmetric $(2, 0)$-tensor which is invertible in the case of Lorentz-type group and is degenerate for the Galilei group. We shall show how these quadratic forms arise in general terms.

1.1.2 Symmetric tensors associated with equivalence classes of reference frames

We consider a decomposition of R_0 into (α_0, Γ_0). By using the selected relativity group identified by the mutual objective existence condition, we generate sets of couples $(\phi_g(\alpha), \phi_g(\Gamma))$.

If $\{\phi_g(\alpha)\}_{g\in G}$ and $\{\phi_g(\Gamma)\}_{g\in G}$ are a base of one-forms and vector fields respectively, we may define a symmetric covariant tensor field S by setting

$$S(\phi_g(\Gamma)) = \phi_g(\alpha)$$

or, equivalently, a symmetric contravariant tensor

$$\sigma(\phi_g(\alpha)) = \phi_g(\Gamma)$$

because of the assumption that we generate a base out of the chosen fiducial pairs (α, Γ).

When the generated vector fields or one-forms are not a base, say α is a fixed one-form under the selected group while $\phi_g(\Gamma)$ are a base of the vector fields, the defined symmetric covariant tensor would be

$$S(\phi_g(\Gamma)) = \phi_g(\alpha) = \alpha$$

and therefore S will be degenerate. This would correspond to the Galilei group. We should notice however that the "spatial part" of this group is the full general linear group in three dimensions $GL(3, \mathbb{R})$.

If $\{\phi_g(\alpha)\}_{g\in G}$ is a base but Γ is a fixed vector field $\phi_g(\Gamma) = \Gamma$, we may define a contra variant symmetric tensor by setting

$$\sigma(\phi_g(\alpha)) = \Gamma.$$

This situation would correspond to the so-called Carroll group. It should be noticed, however, that this group would violate the causality relation as usually understood.

Putting in formulae, we would get a quadratic form in a covariant expression as

$$g = dx_0 \otimes dx_0 - d\vec{x} \otimes d\vec{x} \tag{1.5}$$

or, in contravariant form,

$$g = \frac{\partial}{\partial x_0} \otimes \frac{\partial}{\partial x_0} - \frac{\partial}{\partial \vec{x}} \otimes \frac{\partial}{\partial \vec{x}}. \tag{1.6}$$

By replacing x_0 with ct, we would find in the limit $c \to \infty$ respectively $dt \otimes dt$ and $\frac{\partial}{\partial \vec{x}} \otimes \frac{\partial}{\partial \vec{x}}$. In the limit the two quadratic forms are not anymore inverse of one another. It follows that, while the Poincaré group may be identified either by means of the covariant or the covariant expression, the Galilei group is identified by requiring that it preserve both symmetric forms, the covariant $dt \otimes dt$ and the contravariant one, $\frac{\partial}{\partial \vec{x}} \otimes \frac{\partial}{\partial \vec{x}}$.

1.2 Relativity groups and related subgroups

In this section we describe in some detail the most relevant transformation groups of space-time. Following the general philosophy of the previous section, we start with the inhomogeneous and homogeneous Lorentz group, which have been previously introduced as the relativity groups which leave invariant the quadratic form equivalently represented in covariant and contravariant form by (1.5) and (1.6) respectively. We thus consider the degenerate case of the Galilei group and pass to the analysis of relevant subgroups.

1.2.1 The inhomogeneous Lorentz group

Let us consider $\mathbb{M} = \mathbb{R}^4$, with coordinates x^μ, $\mu = 0, 1, 2, 3$, $x^0 = ct$, x^i ($i = 1, 2, 3$) the usual Cartesian coordinates in $\mathbb{R}^3$ (Latin indices will be used for (pure) spatial coordinates, Greek ones for **space-time** coordinates).

The **Minkowski metric** g on $\mathbb{R}^4$ is the pseudo-Riemannian metric tensor of index 3 and signature -2 given by:

$$\|g_{\mu\nu}\| = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{vmatrix} \tag{1.7}$$

or, in intrinsic tensorial form

$$|dx_0, dx_1, dx_2, dx_3| \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{vmatrix} \begin{vmatrix} dx_0 \\ dx_1 \\ dx_2 \\ dx_3 \end{vmatrix} = (dx_0)^2 - (dx_1)^2 - (dx_2)^2 - (dx_3)^2 \tag{1.8}$$

The inverse of $\|g_{\mu\nu}\|$, $\|g^{\mu\nu}\|$ is usually introduced via:

$$g^{\mu\nu}g_{\nu\rho} = \delta^{\mu}_{\rho} = \begin{cases} 1 & \mu = \rho \\ 0 & \mu \neq \rho \end{cases} \tag{1.9}$$

and it turns out that:

$$g^{\mu\nu} = g_{\mu\nu} \quad \forall \mu, \nu \tag{1.10}$$

In a more intrinsic language, the inverse is associated with the contravariant form of the metric tensor

$$\frac{\partial}{\partial x^0} \otimes \frac{\partial}{\partial x^0} - \frac{\partial}{\partial \vec{x}} \otimes \frac{\partial}{\partial \vec{x}}. \tag{1.11}$$

It is then natural to call the $g_{\mu\nu}$'s the **covariant** components of the metric tensor, the $g^{\mu\nu}$'s the **contravariant** components, and the δ^{μ}_{ν}'s the **mixed ones**.

Definition. 1.2 *The inhomogeneous Lorentz group or* **Poincaré group** *is the group* $\mathfrak{P}$ *of* ***linear inhomogeneous*** *transformations which leave the Lorentz metric invariant.*

They are of of the form:

$$x \mapsto x', \quad x'^{\mu} = \Lambda^{\mu}_{\nu} x^{\nu} + a^{\mu} \tag{1.12}$$

with $a^{\mu} \in \mathbb{R}$, four real parameters identifying the subgroup of space-time translations, whereas $\Lambda \in \mathscr{L}$ is the subgroup of 4×4 real antisymmetric matrices which define the homogeneous Lorentz transformations.

Before describing in details these transformations, let us anticipate the following **Remarks**:

i) As $\mathscr{L}$ has four connected components, and the translations constitute by themselves a connected group, $\mathfrak{P}$ also has four connected components, connected to each other as in (1.48). We might therefore just study the connected component for which $\Lambda \in \mathscr{L}^{\uparrow}_{+}$.

ii) As it takes four parameters to specify a translation, six to specify a 4×4 real antisymmetric matrix, **the Poincaré group is a ten-parameter group**.

Denoting by (Λ, a) an element of the Poincaré group, i.e.:

$$(\Lambda, a) : \; x \mapsto x' = \Lambda x + a \tag{1.13}$$

the group composition law is easily worked out as:

$$(\Lambda, a)(\Lambda', a') = (\Lambda\Lambda', \Lambda a' + a) \tag{1.14}$$

This is an instance of the composition law for *semi-direct products* of groups, which will be described in detail in 1.2.7. From (1.14) we also obtain:

$$(\Lambda, a)^{-1} = (\Lambda^{-1}, -\Lambda^{-1} a) \tag{1.15}$$

and:

$$(\Lambda, a)(\Lambda', a')(\Lambda, a)^{-1} = (\Lambda\Lambda'\Lambda^{-1}, (\,Id - \Lambda\Lambda'\Lambda^{-1})a + \Lambda a') \tag{1.16}$$

For $\Lambda' = 1$, this proves that space-time translations constitute an invariant subgroup of $\mathfrak{P}$, and we can conclude that:

Proposition. 1.1 *The Poincaré group is the semi-direct product of the Lorentz group and the (Abelian) group of space-time translations.*

1.2.2 The homogeneous Lorentz group

The homogeneous component of the Poincaré group is the Lorentz group $\mathscr{L}$. This can be regarded as the subgroup of $GL(4,\mathbb{R})$of the 4×4 real non-singular matrices, constituted by elements which are isometries for the metric g:

$$\mathscr{L} \stackrel{\text{def}}{=} \left\{\Lambda \in GL(4,\mathbb{R}) \mid \Lambda^* g = g\right\} \tag{1.17}$$

That $\mathscr{L}$is a group is obvious from the definition. Explicitly, we may write:

$$\Lambda : \; x \mapsto x' \quad \text{by} \quad x'^{\mu} = \Lambda^{\mu}_{\nu} x^{\nu} \tag{1.18}$$

and the condition $\Lambda^* g = g$ or, equivalently:

$$g(\Lambda x, \Lambda y) = g(x, y) \quad \forall x, y \tag{1.19}$$

becomes:

$$g_{\mu\nu}\Lambda^{\mu}_{\eta}\Lambda^{\nu}_{\rho} = g_{\eta\rho} \tag{1.20}$$

or, in short:

$$\tilde{\Lambda} g \Lambda = g \tag{1.21}$$

The Λ^{μ}_{ν}'s are the "mixed" components of Λ. We may introduce covariant and contravariant components by raising and lowering indices with g i.e.:

$$\Lambda^{\mu\nu} \stackrel{\text{def}}{=} \Lambda^{\mu}_{\rho} g^{\rho\nu} \; ; \quad \Lambda_{\mu\nu} \stackrel{\text{def}}{=} g_{\mu\rho}\Lambda^{\rho}_{\nu}, \tag{1.22}$$

(in intrinsic language, we may associate bilinear forms with linear transformations) and equivalent forms of (1.21) are:

$$\Lambda_{\nu\eta}\Lambda^{\nu\varepsilon} = \delta^{\varepsilon}_{\eta}; \quad \Lambda_{\eta\nu}\Lambda_{\rho\mu}g^{\eta\rho} = g_{\mu\nu}; \quad \Lambda^{\eta\nu}g_{\eta\rho}\Lambda^{\rho\mu} = g^{\mu\nu} \tag{1.23}$$

The group composition law is easily seen to be represented by the ordinary matrix multiplication, i.e.:

$$(\Lambda\Lambda')^{\mu}_{\nu} = \Lambda^{\mu}_{\rho}\Lambda^{\rho}_{\nu} \quad \forall\Lambda, \Lambda' \in \mathscr{L}. \tag{1.24}$$

The bidifferential operator (1.11) may be made to act as a second order differential operator on scalar functions, to give

$$\frac{\partial^2 u}{\partial x_0^2} - \frac{\partial^2 u}{\partial x_1^2} - \frac{\partial^2 u}{\partial x_2^2} - \frac{\partial^2 u}{\partial x_3^2}.$$

This observation shows that the Lorentz group may be defined equivalently as preserving the Lorentz metric in covariant or contravariant form, and as the group which preserves the wave or Klein Gordon equation.

Defining the scalar product in $\mathbb{M}$ as:

$$(x, y) \stackrel{\text{def}}{=} g(x, y) \tag{1.25}$$

the sign of (x, y) will of course be invariant under Lorentz transformations, so the classification of points of $\mathbb{M}$ as:

i) **Space-like**, iff $(x, x) < 0$

ii) **Time-like**, iff $(x, x) > 0$

will have a Lorentz-invariant meaning. Also, the **light cone**:

$$\text{light-cone} \stackrel{\text{def}}{=} \{x \in \mathbb{R}^4 | (x, x) = 0\} \tag{1.26}$$

will be invariant under Lorentz transformations.

Remarks

i) On a general pseudo-Riemannnian manifold $(\mathbb{M}, g)$ we may also classify the tangent vectors at any point as "space-like", "time-like" and "belonging to the light cone", according to whether $g(X, X) \lessgtr 0$ or $g(X, X) = 0$ (X a tangent vector at any given point). The classification will be invariant w.r.t. the group of isometries of g; if ϕ is an isometry, phi_* will map bijectively space (time)-like vectors into space (time)-like ones, and light cones onto light cones.

ii) As g itself can be viewed as a bilinear form associated with a Lorentz transformation (in fact, it is just the identity) and as (1.21) can be rewritten as $\tilde{\Lambda} = g(g\Lambda)^{-1}$ it is clear that $\tilde{\Lambda}$ will be a Lorentz transformation as well. Hence:

$$\Lambda g \tilde{\Lambda} = g \tag{1.27}$$

will hold iff (1.21) does.

One easily infers from (1.21) that:

$$|\det \Lambda| = 1 \quad \forall\Lambda \in \mathscr{L} \tag{1.28}$$

Also, setting $\eta = \rho = 0$ in (1.20), we obtain:

$$1 = (\Lambda^0_0)^2 - \sum_{i=1}^{3} (\Lambda^i_0)^2 \tag{1.29}$$

whence:

$$|\Lambda^0_0| \geq 1 \quad \forall \Lambda \in \mathscr{L} \tag{1.30}$$

The **future cone** V_+ is the subset of $\mathbb{M}$ defined by:

$$V_+ \stackrel{\text{def}}{=} \{x \in \mathbb{R}^4 \mid (x, x) > 0,\ x^0 > 0\} \tag{1.31}$$

(the **past cone** V_- will be defined in a similar way). We then have:

Proposition. 1.2 *A Lorentz transformation Λ will have $\Lambda^0_0 \geq 1$ iff it maps the future cone on to itself.*

Indeed, $\forall x \in \mathbb{R}^4$, we have:

$$x'^0 = \Lambda^0_0 x^0 + \sum_{i=1}^{3} \Lambda^0_i x^i \tag{1.32}$$

But[1]:

$$|\sum_{i=1}^{3} \Lambda^0_i x^i|^2 \leq \sum_i (\Lambda^0_i)^2 \sum_j (x^j)^2 = \left[(\Lambda^0_0)^2 - 1\right] \sum_j (x^j)^2 \tag{1.33}$$

As:

$$(x, x) > 0 \Rightarrow \sum_j (x^j)^2 < (x^0)^2 \tag{1.34}$$

we obtain:

$$|\sum_{i=1}^{3} \Lambda^0_i x^i|^2 \leq \left[(\Lambda^0_0)^2 - 1\right] (x^0)^2 \leq (\Lambda^0_0 x^0)^2 \tag{1.35}$$

Hence, if $x^0 > 0$,

$$\text{Sgn } x'^0 = \text{Sgn } \Lambda^0_0 \tag{1.36}$$

Which proves the Proposition.□

As Prop. 1.2 expresses an invariance property, we have at once that:

[1]We are using here $(\Lambda^0_0)^2 - \sum_{i=1}^{3} (\Lambda^0_i)^2 = 1$, which can be deduced from (1.27) along the same lines as (1.29)

Proposition. 1.3 *The subset:*

$$\mathscr{L}^{\uparrow} = \{\Lambda \in \mathscr{L} \mid \Lambda^0_0 \geq 1\} \tag{1.37}$$

is a subgroup of the full Lorentz group.

$\mathscr{L}^{\uparrow}$ will be called **orthochronous Lorentz group**.

Remark: $\mathscr{L}^{\uparrow}$ will also map the **past cone** onto itself. The fact that the sign of the time coordinate (the "direction" of time) is not changed motivates the name "orthochronous".
When a particular splitting of space-time is selected (and only then) it is possible to introduce the (discrete) operation of "time-reversal" as follows:

Definition. 1.3 *The **time-reversal** is the (Lorentz) transformation:*

$$T: \quad x \mapsto Tx \quad \textit{by} \quad x^0 \mapsto -x^0\,; x^k \mapsto x^k \tag{1.38}$$

One sees at once that:

$$T^2 = 1 \tag{1.39}$$

Hence, $\{T, \mathbf{1}\}$ may be considered to be a subgroup of the full Lorentz group, after a splitting of space-time has been selected, but **not an invariant one**, as, in general, $T\Lambda \neq \Lambda T$. As, however:

$$(T\Lambda)^0_0 = (\Lambda T)^0_0 = -\Lambda^0_0 \tag{1.40}$$

multiplication by T changes orthochronous transformations into **anti**-orthochronous ones. If Λ is an anti-orthochronous transformation, then $T\Lambda \in \mathscr{L}^{\uparrow}$, and Λ can be written uniquely as:

$$\Lambda = T(T\Lambda) \tag{1.41}$$

Hence, we can reconstruct $\mathscr{L}$ as:

$$\begin{gathered} \mathscr{L} = \mathscr{L}^{\uparrow} \cup \mathscr{L}^{\downarrow} \\ \mathscr{L}^{\downarrow} = \{\Lambda \in \mathscr{L} \mid \Lambda^0_0 \leq -1\} \end{gathered} \tag{1.42}$$

with T mapping bijectively $\mathscr{L}^{\uparrow}$ onto $\mathscr{L}^{\downarrow}$

$$\tag{1.43}$$

$\mathscr{L}^{\uparrow}$ (and $\mathscr{L}^{\downarrow}$) can be further analyzed according to the sign of $\det \Lambda$. The subset with $\det \Lambda = 1$ is (of course) a group itself:

Definition. 1.4

$$\mathscr{L}^{\uparrow}_{+} \stackrel{\text{def}}{=} \{\Lambda \in \mathscr{L} \mid \Lambda^0_0 \geq 1,\ \det \Lambda = 1\} \tag{1.44}$$

*Is the **proper orthochronous Lorentz group** (or the **restricted** Lorentz group).*

We next introduce the **space inversion** (or **parity**) as follows. Once a given splitting of space-time has been selected we have:

Definition. 1.5 *The* ***parity*** *is the (Lorentz) transformation:*

$$P:\ x \mapsto Px; \quad (Px)^0 = x^0, \quad (Px)^k = -x^k \tag{1.45}$$

We have:

$$P^2 = \ Id \tag{1.46}$$

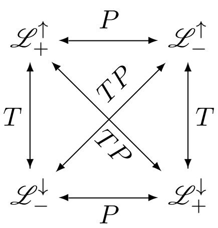
but, in contrast with other cases, as for example the group of rotations, $SO(3)$, $P\Lambda \neq \Lambda P$ in general, so $\{P,\ Id\}$ is not an invariant subgroup. By defining, besides $\mathscr{L}_+^\uparrow$, the subsets:

$$\mathscr{L}_+^\downarrow = \{\Lambda \in \mathscr{L}^\downarrow |\ \det\Lambda = 1\} \tag{1.47a}$$

$$\mathscr{L}_-^\uparrow = \{\Lambda \in \mathscr{L}^\uparrow |\ \det\Lambda = -1\} \tag{1.47b}$$

$$\mathscr{L}_-^\downarrow = \{\Lambda \in \mathscr{L}^\downarrow |\ \det\Lambda = -1\} \tag{1.47c}$$

$\mathscr{L}$ will be the (disjoint) union of $\mathscr{L}_+^\uparrow$ and the three subsets above. The four will be bijectively connected by P, T or PT (=TP) in the manner indicated in the following diagram:



(1.48)

We know already that $\mathscr{L}^\uparrow = \mathscr{L}_+^\uparrow \cup \mathscr{L}_-^\uparrow$. Other subgroups are:

$$\mathscr{L}_+ = \mathscr{L}_+^\uparrow \cup \mathscr{L}_+^\downarrow \quad \text{the } \textbf{proper} \text{ Lorentz group} \tag{1.49}$$

$$\mathscr{L}_0 = \mathscr{L}_+^\uparrow \cup \mathscr{L}_-^\downarrow \quad \text{the } \textbf{orthochronous} \text{ Lorentz group} \tag{1.50}$$

It should be noticed the particular way discrete transformations have been introduced, we have been obliged to introduce preliminarily a splitting of space-time, differently form the elements of the Poincaré group, which are defined independently of any splitting. The reason is that $\mathbf{1}, P, T, PT,$ are a quotient group of the Poincaré group by the connected component to the identity. The immersion of the quotient group as a subgroup requires a "gauge", i.e., a splitting of space-time.

From now on, we may restrict ourselves to the study of the restricted group $\mathscr{L}_+^\uparrow$. There are two important subgroups of $\mathscr{L}_+^\uparrow$ namely:

i) The subgroup $\mathfrak{R}$ of pure **space rotations**:

$$\mathfrak{R} = \{\Lambda \in \mathscr{L} |\ \Lambda_0^0 = 1,\ \Lambda_i^0 = \Lambda_0^i = 0\} \tag{1.51}$$

It is left as an exercise to check that $\mathfrak{R}$ is indeed a group. As:

$$(\Lambda x)^0 = x^0 \quad \forall \Lambda \in \mathfrak{R} \tag{1.52}$$

$\mathfrak{R}$ does indeed contain only (**proper**, because of $\det \Lambda = 1$) space rotations, i.e. $\mathfrak{R} \sim SO(3)$.

ii) The subgroup of **Lorentz boosts along a fixed direction**. This is the subset of all Lorentz transformations connecting two reference frames, S, S', whose space axes remain parallel while S' moves at constant speed v relative to S along a fixed direction. Without loss of generality, the direction of relative motion can be taken as one of the coordinate axes, say the x^1-axis, and we will denote the above subset as $\mathscr{L}_{x^1}$. The familiar transformation laws are then:

$$\begin{aligned} x'^1 &= \frac{x^1 - \beta x^0}{\sqrt{1-\beta^2}} \\ x'^2 &= x^2 \\ x'^3 &= x^3 \\ x'^0 &= \frac{x^0 - \beta x^1}{\sqrt{1-\beta^2}} \\ \beta &= \frac{v}{c} \end{aligned} \tag{1.53}$$

Introducing the "**rapidity**" ξ, defined through:

$$\tanh \xi \stackrel{\text{def}}{=} \beta \tag{1.54}$$

(1.53) can be recasted in the form:

$$x' = \Lambda(\xi) x \qquad \begin{vmatrix} \cosh \xi & -\sinh \xi & 0 & 0 \\ -\sinh \xi & \cosh \xi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} \tag{1.55}$$

It is easily proved that:

$$\Lambda(\xi)\Lambda(\xi') = \Lambda(\xi'') \tag{1.56}$$

with

$$\xi'' = \xi + \xi' \tag{1.57}$$

hence, $\mathscr{L}_{x^1}$ is actually a **one-parameter subgroup** of the Lorentz group. Parameterizing it with the "rapidity" ξ instead of the (in units of c) velocity β exhibits explicitly the one-parameter structure. Also, the composition law (1.57) corresponds to the (more familiar) **composition law for velocities**, i.e.:

$$\beta'' = \frac{\beta + \beta'}{1 + \beta\beta'}. \tag{1.58}$$

The basic theorem concerning $\mathscr{L}_+^\uparrow$ is the following, which we give without proof (for a proof see, e.g., [154], vol. II, §19.4):

Proposition. 1.4 *Any Lorentz transformation $\Lambda \in \mathscr{L}_+^\uparrow$ can be expressed in the form:*

$$\Lambda = R_1 \Lambda(\xi) R_2 \tag{1.59}$$

where $R_1, R_2 \in \mathfrak{R}$ and $\Lambda(\xi) \in \mathscr{L}_{x^1}$.

Hence, $\mathscr{L}_+^\uparrow$ may be thought of as being "generated" by the two subgroups $\mathfrak{R}$ and $\mathscr{L}_{x^1}$. As $SO(3)$ (and hence $\mathfrak{R}$) is connected, and $\mathscr{L}_{x^1}$ is also (trivially) connected, Prop. 1.4 has the immediate corollary that:

Proposition. 1.5 *The restricted Lorentz group $\mathscr{L}_+^\uparrow$ is **connected**.*

With reference to (1.48), we may then conclude that **the (full) Lorentz group has exactly four connected components**, one of them containing the identity and therefore a subgroup.

A general boost (not in the direction of the x^1-axis) can be obtained by first rotating the coordinate axes in such a way that the (rotated) x^1-axis coincides with the direction of the boost, by performing the boost at the required speed (or rapidity), and then by rotating the axes back to the original position. Hence, any boost can be represented as:

$$\Lambda_B = R^{-1} \Lambda(\xi) R \tag{1.60}$$

for some (not unique[2]) $R \in \mathfrak{R}$, and for the given rapidity ξ. Two boosts in the **same** direction can be made to correspond to the **same** R, hence $\Lambda_B \cdot \Lambda'_B$ will again be a boost if Λ_B and Λ'_B are parallel boosts.

This will not however be possible for boosts in **different** directions. Therefore, while (as expected) boosts along any fixed direction are a (one-parameter) subgroup, the subset of **all** the Lorentz boosts is **not** a subgroup[3]. Actually, it can be shown that, in general, the composition of two boosts is equivalent to a boost plus a rotation, so one reaches the (pseudo) paradoxical conclusion that **the composition of three boosts can result in a pure rotation**. Physically, this is at the origin of the so-called "Thomas precession", a relativistic phenomenon known in atomic physics since 1927 (for a discussion of the Thomas precession in classical terms, see, e.g., §11.8 of [92]).

[2]There will be residual freedom of adding arbitrary rotations around the boost direction.
[3]While, we shall see that in the case of Galilei group, the boosts do form a subgroup

1.2.3 The group $T(n)$ of translations in R^n

Space-time translations, T_a, introduced in §1.2.1 are an Abelian subgroup of the Poincaré group, preserving the quadratic form $(\mathrm{d}x^0)^2 - (\mathrm{d}\vec{x})^2$ and acting on space-time coordinates as

$$x^\mu \mapsto x^\mu + a^\mu = T_a(x). \tag{1.61}$$

This definition ca be easily generalized to n dimensions. Choosing for definiteness the Euclidean signature for the metric and unit vectors $\vec{e_i}$, $i = 1 \dots n$ along the coordinate axes of a Cartesian reference frame, every translation is uniquely represented by the transformation:

$$\mathbb{R}^n \ni \vec{r} \mapsto \vec{r} + \vec{a} = T_{\vec{a}}(\vec{r}), \qquad \vec{a} = a^i \vec{e_i} \in \mathbb{R}^n \tag{1.62}$$

It follows then easily that $T(n)$ is an Abelian group. It is also a manifold, with the a^i's as local coordinates (of a one-chart atlas, i.e. $T(n) \approx \mathbb{R}^n$ as a manifold). If we consider the one-parameter groups of translations along the coordinate axes, then any other translation can be uniquely represented (see (1.62)) as the composition of n such translations. The corresponding infinitesimal generators, $X_{(i)}$ are nothing but the natural base in $\mathbb{R}^n$, i.e.:

$$X_{(i)} = \frac{\partial}{\partial x^i}, \quad i = 1, \dots, n \tag{1.63}$$

In intrinsic terms, the infinitesimal generators of the translation group are solutions of the equation $L_{X_{(i)}} \sum_j dx^j \otimes dx^j = 0$, where

$$g = \sum_j dx^j \otimes dx^j \tag{1.64}$$

is the standard Euclidean metric in covariant form and $L_{X_{(i)}}$ is the Lie derivative with respect to the vector field X_i (see Chapter 24 for a definition). Therefore translations preserve the quadratic Euclidean form. The definition easily generalizes to a metric with Lorentzian signature.

1.2.4 The groups $SO(3)$ and $SU(2)$.

The group $SO(3$ has already emerged as the subgroup of spatial rotations of the Lorentz group. We shall see that $SU(2)$ is the **universal covering** of $SO(3)$.

Let $GL(n, \mathbb{K})$, $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ denote the group of $n \times n$ nonsingular matrices with $\mathbb{K}$-valued entries, and, for $A \in \mathbb{G}L(n, \mathbb{K})$, $\tilde{A}$ denote the transpose of A, $A^\dagger$ the adjoint of A (both defined with respect to the standard scalar product in $\mathbb{R}^n$, or $\mathbb{C}^n$). Then:

Definition. 1.6 *The groups $SO(3)$ and $SU(2)$ are defined by:*

$$SO(3) = \left\{ R \in GL(3, \mathbb{R}) \mid R\tilde{R} = Id, \det R = 1 \right\} \tag{1.65}$$

and:

$$SU(2) = \left\{ U \in GL(2, \mathbb{C}) \mid UU^\dagger = Id, \det U = 1 \right\} \tag{1.66}$$

We remark that $SO(n)$ is, for any n, the connected component of $O(n)$ containing the identity, and is a **submanifold** of $\mathbb{R}^{n^2}$ of dimension $\frac{n(n-1)}{2}$ defined by the constraint $R\tilde{R} = Id.$. Hence, $SO(3)$ can be viewed as a three-dimensional submanifold of $\mathbb{R}^9$. In the same way, $U(n)$, the group of complex, unitary $n \times n$ matrices, can be viewed as a submanifold of $\mathbb{C}^{n^2} \sim \mathbb{R}^{2n^2}$, of complex dimension n^2. As $UU^* = 1$ only implies $|\det U| = 1$, the additional condition does not simply select one component ($U(n)$ is actually connected), but it is an extra condition, and $SU(n)$ is a connected submanifold of $\mathbb{R}^{2n^2}$ of dimension $n^2 - 1$. Hence, both $SO(3)$ and $SU(2)$, as manifolds, are three-dimensional. In other words, every element of $SO(3)$ ($SU(2)$) can be uniquely specified by three independent parameters. It will be seen later that the group manifolds of both groups are more conveniently viewed as (three-dimensional) manifolds embedded in $\mathbb{R}^4$.

Let us start by studying $SO(3)$. Given the standard Euclidean metric on $\mathbb{R}^3$, Eq. (1.64) and the associated (volume-form and) orientation, $SO(3)$ is the group of orientation-preserving linear maps which leave (1.64) unchanged. If we consider the homogeneous first order differential operator $X_A = x^j A_j^k \frac{\partial}{\partial x^k}$, the invariance requirement amounts to $L_{X_A} \sum_j (dx^j)^2 = 0$.

The classical Euler's theorem, stating that every rotation leaves one direction in space unchanged, is a consequence of the following:

Proposition. 1.6 *If $A \in GL(3, \mathbb{R})$, $A\tilde{A} = Id$ and $\lambda_1, \lambda_2, \lambda_3$ are its eigenvalues, then, besides $\lambda_1 \cdot \lambda_2 \cdot \lambda_3 = \det A$, λ_j^{-1} $(j = 1, 2, 3)$ and $\bar{\lambda}_j$ are eigenvalues iff λ_j is, and $|\lambda_j| = 1$.*

The proof is easy and is left as an exercise.

For $\det A = 1$, one of the eigenvalues must be equal to one, say, λ_3, and $\lambda_1 = \bar{\lambda}_2 = \exp(i\varphi)$ for some $\varphi \in \mathbb{R}$. The eigenvector corresponding to $\lambda_3 = 1$ will be a real vector (hence in $\mathbb{R}^3$), and will be left unchanged (Euler's theorem).

Hence, to every pair $(\vec{n}, \Theta)$, $||\vec{n}|| = 1$ and Θ an angle (with standard convention $\Theta > 0$ (< 0) for anticlockwise (clockwise) rotations around $\vec{n}$) there will correspond a well-defined element of $SO(3)$, $R(\vec{n}, \Theta)$. There are two ambiguities in this representation:

i) $\Theta = 0$ will give the identity, $\forall \vec{n}$;

ii) $(\vec{n}, \Theta)$, $(-\vec{n}, -\Theta)$ will correspond to the same rotation.

They may be partly avoided by lumping together $\vec{n}$ and Θ in the single three-dimensional vector and restricting Θ to the interval $[-\pi, \pi]$:

$$\vec{\Theta} \stackrel{\text{def}}{=} \vec{n}\Theta \tag{1.67}$$

Again, however, two opposite vectors $\vec{\Theta}$, $\vec{\Theta}'$, with $||\vec{\Theta}|| = ||\vec{\Theta}'|| = \pi$ will represent the same rotation. We may therefore put elements of $SO(3)$ in a one-to-one correspondence with points of the set:

$$K \stackrel{\text{def}}{=} \left\{ \vec{\Theta} \in \mathbb{R}^3 \mid ||\vec{\Theta}|| \leq \pi \text{ ; antipodal points on } ||\vec{\Theta}|| = \pi \text{ identified} \right\} \tag{1.68}$$

This is a useful representation of $SO(3)$ for computational purposes. However, K is rather awkward as a manifold, so we will introduce later another coordinatization for $SO(3)$ in order to obtain a simpler description of its group manifold.

Remark: the classical coordinatization of the matrices of $SO(3)$ is in terms of the **Euler angles**. For our (limited) purposes, however, they will not be needed, so we will not introduce them. For a discussion of $SO(3)$ in terms of Euler's angles, see, e.g. [72], Chap. 4, or [172] Chap. 15. For a discussion of the group manifold in terms of Euler's angles, see [38] Problem II.5.

Let's fix now $\vec{n}$, and consider $R(\vec{n}, t)$, $t \in \mathbb{R}$, by setting:

$$R(\vec{n}, t) \stackrel{\text{def}}{=} R(\vec{n}, \Theta), \quad \Theta = t \mod [-\pi, \pi] \tag{1.69}$$

As t varies, we may visualize the representative point in K as sweeping a diameter, jumping to the antipodal point as soon as t reaches π (or $-\pi$), and then repeating the process again and again. Because of the identification of antipodal points, $R(\vec{n}, t)$ will generate a continuous (actually, a smooth) curve in K. Also, as rotations around the same axis combine by addition of the corresponding angles:

$$R(\vec{n}, t) \cdot R(\vec{n}, t') = R(\vec{n}, t + t') \tag{1.70}$$

$\{R(\vec{n}, t)\}_{t \in \mathbb{R}}$ will be a one-parameter group of diffeomorphisms of $\mathbb{R}^3$, and, at the same time, a one-parameter subgroup of $SO(3)$. By defining $SO(2)$ along the same line as $SO(3)$, it is easily seen that any such one-parameter subgroup will be isomorphic to $SO(2)$. It is also clear that every element $R \in SO(3)$, $R \neq Id$, will belong to just one such subgroup.

Let us study then in detail the subgroup $\{R(\vec{n}, t)\}$. If $\vec{x} \equiv (x^1, x^2, x^3) \in \mathbb{R}^3$, the infinitesimal generator will be the vector field:

$$X \in \mathfrak{X}(\mathbb{R}^3), \quad X(\vec{x}) \stackrel{\text{def}}{=} \frac{\mathrm{d}}{\mathrm{d}\,t} R(\vec{n}, t) \cdot \vec{x}\Big|_{t=0} \tag{1.71}$$

Then, with:

$$A(\vec{n}) \stackrel{\text{def}}{=} \frac{\mathrm{d}}{\mathrm{d}\,t} R(\vec{n}, t)\Big|_{t=0} \tag{1.72}$$

the orthogonality condition on R implies **skew-symmetry** of A:

$$A(\vec{n}) + \tilde{A}(\vec{n}) = 0 \tag{1.73}$$

It is known (see, e.g.: [9], Chapter 3) that on any (finite-dimensional) linear vector space, a first-order ordinary differential equation of the form:

$$\frac{\mathrm{d}}{\mathrm{d}\,t} \vec{x}(t) = A \cdot \vec{x}(t), \quad \vec{x}(0) = \vec{x} \tag{1.74}$$

(A a matrix independent of $\vec{x}$) integrates to:

$$\vec{x}(t) = \exp[At] \cdot \vec{x} \tag{1.75}$$

Also:

$$\widetilde{(\exp A)} = \exp \tilde{A}, \quad (\exp A)\widetilde{(\exp A)} = \exp(A + \tilde{A}) = \mathbf{1} \tag{1.76}$$

where we have used $\frac{d}{dt}(R\tilde{R}) = 0$ to derive $A + \tilde{A} = 0$. We then conclude that:

Proposition. 1.7 *To any one-parameter subgroup $R(\vec{n}, t)$ of SO(3) there is associated a skew-symmetric matrix $A(\vec{n})$ s.t.*

$$R(\vec{n}, t) = \exp\Big[tA(\vec{n})\Big] \tag{1.77}$$

Viceversa, any skew-symmetric matrix A will generate a one-parameter subgroup via Eq. (1.77).

Remark: A being skew-symmetric on an odd-dimensional space, it is necessarily degenerate: $\det A = 0$, implying that one of its eigenvalues vanish. the corresponding eigenvector will be left invariant by $\exp[tA]$ and it will single out the axis of rotation, i.e. $\vec{n}$.

Infinitesimal rotations around a fixed axis will correspond therefore to vector fields of the form:

$$X(\vec{x}) = A^i_k q^k \frac{\partial}{\partial\, q^i} \,, \quad A^i_k + A^k_i = 0 \tag{1.78}$$

In particular, infinitesimal rotations around the coordinate axes will be generated by:

$$A_1 = \begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{vmatrix} \quad A_2 = \begin{vmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{vmatrix} \quad A_3 = \begin{vmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix} \tag{1.79}$$

respectively.

Exercise. Prove (1.79). Prove also that A_i's form a base for the space of 3x3 skew-symmetric matrices, and that:

$$A(\vec{n}) = \vec{n} \cdot \vec{A} \,, \quad \vec{A} \equiv (A_1, A_2, A_3). \tag{1.80}$$

Starting with any vector in $|R^3$, the action of $SO(3)$ on it will generate a two-dimensional manifold characterized by $\vec{r} \cdot \vec{r} = r^2$, i.e. a sphere S^2 which is not linear anymore. However, as, in $\mathbb{R}^3$, $SO(3)$ maps every sphere centered at the origin onto itself, there will be a well defined **action** of $SO(3)$ on S^2, i.e., in particular, to every one-parameter subgroup of $SO(3)$ there will correspond a one-parameter group of diffeomorphisms of S^2.

We now turn to $SU(2)$. Every $U \in SU(2)$, being a unitary matrix, can be represented in the form:

$$U = \exp(iH) \tag{1.81}$$

with H hermitian:

$$H = H^\dagger \tag{1.82}$$

Remark: Unitary matrices can be always represented in the form $A = \exp B$ for some B (the "logarithm" of A). Indeed, unitary matrices can be always brought into diagonal form; the proof relies thus on the eigenvalues being complex numbers of unit modulus. The details of the proof are left as an exercise, as well the proof that:

$$\det U = \exp(i \operatorname{Tr} H) \tag{1.83}$$

$\det U = 1$ will imply $\operatorname{Tr} H = 2k\pi,\ k \in \mathbb{Z}$. By subtracting an appropriate multiple of the identity, we can, without loss of generality, set $\operatorname{Tr} H = 0$.

It is well known that the unit 2x2 matrix σ_0 and the three Pauli matrices $\sigma_j,\ j = 1, 2, 3$, subject to:

$$\sigma^\dagger_= \sigma_j \ , \quad \sigma_j \sigma_k = i\epsilon_{jkl}\sigma_l + Id\delta_{jk} \tag{1.84}$$

yield a base in the linear vector space of (complex) 2x2 matrices.

We conclude that:

Proposition. 1.8 *Every $U \in SU(2)$ can be uniquely represented as:*

$$U = \exp(iH) \tag{1.85}$$

*with H **Hermitian and traceless**. H is **uniquely** given in the form:*

$$H = \vec{x} \cdot \vec{\sigma} \ , \quad \vec{x} \in \mathbb{R}^3 \tag{1.86}$$

We may always choose a unit vector $\vec{n}$ and an angle θ in such a way that: $\vec{x} = -\frac{\theta}{2}\vec{n}$. Hence, we have the representation:

$$U = \exp\left[-i\frac{\theta}{2}\vec{n} \cdot \vec{\sigma}\right] \tag{1.87}$$

for the elements of $SU(2)$. By expanding the exponential, we also have:

$$U = \exp\left[-i\frac{\theta}{2}\vec{n} \cdot \vec{\sigma}\right] = \sigma_0 \cos\left(\frac{\theta}{2}\right) - i(\vec{n} \cdot \vec{\sigma}) \sin\left(\frac{\theta}{2}\right) \tag{1.88}$$

Remark: The representation (1.88) is subject to much the same ambiguities as those discussed for the similar representation of $SO(3)$ in terms of $\vec{n}$ and θ. They can be treated the same way introducing the vector $\vec{\theta} = \vec{n}\theta \in \mathbb{R}^3$ and, because of the factor $\frac{1}{2}$ in (1.88), by restricting it to the spherical ball $\|\theta\| \leq 2\pi$. However, in this case, $U = -\, Id$ whenever $\|\theta\| = 2\pi$, so **all** points on the surface of the ball have to be identified.

Relationship between $SU(2)$ and $SO(3)$

Setting, for any $\vec{x} \in \mathbb{R}^3$

$$H(\vec{x}) = \vec{x} \cdot \vec{\sigma} \tag{1.89}$$

we have at once:

$$\det H(\vec{x}) = -\|\vec{x}\|^2 \tag{1.90}$$

For $U \in SU(2)$, $UH(\vec{x})U^*$ will be again hermitian and traceless, i.e.:

$$UH(\vec{x})U^\dagger = H(\vec{x}') \tag{1.91}$$

for a uniquely determined $\vec{x}' \in \mathbb{R}^3$ linear in $\vec{x}$. Moreover, $\det(UHU^*) = \det H$ implies: $\|\vec{x}'\| = \|\vec{x}\|$. Hence, there is a unique orthogonal transformation $R:\ \mathbb{R}^3 \to \mathbb{R}^3$ s.t.:

$$\vec{x}' = R \cdot \vec{x}\ , \quad R\tilde{R} = \ Id \quad \text{i.e. } \mathbb{R} \in O(3) \tag{1.92}$$

As U can be varied with continuity to meet the identity in $SU(2)$, R must belong to the component of $O(3)$ containing the identity, i.e. $R \in SO(3)$. Let now $U,\ U' \in SU(2)$ determine the same $R \in SO(3)$.

With:

$$\gamma \overset{\text{def}}{=} U'^*U \tag{1.93}$$

we obtain:

$$\gamma H(\vec{x}) = H(\vec{x})\gamma \quad \forall \vec{x} \tag{1.94}$$

In particular, γ will commute with the Pauli matrices, and hence, by Schur's lemma, it will be a multiple of the identity. As $\gamma \in SU(2)$, and hence $\det \gamma = 1$, it follows that $\gamma = \pm\, Id$, i.e., $U' = \pm U$. Hence:

Proposition. 1.9 *There is a (two-to-one) homomorphism:*

$$\pi \ :\quad SU(2) \to SO(3) \tag{1.95}$$

by which every pair $(U, -U)$ of elements of $SU(2)$ uniquely defines an element of $SO(3)$, In other words:

$$Ker\,\pi = \{-\,Id,\ Id\} \tag{1.96}$$

with Id the identity element in $SU(2)$.

Exercise. Using (1.88) and (1.91), show that:

$$\vec{x}' = \vec{x}\cos\theta + \vec{n}(\vec{x}\cdot\vec{n})(\ Id - \cos\theta) + (\vec{n}\times\vec{x})\sin\theta \tag{1.97}$$

and hence that $(\vec{n}, \theta)$ in (1.88) correspond to the parameters in $SO(3)$. Work out explicitly the form of the rotation matrix in terms of θ and $\vec{n}$ (the directions cosines of the rotation axis).

Group manifolds of $SO(3)$ and $SU(2)$

In order to see the structure of the group manifolds, we introduce another coordinatization for $SU(2)$. Observe that any matrix of the form (1.88) may be written as:

$$U = \begin{vmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{vmatrix} \tag{1.98}$$

Explicitly:

$$\alpha = \cos\left(\frac{\theta}{2}\right) - in_3 \sin\left(\frac{\theta}{2}\right) \quad ; \quad \beta = -\left(n_2 + in_1\right) \sin\left(\frac{\theta}{2}\right) \tag{1.99}$$

and $\|\vec{n}\| = 1$ implies:

$$|\alpha|^2 + |\beta|^2 = 1 \tag{1.100}$$

Viceversa, whenever two complex numbers α, β are given which satisfy (1.100), the corresponding U, (1.98), will belong to $SU(2)$. The complex numbers satisfying (1.100) give therefore a (one-to-one) coordinatization of $SU(2)$. (1.100) can be seen as **the equation defining the unit sphere in** $\mathbb{R}^4$. In view of Prop. 26.4, the same coordinatization will hold for $SO(3)$, provided (α, β) and $(-\alpha, -\beta)$, i.e. antipodal points on the unit sphere, are identified. Therefore:

Proposition. 1.10 *The group manifold of $SU(2)$ may be taken as the* ***unit sphere*** *S^3 (immersed in $\mathbb{R}^4$). The group manifold of $SO(3)$ may be taken as S^3 with identification of the antipodal points, i.e.* ***the projective space*** *$\mathbb{P}^3$.*

Remarks: (i) As already noted, the group manifolds of both groups are rather awkward-looking when considered as subsets (submanifolds of) $\mathbb{R}^3$. Instead, by suitably enlarging (from $\mathbb{R}^3$ to $\mathbb{R}^4$) the space in which they are immersed makes them look definitely more manageable.

(ii) If the representation (1.98) of the elements of $SU(2)$ is inserted into (1.91) and one works out, along the same lines as in the above Exercise, the form of the rotation matrix, one finds a representation of $R(\vec{n}, \theta)$ in terms of the so-called **Cayley-Klein parameters** (Goldstein, cit.).

(iii) S^3, both without and with identification of the antipodal points, is a compact, connected manifold. In the second case, however, **it is not simply connected**. We recall that a (arcwise connected) topological space is **simply connected** iff every loop based at any of its points (i.e. a closed, continuous curve starting and ending at the given point) can be continuously shrunk to the point itself (i.e. it is **homotopic** to the trivial loop consisting of single point). Equivalently, any two curves connecting any two points can be continuously deformed into (or are **homotopic** to) each other keeping the end points fixed.

That $\mathbb{P}^3$ is **not** simply connected can be seen by using (only for the sake of drawing figures) its representation (1.64). the half-meridian from N to S (Fig. 26.4) **is** a loop because of the

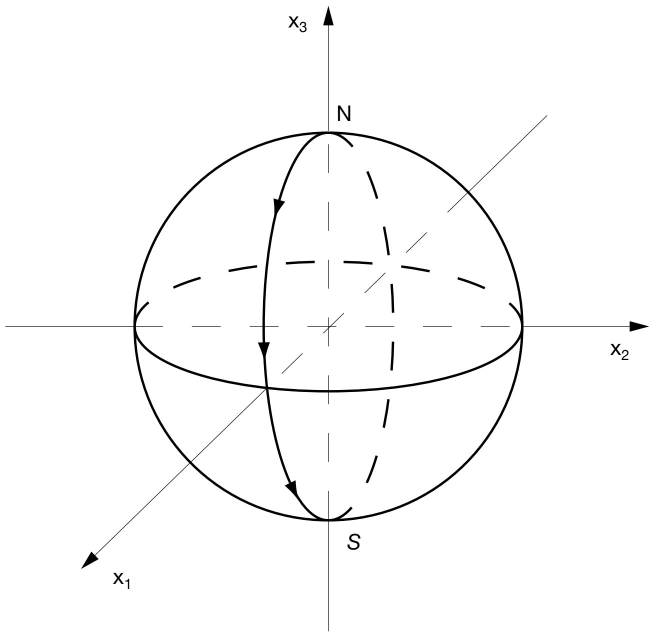


Figure 1.1:

identification $N \sim S$, but cannot obviously be shrunk to a point. However, the "double loop" of Fig. 26.5 can be homotopically shrunk to a point (for example, by "disentangling" it from N and then shrinking the corresponding curve).

We will express this fact by saying that $SO(3)$ is **doubly connected**. Whenever there is a homomorphism between a (topological) simply-connected group and a nonsimply-connected one, we say that the former is a **universal covering group** for the latter. The universal covering group can be shown to be unique up to isomorphisms. Hence: $SU(2)$ **is the universal covering group of** $SO(3)$ (and it gives a double covering, corresponding to the double-connectedness of $SO(3)$).

These are only elementary remarks. More precise considerations of homotopy theory in connection with (Lie) groups and manifolds will be taken up later on.

1.2.5 Relationship between $\mathscr{L}_+^\uparrow$ and $SL(2,\mathbb{C})$. Group manifolds.

Just as $SO(3)$ has a close relationship with $SU(2)$, so the restricted Lorentz group has a similar relationship with the group $SL(2,\mathbb{C})$:

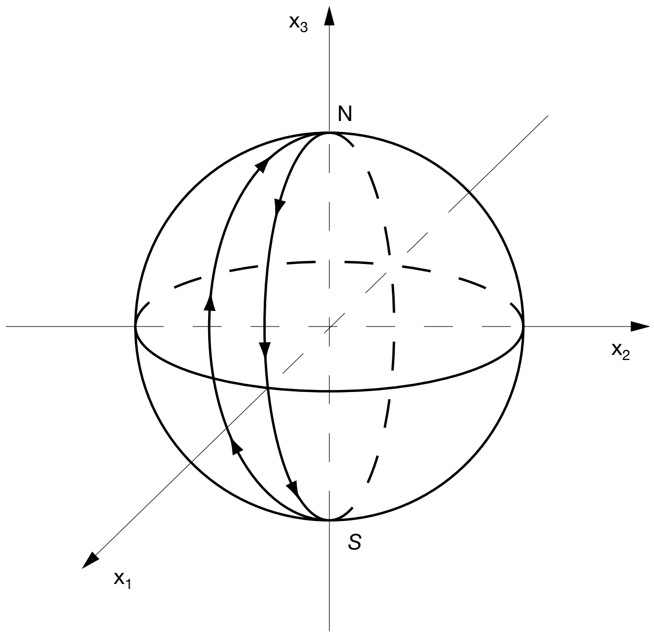


Figure 1.2:

Definition. 1.7 *$SL(2,\mathbb{C})$ is the subgroup of $GL(2,\mathbb{C})$ of unimodular matrices:*

$$SL(2,\mathbb{C}) = \{A \in GL(2,\mathbb{C}) |\ \det A = 1\} \tag{1.101}$$

Setting:

$$\begin{aligned} \tau_0 &= \sigma_0 \qquad \text{(identity 2x2 matrix)} \\ \tau_k &= \sigma_k \qquad \text{the Pauli matrices} \end{aligned} \tag{1.102}$$

to every $x \in \mathbb{R}^4$ we can associate the **hermitian matrix**:

$$\Xi = x^\mu \tau_\mu \tag{1.103}$$

viceversa, any hermitian 2x2 matrix will be of the form (1.103), and will determine x^μ via:

$$x^\mu = \frac{1}{2}\text{Tr}\ (\tau_\mu \Xi) \tag{1.104}$$

Moreover:

$$\det \Xi = (x^0)^2 - \sum_{j=1}^{3} (x^j)^2 = x^\mu x_\mu \tag{1.105}$$

If now $A \in SL(2, \mathbb{C})$, then:

$$\Xi' \stackrel{\text{def}}{=} A \Xi A^\dagger \tag{1.106}$$

will again be a Hermitian matrix, and $\det \Xi' = \det \Xi$. Hence, to every $A \in SL(2, \mathbb{C})$ there will correspond a unique Lorentz transformation Λ. Writing $\Xi(x)$ for the matrix (1.103), we will then have:

$$\Xi' = \Xi(\Lambda x) \tag{1.107}$$

If now A_1 determines Λ_1 and A_2 determines Λ_2, it is clear from (1.106,1.107) that $A_1 \cdot A_2$ will determine $\Lambda_1 \Lambda_2$, i.e. the correspondence $A \to \Lambda$ defined through (1.106) is a homomorphism. Some simple algebra shows that:

$$A \tau_\rho A^\dagger = \tau_\mu \Lambda^\mu_\rho \tag{1.108}$$

which gives a direct way for the identification of the matrix Λ representing the Lorentz transformation associated to Λ[4]. We remark that $SL(2, \mathbb{C})$ is a **connected** (actually simply-connected) group. Hence, A, in (1.106) can be varied with continuity to meet the identity in $SL(2, \mathbb{C})$. It follows that Λ must belong to the connected component of $\mathscr{L}$containing the identity, i.e. $\mathscr{L}^\uparrow_+$. We will give without proof the final part of the following:

Proposition. 1.11 *There is a homomorphism*

$$\pi : \; SL(2, \mathbb{C}) \to \mathscr{L}^\uparrow_+ \tag{1.109}$$

defined by Eqs. (1.107),(1.108). The homomorphism is ***two-to-one****, with kernel given by* $(\, Id, -\, Id)$ *(*Id *the identity in* $SL(2, \mathbb{C})$*), and* ***onto****.*

$SL(2, \mathbb{C})$, as a manifold, can be viewed as a six-dimensional submanifold of $\mathbb{R}^8 (\sim \mathbb{C}^4)$ defined by the two real equations corresponding to the (complex) equation $\det A = \; Id$. $\mathscr{L}^\uparrow_+$ can be viewed as the same manifold provided every point $A \in SL(2, \mathbb{C})$ is identified with its opposite, $-A$. As the defining equations do not single out a bounded subset of $\mathbb{R}^3$, **neither** $SL(2, \mathbb{C})$ **nor** $\mathscr{L}^\uparrow_+$ **(and, a fortiori,** $\mathscr{L}$**) are compact groups**.

Similarly to the case of the groups $SO(3)$ and $SU(2)$, $\mathscr{L}^\uparrow_+$ is doubly-connected, and $SL(2, \mathbb{C})$ is its universal covering group.

The Lorentz algebra.

We recall that the Lorentz group preserves the Minkowski metric η of signature (1,3) in a four-dimensional vector space V. More generally, given a vector space V with a metric g we consider the associated “rotation" algebra to be the algebra of $End\, V$ which satisfies the condition

$$g(Av_1, v_2) + g(v_1, Av_2) = 0$$

[4]Explicitly: $\Lambda^\mu_\rho = \frac{1}{2} \mathrm{Tr}\, [\tau_\mu A \tau_\rho A^*]$

. Indeed from $g(e^{sA}v_1, e^{sA}v_2) = g(v_1, v_2)$ we get the previous equation by differentiation and evaluation at $s = 0$. The map

$$\mathbb{I} \otimes g : V \times V \to V \otimes V^* \equiv End\, V$$

defines a linear isomorphism between $\Lambda^2 V$ and the "rotation" algebra. Indeed we set

$$v_1 \wedge_g v_2 = v_1 \otimes g(v_2) - v_2 \otimes g(v_1) = (\mathbb{I} \otimes g)(v_1 \wedge v_2) \tag{1.110}$$

for any $v_1, v_2 \in V$.

We can identify the Lorentz algebra $\mathfrak{l} \equiv \mathfrak{g}_{\mathscr{L}}$ with its dual $\mathfrak{l}^*$. The spin variable S will then take values in $\mathfrak{l}$ and we use the scalar product

$$\langle s|s\rangle = \frac{1}{2}\,\mathrm{Tr}\, S^2.$$

Any time-like vector $u \in V$ such that $\eta(u, u) = 1$, defines an orthonormal decomposition of $\mathfrak{l}$ into rotations and boosts $\mathfrak{l} = \mathfrak{l}_u \oplus (\mathfrak{l}_u)^\perp$

$$s = (s - su \wedge_\eta u) + su \wedge_\eta u \quad s \in \mathfrak{l}. \tag{1.111}$$

The boost part $su \wedge_\eta u$ is encoded in $v := su$ belonging to $u^\perp$, the orthogonal complement of u. The rotation part is also encoded in a vector $\gamma \in u^\perp$, such that

$$(s - su \wedge_\eta u)x = \gamma \wedge x, \quad x \in u^\perp \tag{1.112}$$

where $\wedge$ is the three dimensional vector product in $\mathbb{R}^3$ (we have fixed an overall orientation). Therefore we can represent $s \in \mathfrak{l}$ by a pair of vectors (γ, v), where $(\gamma, v) \in u^\perp$. Using the bijection between $\mathfrak{l}_u$ and $(\mathfrak{l}_u)^\perp$ (both isomorphic to $u^\perp$) one can see that $\mathfrak{l} \simeq (\mathfrak{l}_u)^{\mathbb{C}}$.

The appropriate complex structure on the Lie algebra is given by the following multiplication

$$J(\gamma, v) = (-v, \gamma). \tag{1.113}$$

Thus γ is computed from s as follows

$$\gamma = (Js)u \quad v = su. \tag{1.114}$$

In terms of components γ, v the Killing form reads

$$\langle s|s\rangle = \eta(\gamma, \gamma) - \eta(v, v) = -(\gamma^2 - v^2) \tag{1.115}$$

where $\gamma^2 = -\eta(\gamma, \gamma)$ denotes the positive definite metric in the three-dimensional space. We have $\langle Js|s\rangle = -2g(\gamma, v) = 2\vec{\gamma} \cdot \vec{v}$.

1.2.6 The Galilei group

The Galilei group is the group of transformations on $\mathbb{R}^4$ which preserve separately the quadratic forms $dt \otimes dt$ and $\frac{\partial}{\partial \vec{x}} \otimes \frac{\partial}{\partial \vec{x}}$, which, as we have already discussed, represent the non-relativistic limit of the Minkowskian metric tensor, in covariant and contravariant form respectively. By using the language of mechanics, we consider $Q = \mathbb{R}^3$ as the configuration space of a mechanical system (typically, a point particle), and the "enlarged configuration space" (or "space-time") $\mathbb{M} = Q \times \mathbb{R} = \mathbb{R}^4$ ($\mathbb{R}$ =the time axis). In the framework of Newtonian mechanics, two frames S, S', in $\mathbb{R}^3$ are equivalent (i.e. the laws of motion can be written in the same form in both frames) iff:

i) the two frames move at **constant** speed relative to each other, and;

ii) time-intervals separating space-time points are the same in both frames, they preserve simultaneity leaves.

The resulting group is **the Galilei group**.

Let now $(\vec{x}, t)$, $(\vec{x}', t')$ be the coordinates of two space-time points w.r.t. to S and S', each one supplemented by its own time axis. In view of ii) above, t and t' can differ at most by a constant, dt being an invariant. At any fixed time, on each simultaneity leaf, excluding for the moment spatial inversions, which can be introduced separately later in a direct-product fashion, S and S' will be connected by a transformation of the so called Euclidean group $E(3)$ (the semi-direct product of rotations and translations), identified, as we shall see, by requiring the invariance of $\frac{\partial}{\partial \vec{x}} \otimes \frac{\partial}{\partial \vec{x}}$. Once this has been accomplished, we can "boost" the resulting frame at constant speed in order that it coincides with S' at all times. Hence, the most general transformation of the Galilei group will be identified by ten parameters (or, as we will say, **the group is a ten-parameter group**):

(a) $(\vec{n}, \theta, T_{\vec{a}})$ identifying the element $(R, T_{\vec{a}})$ $(R = R(\vec{n}, \theta))$ of $E(3)$;

(b) $\vec{v}$, the boost velocity, which may also be thought of as an element of $T(3)$;

(c) $\tau \in T(1)$ the shift in the general origin of times.

The most general transformation of the Galilei group, which we shall indicate as $(R, T_{\vec{a}}, \vec{v}, \tau)$, will then have the following action on $\mathbb{R}^4 = \mathbb{M}$:

$$(R, T_{\vec{a}}, \vec{v}, \tau) \ : \quad (\vec{x}, t) \mapsto (\vec{x}' = R\vec{x} + \vec{a} + \vec{v}t, t' = t + \tau) \tag{1.116}$$

By working out explicitly the effect of two successive Galilei transformations, we find the group composition law as:

$$(R, \vec{a}, \vec{v}, \tau)(R', \vec{a}', \vec{v}', \tau') = (RR', R\vec{a}' + \vec{a} + \vec{v}\tau', R\vec{v}' + \vec{v}, \tau + \tau') \tag{1.117}$$

In particular, then:

$$(R, \vec{a}, \vec{v}, \tau)^{-1} = (R^{-1}, R^{-1}(\vec{v}\tau - \vec{a}), -R^{-1}\vec{v}, -\tau) \tag{1.118}$$

and also:

$$\begin{aligned} (R, \vec{a}, \vec{v}, \tau)(R', \vec{a}', \vec{v}', \tau')(R, \vec{a}, \vec{v}, \tau)^{-1} = (RR'R^{-1}, \\ , (\,Id - RR'R^{-1})(\vec{a} - \vec{v}\tau) + R(\vec{a}' - \vec{v}'\tau) + \vec{v}\tau', (\,Id - RR'R^{-1})\vec{v} + R\vec{v}', \tau') \end{aligned} \tag{1.119}$$

The Galilei group has several subgroups, namely:

(a) the subgroup of (proper) rotations $SO(3)$, made out of elements of the form:

$$(R, 0, 0, 0) \tag{1.120}$$

(b) the three-dimensional subgroup of the **Galilei boosts**, with elements of the form:

$$(Id, 0, \vec{v}, 0) \tag{1.121}$$

which we will call $T^{(v)}(3)$

(c) the (one-dimensional) group of time translations, $T(1)$, with elements of the form:

$$(Id, 0, 0, \tau) \tag{1.122}$$

(d) the (three-dimensional) subgroup $T(3)$ of space translations, with elements of the form:

$$(Id, \vec{a}, 0, 0) \tag{1.123}$$

Exercise. Using (1.117), check that (a) to (d) are indeed subgroups of the Galilei group. Using (1.119), check that none of them is a normal subgroup. Show that, for group (b), (1.117) yields the usual (nonrelativistic) composition law for velocities.

Remark. The time-displacement enters in a particularly simple way into the group composition law. Moreover, the subset of the Galilei group corresponding to $\tau = 0$ is seen at once, from (1.117-1.119) to be a (nine-dimensional) **invariant** subgroup. This reflects the special role played by time in nonrelativistic mechanics: due to the invariance of dt, the only freedom is a shift in the origin of its measurement. Apart from this, time has an absolute character, i. e. the simultaneity foliation is preserved.

Repeated use (the details are left as an exercise) of Eq. (1.117) shows that the elements of $SO(3)$ and $T^{(v)}(3)$ combine in a **semidirect way** (see 1.2.7) to yield the subgroup:

$$E^{(v)}(3) \stackrel{\text{def}}{=} SO(3) \ltimes T^{(v)}(3) \tag{1.124}$$

In the same manner, one shows that $T(1)$ and $T(3)$ combine in a **direct** way to give the four-dimensional Abelian group of **space-time translations**:

$$T(4) \stackrel{\text{def}}{=} T(3) \times T(1) \tag{1.125}$$

Finally, the full Galilei group can be reconstructed as the semidirect product $E^{(v)}(3) \ltimes T(4)$, or, in a more detail:

Proposition. 1.12 *The Galilei group can be represented as:*

$$\left[SO(3) \ltimes T^{(v)}(3)\right] \ltimes \left[T(3) \times T(1)\right] \tag{1.126}$$

As an exercise, let us check that $T(4)$ is indeed an invariant subgroup. An element of $T(4)$ is of the form $(\,Id, T_{\vec{a}'}, 0, \tau')$, and, from (1.119), we have:

$$(R,\vec{a},\vec{v},\tau)(\,Id,\vec{a}',0,\tau')(R,\vec{a},\vec{v},\tau)^{-1} = \\ (\,Id, R\vec{a}' + \vec{v}\tau', 0, \tau') \in T(4) \tag{1.127}$$

as it should. Instead, $E^{(v)}(3)$ will be a subgroup, but not an invariant one.

Remark: It is easy to check using again (1.117), that $T^{(v)}(3)$ and $T(3)$ combine in a direct way, but do not give rise to an invariant subgroup. These two groups yield **another decomposition of Galilei group as semidirect product**, namely as:

$$\Big[SO(3) \times T(1)\Big] \times \Big[T^{(v)}(3) \times T(3)\Big] \tag{1.128}$$

The proof is left as an exercise.

1.2.7 Direct and semidirect products of groups.

Let G and G' be two groups, with elements $a, b, \dots$ and $a', b', \dots$ respectively. The **direct product**, $G \times G'$, of G and G', is defined in the following way:

Definition. 1.8 *The direct product $G \times G'$ is defined as the group whose elements are the ordered pairs $(a, a'),\ a \in G,\ a' \in G'$, with composition law:*

$$(a, a') \cdot (b, b') \overset{\text{def}}{=} (ab, a'b') \tag{1.129}$$

Remarks: (i) to prove that (1.129) is indeed a (well-defined) group-composition law is a trivial exercise. In particular, the inverse element of (a, a') is $(a, a')^{-1} = (a^{-1}, a'^{-1})$.

(ii) Considering G and G' (only) as sets, $G \times G'$ is also, as a set, the direct product of G and G' (again as sets).

(iii) Consider the subset of elements of $G \times G'$ of the form (e, a') e= the unit element in G. Such a subset can be considered as the direct product group $\{e\} \times G'$, and is clearly isomorphic to G'. Moreover:

$$(a, a') \cdot (e, c') \cdot (a, a')^{-1} = (e, a'c'a'^{-1}) \tag{1.130}$$

a, a', c'. Similar conclusions will hold for the subset of elements of the form (a, e'), e' = the identity in G'. So:

Proposition. 1.13 *The subsets $\big\{(e, a')\big\}_{a' \in G'}$, $\big\{(a, e')\big\}_{a \in G}$ of $G \times G'$ are **normal subgroups** of $G \times G'$*

It can be proven that whenever one has a group G, and a **normal** subgroup H, i.e., in short, $gH = Hg\ \forall g \in G$ (for every $h \in H$, one can find an $h' \in H$ s.t. $gh = h'g$), then the quotient of G w.r.t. the equivalence relation:

$$g \sim g' \Leftrightarrow gH = g'H \tag{1.131}$$

i.e. the set $\{S_g\}$ of equivalence classes under (1.131), also called G/H, can be given a group structure with

$$S_g \cdot S_{g'} = S_{gg'} \tag{1.132}$$

(Note: the identity in G/H will be H itself) G/H will be called the **quotient** group of G w.r.t. H. Then, considering $G \times G'$, G (G') can be considered as (or to be isomorphic to) the quotient group of $G \times G'$ w.r.t. the invariant subgroup $\{e\} \otimes G'$ ($G \times \{e'\}$). Identifying also the latter with G' and G, we write:

$$G \sim G \times G'/G' \quad\quad G' \sim G \otimes G'/G \tag{1.133}$$

Suppose now that there is an **action** of G on G' which is an **automorphism** of G' itself. By this we mean that there exists a map:

$$\tau \ : \quad G \times G' \to G' \tag{1.134}$$

by:

$$(g, g') \mapsto \tau(g, g') \in G' \tag{1.135}$$

s.t., for fixed $g \in G$:

$$\tau_g(\cdot) \stackrel{\text{def}}{=} \tau(g, \cdot)\ , \quad \tau_g\ : G' \to G' \tag{1.136}$$

satisfies

$$\tau_g(g'h'^{-1}) = \tau_g(g')\tau_g(h'^{-1}) \quad \forall g', h' \in G' \tag{1.137}$$

(this implies that $\tau_g(\cdot)$ satisfies the group properties on G', i.e. $\tau_g(g' \cdot h') = \tau_g(g') \cdot \tau_g(h')$, $\tau_g(e') = e'$, $\tau_g(g'^{-1}) = \tau_g^{-1}(g')$) and

$$\tau_{g_1 g_2} = \tau_{g_1} \cdot \tau_{g_2} \quad \forall g_1, g_2 \in G \tag{1.138}$$

We may consider now the **semidirect** product of G and G', $G \ltimes G'$, defined as follows:

Definition. 1.9 *If there is an action of G on G' which is also an automorphism of G', the* ***semidirect*** *product, $G \ltimes G'$ of G and G', is defined by the group which, as a set, coincides with the topological (direct) product of G and G', and with the composition law:*

$$(a, a') \cdot (b, b') \stackrel{\text{def}}{=} \Big(ab, a' \cdot \tau_a(b')\Big) \tag{1.139}$$

Exercise: (see, e.g. [172], Chap. 15) Check that (1.139) is indeed a group-composition law. In particular, check that the inverse of (a, a') is:

$$(a, a')^{-1} = \left(a^{-1}, \tau_{a^{-1}}(a'^{-1})\right) \tag{1.140}$$

Remark: Direct products are particular cases of the semidirect ones, and are recovered when $\tau_g = \text{Id}$, i.e. $\tau_g(g') = g' \ \forall g \in G, g' \in G'$ (hence: $\tau_g = \tau_e \ , \ \forall g$)

Consider now the subsets of $G \ltimes G'$ of the form $\left\{(g, e')\right\}_{g \in G}$ and $\left\{(e, g')\right\}_{g' \in G'}$ respectively. As $\tau_g(e') = e' \ \forall g$ and $\tau_e(g') = g' \ \forall g'$, both sets are subgroups of $G \ltimes G'$, isomorphic to G and G' respectively. With some abuse of terminology, we will identify them with G and G'. It is easily seen that:

$$(a, a')(e, g')(a, a')^{-1} = (e, a'\tau_a(g')a'^{-1}) \tag{1.141}$$

while:

$$(a, a')(g, e')(a, a')^{-1} = (aga', a'\tau_{aga^{-1}}(a'^{-1})) \tag{1.142}$$

Hence:

Proposition. 1.14 *In the semidirect product $G \ltimes G'$, both G and G' are subgroups of $G \ltimes G'$, but, unless the semidirect product "collapses" into a direct one, only G' is a normal subgroup.*

In the case of both direct **and** semidirect products, the group manifold may be identified with the **product manifold** $G \times G'$.

These rather long formal preliminaries may seem somehow disproportionate for the analysis of the Euclidean group which we are going to perform now. They are however useful for the study of other composite groups considered in this section.

1.2.8 The Euclidean group $E(3)$ and related groups.

Consider the transformations of $\mathbb{R}^3$ made up of a translation $T_{\vec{a}}$ (hence $T_{\vec{a}} \in T(3)$) and of a (proper) rotation R (hence $R \in SO(3)$), i.e. the set:

$$\left\{(R, T_{\vec{a}})\right\}, \quad (R, T_{\vec{a}}) : \ \vec{x} \mapsto R \cdot \vec{x} + \vec{a} \tag{1.143}$$

Clearly:

$$(R, T_{\vec{a}}) \cdot (R', T_{\vec{a}'}) : \ \vec{x} \mapsto R \cdot R' \cdot \vec{x} + R\vec{a'} + \vec{a} \tag{1.144}$$

Hence $\left\{(R, T_{\vec{a}})\right\}$ is a group with the group composition law:

$$(R, T_{\vec{a}}) \cdot (R', T_{\vec{a}'}) = (RR', T_{\vec{a}} + R\vec{a}') \tag{1.145}$$

It is **the Euclidean group** $E(3)$. Now, because $T(3) \sim \mathbb{R}^3$, $SO(3)$ has an action on $T(3)$ which is also an automorphism via

$$\vec{a} \mapsto R\vec{a} \tag{1.146}$$

(which we have already implicitly used), and: $\vec{a} + R\vec{a}' \equiv \vec{a} \cdot R\vec{a}'$ with the composition law of $T(3)$. Hence:

Proposition. 1.15 *The Euclidean group is the semidirect product of $SO(3)$ and $T(3)$:*

$$E(3) = SO(3) \ltimes T(3) \tag{1.147}$$

As a manifold, $E(3)$ can be given a coordinatization in terms of that of the product manifold $SO(3) \ltimes T(3)$. It will be seen in Section 27.7 however, that this is not a "canonical" coordinatization in the sense of Lie groups.

We now consider some related semidirect product groups:

- The group $O(3) \ltimes T(3)$ is the most general group leaving the Euclidean distance in $\mathbb{R}^3$:

  $$\mathrm{d}(\vec{x}, \vec{y}) \overset{\text{def}}{=} \|\vec{x} - \vec{y}\| \tag{1.148}$$

  unchanged. This is the vector space version of the tensorial form provided in Eq. (1.64). In the spirit of Klein's "Erlangen Programme", according to which a geometry is the study of those properties which are invariant under a given group of transformations, $O(3) \times T(3)$ **is the group characterizing Euclidean geometry in** $\mathbb{R}^3$.

- One may consider, more generally, the semidirect product $GL(3, \mathbb{R}) \ltimes T(3)$. This is the **group of affine motions** of $\mathbb{R}^3$ (or "**affine group**", for short). The composition law is again (1.145), where now $R \in GL(3, \mathbb{R})$. One can also mention the general affine group $GL(n, \mathbb{R}) \ltimes T(n)$. This means that $\mathbb{R}^n$ is considered as an affine space, instead of a vector space. Indeed in Eq. (1.148) the 'origin' of the vectors is irrelevant for the definition of distance.

Remark: $GL(n, \mathbb{R})$ has dimension n^2. Hence, the general affine group is an $n(n+1)$-dimensional group, while $O(n) \ltimes T(n)$ (and also: $E(n) \overset{\text{def}}{=} SO(n) \ltimes T(n)$) are $n(n+1)/2$-dimensional.

According to Prop. 26.7, $T(3)$ **is an invariant subgroup of** $E(3)$. According to (1.140) we have, for elements of $E(3)$:

$$(R, T_{\vec{a}})^{-1} = (R^{-1}, -T_{R^{-1}\vec{a}}), \qquad R^{-1} = \tilde{R} \tag{1.149}$$

using (1.149), a direct proof of invariance of $T(3)$ is straightforward, and one obtains:

$$(R, T_{\vec{a}})(\,Id, \vec{b})(R, T_{\vec{a}})^{-1} = (\,Id, R\vec{b}) \tag{1.150}$$

(cfr. also (1.141)) □.

Exercise. The general affine group is an open subset, hence a submanifold, of $\mathbb{R}^{n(n+1)}$ with the standard topology. Prove that, say, $E(n)$ is a **closed** submanifold of $GL(n, \mathbb{R}) \times T(n)$.

1.2.9 Infinitesimal generators as vector fields

In the previous sections we have already introduced the concept of infinitesimal generators of a space-time transformation. In this section we wish to introduce the Poincaré infinitesimal generators as vector fields acting on space-time and derive the Galilei generators in the limit $c \to \infty$ with c the speed of light.

The Poincaré group has been introduced as the ten-dimensional group of transformations which leave invariant the quadratic form corresponding to the Minkowski metric. At the infinitesimal level this request corresponds to require that the Lie derivative of the metric tensor with respect to the vector fields generating Poincaré transformations be zero

$$L_{X_{(a)}} g = 0 \tag{1.151}$$

where g is the metric tensor which we have already seen many times in this chapter, given in covariant and contravariant form respectively by

$$g = \eta_{\mu\nu} dx^\mu \otimes dx^\nu = \eta^{\mu\nu} \frac{\partial}{\partial x^\mu} \otimes \frac{\partial}{\partial x^\nu} \tag{1.152}$$

with $\eta_{\mu\nu} = \eta^{\mu\nu}\mathrm{diag}(1, -1, -1, -1)$, while $X_{(a)} = X^\mu_{(a)}(x)\frac{\partial}{\partial x^\mu}, a = 1, ..., 10$ are the unknown vector fields to be determined. To compute the Lie derivative (see §24) we apply the Leibnitz rule to Eq. (1.151) and observe that

$$L_{X_{(a)}} \eta^{\mu\nu} = X^\rho_{(a)} \frac{\partial}{\partial x^\rho} \eta^{\mu\nu} = 0 \tag{1.153}$$

$$L_{X_{(a)}} \frac{\partial}{\partial x^\mu} = [X^\rho_{(a)} \frac{\partial}{\partial x^\rho}, \frac{\partial}{\partial x^\mu}] = -\frac{\partial X^\rho_{(a)}(x)}{\partial x^\mu} \frac{\partial}{\partial x^\rho} \tag{1.154}$$

$$L_{X_{(a)}} dx^\mu = dX^\mu_{(a)}(x) \tag{1.155}$$

which yield

$$\eta^{\mu\beta} \partial_\mu X^\alpha_{(a)} + \eta^{\alpha\nu} \partial_\nu X^\beta_{(a)} = 0 \tag{1.156}$$

or equivalently

$$\eta_{\mu\beta} \partial_\alpha X^\mu_{(a)} + \eta_{\alpha\nu} \partial_\beta X^\nu_{(a)} = 0 \tag{1.157}$$

The solutions may be verified to be

$$X_{(\mu)} = \partial_\mu \quad \text{with } \mu = 0, ..., 3 \tag{1.158}$$

$$X_{(jk)} = x_j \partial_k - x_k \partial_j \quad \text{with j, k} = 1, 3 \tag{1.159}$$

$$X_{(0,j)} = x_0 \partial_j + x_j \partial_0 \quad \text{with j} = 1, ..., 3 \tag{1.160}$$