Soluzione di integrali di funzioni fratte

Integrali funzioni fratte/razionali

1

c) $\int \frac{2x^2-3x+7}{x-5}\,dx$

$\frac{M(x)}{N(x)} \Rightarrow M(x) \;\underline{|\; N(x)}$
$\qquad R(x) \quad Q(x)$

$$
\begin{array}{rrr|l}
2x^2 & -3x & +7 & x-5 \\
\cline{4-4}
-2x^2 & +10x & & 2x+7 \\
\cline{1-3}
= & 7x & +7 & \\
 & -7x & +35 & \\
\cline{2-3}
 & = & +42 &
\end{array}
$$

$M(x) = N(x)\cdot Q(x) + R(x)$

$\int \frac{M(x)}{N(x)} = \int Q(x) + \int \frac{R(x)}{N(x)}$

$$\int \frac{2x^2-3x+7}{x-5}\,dx = \int (2x+7)\,dx + \int \frac{42}{2x+7}\,dx =$$

$$= \frac{\cancel{2}x^2}{\cancel{2}} + 7x + 42\ln|2x+7| + C$$

b) $\int \frac{3x-4}{x^2-6x+8}\,dx$

$x_{1,2} = \frac{6 \mp \sqrt{36-32}}{2}$ $\quad x_1 = \frac{6-2}{2} = 2 \quad x_2 = \frac{6+2}{2} = 4$

$$x^2-6x+8 = (x-2)(x-4)$$

$$\frac{3x-4}{x^2-6x+8} = \frac{A}{x-2} + \frac{B}{x-4} = \frac{A(x-4)+B(x-2)}{(x-4)(x-2)} =$$

$$= \frac{Ax-4A+Bx-2B}{(x-4)(x-2)} = \frac{(A+B)x-4A-2B}{(x-4)(x-2)}.$$

$$\begin{cases} A+B = 3 \\ -4A-2B = -4 \end{cases}$$

$-3A - B = -1 \Rightarrow -B = -1+3A \Rightarrow B = 1-3A$

$A+1-3A = 3 \rightarrow -2A = 2 \rightarrow \boxed{A=-1}$

$B = 1+3 = 4$

$$\frac{3x-4}{x^2-6x+8} = \frac{-1}{x-2} + \frac{4}{x-4}$$ (2)

$$\int f(x)\,dx = -\ln|x-2| + 4\ln|x-4| + C$$

d) $$\int \frac{9x+8}{x^3+2x^2+x+2}\,dx$$

$$\begin{array}{l|l} x^3+2x^2+x+2 & x+2 \\ \hline -x^3-2x^2 & x^2+1 \\ = \quad x+2 & \\ -x-2 & \end{array}$$

$$\frac{9x+8}{x^3+2x^2+x+2} = \frac{A}{x+2} + \frac{Bx+C}{x^2+1} =$$

$$= \frac{A(x^2+1)+(Bx+C)(x+2)}{(x+2)(x^2+1)} =$$

$$= \frac{(A+B)x^2+(2B+C)x+A+2C}{(\;)\cdot(\;)} =$$

$$\begin{cases} A+B=0 \\ 2B+C=9 \\ A+2C=8 \end{cases} \Rightarrow \begin{cases} A=-2 \\ B=2 \\ C=5 \end{cases}$$

③

$$\int \frac{9x+8}{x^3+2x^2+x+2}\,dx = \int \left(\frac{-2}{x+2} + \frac{2x+5}{x^2+1} \right) dx =$$

$$= -2\ln|x+2| + \int \frac{2x}{x^2+1}\,dx + 5\int \frac{1}{x^2+1}\,dx =$$

$$= -2\ln|x+2| + 2\ln|x^2+1| + 5\,\operatorname{arctg} x + C$$

$$\int \frac{2x}{x^2+1}\,dx \Rightarrow x^2+1 = t \Rightarrow 2x\,dx = dt$$

$$\int \frac{1}{t}\,dt = \ln|t| \Rightarrow \ln|x^2+1|$$

e) $\int \frac{x^5-3x^4+x+3}{x^2-1}\,dx$

$$\begin{array}{l|l} x^5-3x^4+x+3 & x^2-1 \\ \hline -x^5 \quad +x^3 & x^3-3x^2+x-3 \\ = -3x^4+x^3+x+3 & \\ +3x^4 \quad -3x^2 & \\ = x^3-3x^2+x+3 & \\ -x^3 \quad +x & \\ = -3x^2+2x+3 & \\ +3x^2 \quad -3 & \\ = +2x = & \end{array}$$

$$\int \frac{x^5-3x^4+x+3}{x^2-1}\,dx = \int (x^3-3x^2+x-3)\,dx + \int \frac{2x}{x^2-1}\,dx =$$

$$= \frac{x^4}{4} - \frac{\cancel{3}x^3}{\cancel{3}} + \frac{x^2}{2} - 3x + 2\ln|x^2-1| + C$$

④

g) $\int \frac{e^x}{e^{2x}-3e^x+2}dx \to e^x = t \to x = \ln t$

$dx = \frac{1}{t}dt$

$\int \frac{\cancel{t}}{t^2-3t+2} \cdot \frac{1}{\cancel{t}} dt = \int \frac{1}{t^2-3t+2}dt$

$t^2-3t+2 \to t_{1,2} = \frac{3 \mp \sqrt{9-8}}{2}$ — $t_1 = \frac{3-1}{2} = 1$, $t_2 = \frac{3+1}{2} = 2$

$\int \frac{1}{t^2-3t+2}dt = \int \frac{A}{t-1} + \frac{B}{t-2} = \frac{A(t-2)+B(t-1)}{(t-1)(t-2)} =$

$= \frac{At-2A+Bt-B}{(\ )(\ )} = \frac{(A+B)t-2A-B}{(\ )\cdot(\ )} \Rightarrow$

$\begin{cases} A+B=0 \\ -2A-B=1 \end{cases} \Rightarrow -A = 1 \to \begin{cases} A=-1 \\ B=1 \end{cases}$

$-A = 1$

$\int \frac{1}{t^2-3t+2}dt = -\int \frac{1}{t-1}dt + \int \frac{1}{t-2}dt =$

$= -\ln|t-1| + \ln|t-2| + C =$

$= -\ln|e^x-1| + \ln|e^x-2| + C$

h) $\int \frac{x+\sqrt{x-1}}{x-5}dx \to \sqrt{x-1} = t \to x = t^2+1 \to dx = 2t\,dt$

$= \int \frac{t^2+1+t}{t^2+1-5} 2t\,dt = 2\int \frac{t^3+t^2+t}{t^2-4}dt$

$$\frac{t^3+t^2+t}{t^2-4} \to \begin{array}{l|l} t^3+t^2+t & t^2-4 \\ \hline -t^3 \quad +4t & t+1 \\ = t^2+5t & \\ -t^2 \quad +4 & \\ = 5t+4 & \end{array}$$

$$\int \frac{t^3+t^2+t}{t^2-4}\,dt = 2\left[\int (t+1)\,dt + \int \frac{5t+4}{t^2-4}\,dt\right] =$$

$$= 2\frac{t^2}{2} + 2t + 2\int \frac{5t+4}{(t+2)(t-2)}\,dt$$

$$\int \frac{5t+4}{(t+2)(t-2)}\,dt \Rightarrow \frac{5t+4}{(t+2)(t-2)} = \frac{A}{t+2} + \frac{B}{t-2} =$$

$$= \frac{A(t-2)+B(t+2)}{t^2-4} = \frac{At-2A+Bt+2B}{t^2-4} \Rightarrow$$

$$\frac{(A+B)t-2A+2B}{t^2-4}$$

$$\begin{cases} A+B=5 \to A=5-B \\ -2A+2B=4 \end{cases} \Rightarrow -2(5-B)+2B=4 \to -10+2B+2B=4 \Rightarrow$$

$$4B=14 \to B=\frac{14}{4}=\frac{7}{2}$$

$$A=5-\frac{7}{2}=\frac{3}{2}$$

$$\int \frac{5t+4}{(t+2)(t-2)}\,dt = \int \frac{3}{2}\frac{1}{t+2}\,dt + \int \frac{7}{2}\frac{1}{t-2}\,dt = \frac{3}{2}\ln|t+2| + \frac{7}{2}\ln|t-2|$$

$$\int \frac{x+\sqrt{x-1}}{x-5}\,dx = \cancel{2\int (t+1)\,dt + 2\int \frac{7}{2}}$$ ⑥

$$t^2+2t+7\ln|t-2|+3\ln|t+2|+c =$$

$$= x-1+2\sqrt{x-1}+7\ln\left|\sqrt{x-1}-2\right|+3\ln\left|\sqrt{x-1}+2\right|+c$$

i) $\int \frac{2}{(1+tg\,x)^2}\,dx \qquad tg\,x = t \rightarrow x = arctg\,x \rightarrow dx = \frac{1}{1+t^2}\,dt$

$$\hookrightarrow \int \frac{2}{(1+t)^2}\cdot\frac{1}{1+t^2}\,dt$$

$$(1+t)^2 = 1+2t+t^2 \rightarrow t_{1,2} = \frac{-2\mp\sqrt{4-4}}{2} = -1$$

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$$\frac{2}{1+t^2} = \frac{A}{1+t}+\frac{B}{(1+t)^2}$$

$\frac{1}{1+t^2} \rightarrow$ RADICI Complesse $\Rightarrow$

$$\frac{1}{1+t^2} = \frac{Ct+D}{1+t^2}$$

$$\Downarrow$$

$$\int \frac{2}{(1+t)^2}\cdot\frac{1}{1+t^2} = \frac{A}{1+t}+\frac{B}{(1+t)^2}+\frac{Ct+D}{1+t^2}$$

$$\Downarrow$$

$$A=1 \quad B=1 \quad C=-1 \quad D=0$$

$$\int \frac{2}{(1+tg\,x)^2}\,dx = \int \frac{1}{1+t}\,dt + \int \frac{1}{(1+t)^2}\,dt - \int \frac{t}{1+t^2}\,dt$$

$$\hookrightarrow \int (1+t)^{-2}\,dt = \frac{(1+t)^{-2+1}}{-2+1} = -\frac{1}{1+t}$$

$$= \ln|1+t| - \frac{1}{1+t} - \frac{1}{2}\ln(1+t^2) + c$$

$$\int \frac{t}{1+t^2}\,dt \;\cancel{= \frac{1}{2}\int \frac{2t}{1+t^2}\,dt \Rightarrow}$$

$$1+t^2 = u \rightarrow 2t\,dt = du \rightarrow t\,dt = \frac{1}{2}\,du$$

$$\frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln u \Rightarrow \frac{1}{2}\ln|1+t^2|$$

$$\int \frac{2}{(1+tg\,x)^2}\,dx = \int \frac{1}{1+t}\,dt + \int \frac{1}{1+t^2}\,dt - \int \frac{t}{1+t^2}\,dt =$$

$$= \ln|1+t| - \frac{1}{1+t} - \frac{1}{2}\ln(1+t^2) + c =$$

$$= \ln|1+tg\,x| - \frac{1}{1+tg\,x} - \frac{1}{2}\ln(1+tg^2 x) + c$$