

This appendix in mathematics is intended as a brief review of operations and methods. Early in this course, you should be totally familiar with basic algebraic techniques, analytic geometry, and trigonometry. The sections on differential and integral calculus are more detailed and are intended for students who have difficulty applying calculus concepts to physical situations.

B.1 Scientific Notation

Many quantities used by scientists often have very large or very small values. The speed of light, for example, is about 300 000 000 m/s, and the ink required to make the dot over an *i* in this textbook has a mass of about 0.000 000 001 kg. Obviously, it is very cumbersome to read, write, and keep track of such numbers. We avoid this problem by using a method incorporating powers of the number 10:

$$10^0 = 1$$

$$10^1 = 10$$

$$10^2 = 10 \times 10 = 100$$

$$10^3 = 10 \times 10 \times 10 = 1\,000$$

$$10^4 = 10 \times 10 \times 10 \times 10 = 10\,000$$

$$10^5 = 10 \times 10 \times 10 \times 10 \times 10 = 100\,000$$

and so on. The number of zeros corresponds to the power to which ten is raised, called the **exponent** of ten. For example, the speed of light, 300 000 000 m/s, can be expressed as 3.00×10^8 m/s.

In this method, some representative numbers smaller than unity are the following:

$$10^{-1} = \frac{1}{10} = 0.1$$

$$10^{-2} = \frac{1}{10 \times 10} = 0.01$$

$$10^{-3} = \frac{1}{10 \times 10 \times 10} = 0.001$$

$$10^{-4} = \frac{1}{10 \times 10 \times 10 \times 10} = 0.000\,1$$

$$10^{-5} = \frac{1}{10 \times 10 \times 10 \times 10 \times 10} = 0.000\,01$$

In these cases, the number of places the decimal point is to the left of the digit 1 equals the value of the (negative) exponent. Numbers expressed as some power of ten multiplied by another number between one and ten are said to be in **scientific notation**. For example, the scientific notation for 5 943 000 000 is 5.943×10^9 and that for 0.000 083 2 is 8.32×10^{-5} .

When numbers expressed in scientific notation are being multiplied, the following general rule is very useful:

$$10^n \times 10^m = 10^{n+m} \quad (\text{B.1})$$

where n and m can be *any* numbers (not necessarily integers). For example, $10^2 \times 10^5 = 10^7$. The rule also applies if one of the exponents is negative: $10^3 \times 10^{-8} = 10^{-5}$.

When dividing numbers expressed in scientific notation, note that

$$\frac{10^n}{10^m} = 10^n \times 10^{-m} = 10^{n-m} \quad (\text{B.2})$$

Exercises

With help from the preceding rules, verify the answers to the following equations:

- $86\,400 = 8.64 \times 10^4$
- $9\,816\,762.5 = 9.816\,762\,5 \times 10^6$
- $0.000\,000\,039\,8 = 3.98 \times 10^{-8}$
- $(4.0 \times 10^8)(9.0 \times 10^9) = 3.6 \times 10^{18}$
- $(3.0 \times 10^7)(6.0 \times 10^{-12}) = 1.8 \times 10^{-4}$
- $\frac{75 \times 10^{-11}}{5.0 \times 10^{-3}} = 1.5 \times 10^{-7}$
- $\frac{(3 \times 10^6)(8 \times 10^{-2})}{(2 \times 10^{17})(6 \times 10^5)} = 2 \times 10^{-18}$

B.2 Algebra

Some Basic Rules

When algebraic operations are performed, the laws of arithmetic apply. Symbols such as x , y , and z are usually used to represent unspecified quantities, called the **unknowns**.

First, consider the equation

$$8x = 32$$

If we wish to solve for x , we can divide (or multiply) each side of the equation by the same factor without destroying the equality. In this case, if we divide both sides by 8, we have

$$\frac{8x}{8} = \frac{32}{8}$$

$$x = 4$$

Next consider the equation

$$x + 2 = 8$$

In this type of expression, we can add or subtract the same quantity from each side. If we subtract 2 from each side, we have

$$x + 2 - 2 = 8 - 2$$

$$x = 6$$

In general, if $x + a = b$, then $x = b - a$.

Now consider the equation

$$\frac{x}{5} = 9$$

If we multiply each side by 5, we are left with x on the left by itself and 45 on the right:

$$\left(\frac{x}{5}\right)(5) = 9 \times 5$$

$$x = 45$$

In all cases, *whatever operation is performed on the left side of the equality must also be performed on the right side.*

The following rules for multiplying, dividing, adding, and subtracting fractions should be recalled, where a , b , c , and d are four numbers:

	Rule	Example
Multiplying	$\left(\frac{a}{b}\right)\left(\frac{c}{d}\right) = \frac{ac}{bd}$	$\left(\frac{2}{3}\right)\left(\frac{4}{5}\right) = \frac{8}{15}$
Dividing	$\frac{(a/b)}{(c/d)} = \frac{ad}{bc}$	$\frac{2/3}{4/5} = \frac{(2)(5)}{(4)(3)} = \frac{10}{12}$
Adding	$\frac{a}{b} \pm \frac{c}{d} = \frac{ad \pm bc}{bd}$	$\frac{2}{3} - \frac{4}{5} = \frac{(2)(5) - (4)(3)}{(3)(5)} = -\frac{2}{15}$

Exercises

In the following exercises, solve for x .

Answers

- $a = \frac{1}{1+x}$ $x = \frac{1-a}{a}$
- $3x - 5 = 13$ $x = 6$
- $ax - 5 = bx + 2$ $x = \frac{7}{a-b}$
- $\frac{5}{2x+6} = \frac{3}{4x+8}$ $x = -\frac{11}{7}$

Powers

When powers of a given quantity x are multiplied, the following rule applies:

$$x^n x^m = x^{n+m} \tag{B.3}$$

For example, $x^2 x^4 = x^{2+4} = x^6$.

When dividing the powers of a given quantity, the rule is

$$\frac{x^n}{x^m} = x^{n-m} \tag{B.4}$$

For example, $x^8/x^2 = x^{8-2} = x^6$.

A power that is a fraction, such as $\frac{1}{3}$, corresponds to a root as follows:

$$x^{1/n} = \sqrt[n]{x} \tag{B.5}$$

For example, $4^{1/3} = \sqrt[3]{4} = 1.5874$. (A scientific calculator is useful for such calculations.)

Finally, any quantity x^n raised to the m th power is

$$(x^n)^m = x^{nm} \tag{B.6}$$

Table B.1 summarizes the rules of exponents.

TABLE B.1

Rules of Exponents
$x^0 = 1$
$x^1 = x$
$x^n x^m = x^{n+m}$
$x^n/x^m = x^{n-m}$
$x^{1/n} = \sqrt[n]{x}$
$(x^n)^m = x^{nm}$

Exercises

Verify the following equations:

- $3^2 \times 3^3 = 243$
- $x^5 x^{-8} = x^{-3}$
- $x^{10}/x^{-5} = x^{15}$
- $5^{1/3} = 1.709976$ (Use your calculator.)
- $60^{1/4} = 2.783158$ (Use your calculator.)
- $(x^4)^3 = x^{12}$

Factoring

Some useful formulas for factoring an equation are the following:

$$ax + ay + az = a(x + y + z) \quad \text{common factor}$$

$$a^2 + 2ab + b^2 = (a + b)^2 \quad \text{perfect square}$$

$$a^2 - b^2 = (a + b)(a - b) \quad \text{differences of squares}$$

Quadratic Equations

The general form of a quadratic equation is

$$ax^2 + bx + c = 0 \quad \text{(B.7)}$$

where x is the unknown quantity and a , b , and c are numerical factors referred to as **coefficients** of the equation. This equation has two roots, given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad \text{(B.8)}$$

If $b^2 \geq 4ac$, the roots are real.

EXAMPLE B.1

The equation $x^2 + 5x + 4 = 0$ has the following roots corresponding to the two signs of the square-root term:

$$x = \frac{-5 \pm \sqrt{5^2 - (4)(1)(4)}}{2(1)} = \frac{-5 \pm \sqrt{9}}{2} = \frac{-5 \pm 3}{2}$$

$$x_+ = \frac{-5 + 3}{2} = -1 \quad x_- = \frac{-5 - 3}{2} = -4$$

where x_+ refers to the root corresponding to the positive sign and x_- refers to the root corresponding to the negative sign.

Exercises

Solve the following quadratic equations:

Answers

1. $x^2 + 2x - 3 = 0$ $x_+ = 1$ $x_- = -3$
2. $2x^2 - 5x + 2 = 0$ $x_+ = 2$ $x_- = \frac{1}{2}$
3. $2x^2 - 4x - 9 = 0$ $x_+ = 1 + \sqrt{22}/2$ $x_- = 1 - \sqrt{22}/2$

Linear Equations

A linear equation has the general form

$$y = mx + b \quad \text{(B.9)}$$

where m and b are constants. This equation is referred to as linear because the graph of y versus x is a straight line as shown in Figure B.1. The constant b , called the **y-intercept**, represents the value of y at which the straight line intersects the y axis. The constant m is equal to the **slope** of the straight line. If any two points on the straight line are specified by the coordinates (x_1, y_1) and (x_2, y_2) as in Figure B.1, the slope of the straight line can be expressed as

$$\text{Slope} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\Delta y}{\Delta x} \quad \text{(B.10)}$$

Note that m and b can have either positive or negative values. If $m > 0$, the straight line has a *positive* slope as in Figure B.1. If $m < 0$, the straight line has a *negative* slope. In Figure B.1, both m and b are positive. Three other possible situations are shown in Figure B.2.

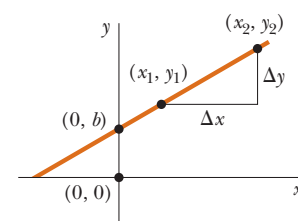


Figure B.1 A straight line graphed on an xy coordinate system. The slope of the line is the ratio of Δy to Δx .

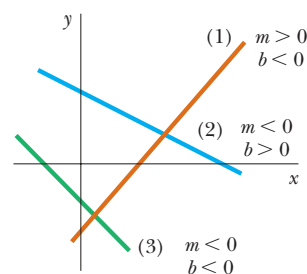


Figure B.2 The brown line has a positive slope and a negative y -intercept. The blue line has a negative slope and a positive y -intercept. The green line has a negative slope and a negative y -intercept.

Exercises

1. Draw graphs of the following straight lines: (a) $y = 5x + 3$ (b) $y = -2x + 4$ (c) $y = -3x - 6$

2. Find the slopes of the straight lines described in Exercise 1.

Answers (a) 5 (b) -2 (c) -3

3. Find the slopes of the straight lines that pass through the following sets of points: (a) $(0, -4)$ and $(4, 2)$ (b) $(0, 0)$ and $(2, -5)$ (c) $(-5, 2)$ and $(4, -2)$

Answers (a) $\frac{3}{2}$ (b) $-\frac{5}{2}$ (c) $-\frac{4}{9}$

Solving Simultaneous Linear Equations

Consider the equation $3x + 5y = 15$, which has two unknowns, x and y . Such an equation does not have a unique solution. For example, $(x = 0, y = 3)$, $(x = 5, y = 0)$, and $(x = 2, y = \frac{9}{5})$ are all solutions to this equation.

If a problem has two unknowns, a unique solution is possible only if we have *two* equations. In general, if a problem has n unknowns, its solution requires n equations. To solve two simultaneous equations involving two unknowns, x and y , we solve one of the equations for x in terms of y and substitute this expression into the other equation.

EXAMPLE B.2

Solve the two simultaneous equations

$$(1) \quad 5x + y = -8$$

$$(2) \quad 2x - 2y = 4$$

Solution From Equation (2), $x = y + 2$. Substitution of this equation into Equation (1) gives

$$5(y + 2) + y = -8$$

$$6y = -18$$

$$y = -3$$

$$x = y + 2 = -1$$

Alternative Solution Multiply each term in Equation (1) by the factor 2 and add the result to Equation (2):

$$10x + 2y = -16$$

$$2x - 2y = 4$$

$$12x = -12$$

$$x = -1$$

$$y = x - 2 = -3$$

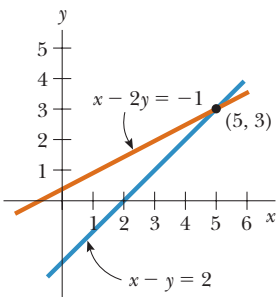


Figure B.3 A graphical solution for two linear equations.

Two linear equations containing two unknowns can also be solved by a graphical method. If the straight lines corresponding to the two equations are plotted in a conventional coordinate system, the intersection of the two lines represents the solution. For example, consider the two equations

$$x - y = 2$$

$$x - 2y = -1$$

These equations are plotted in Figure B.3. The intersection of the two lines has the coordinates $x = 5$ and $y = 3$, which represents the solution to the equations. You should check this solution by the analytical technique discussed earlier.

Exercises

Solve the following pairs of simultaneous equations involving two unknowns:

Answers

- $x + y = 8$ $x = 5, y = 3$
 $x - y = 2$
- $98 - T = 10a$ $T = 65, a = 3.27$
 $T - 49 = 5a$
- $6x + 2y = 6$ $x = 2, y = -3$
 $8x - 4y = 28$

Logarithms

Suppose a quantity x is expressed as a power of some quantity a :

$$x = a^y \quad (\text{B.11})$$

The number a is called the **base** number. The **logarithm** of x with respect to the base a is equal to the exponent to which the base must be raised to satisfy the expression $x = a^y$:

$$y = \log_a x \quad (\text{B.12})$$

Conversely, the **antilogarithm** of y is the number x :

$$x = \text{antilog}_a y \quad (\text{B.13})$$

In practice, the two bases most often used are base 10, called the *common* logarithm base, and base $e = 2.718\ 282$, called Euler's constant or the *natural* logarithm base. When common logarithms are used,

$$y = \log_{10} x \quad (\text{or } x = 10^y) \quad (\text{B.14})$$

When natural logarithms are used,

$$y = \ln x \quad (\text{or } x = e^y) \quad (\text{B.15})$$

For example, $\log_{10} 52 = 1.716$, so $\text{antilog}_{10} 1.716 = 10^{1.716} = 52$. Likewise, $\ln 52 = 3.951$, so $\text{antiln } 3.951 = e^{3.951} = 52$.

In general, note you can convert between base 10 and base e with the equality

$$\ln x = (2.302\ 585) \log_{10} x \quad (\text{B.16})$$

Finally, some useful properties of logarithms are the following:

$$\left. \begin{aligned} \log(ab) &= \log a + \log b \\ \log(a/b) &= \log a - \log b \\ \log(a^n) &= n \log a \end{aligned} \right\} \text{any base}$$

$$\ln e = 1$$

$$\ln e^a = a$$

$$\ln \left(\frac{1}{a} \right) = -\ln a$$

B.3 Geometry

The **distance** d between two points having coordinates (x_1, y_1) and (x_2, y_2) is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \quad (\text{B.17})$$

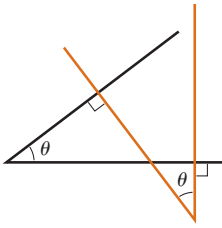


Figure B.4 The angles are equal because their sides are perpendicular.

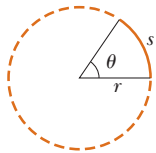


Figure B.5 The angle θ in radians is the ratio of the arc length s to the radius r of the circle.

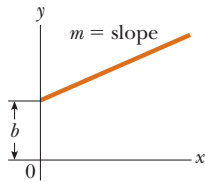


Figure B.6 A straight line with a slope of m and a y -intercept of b .

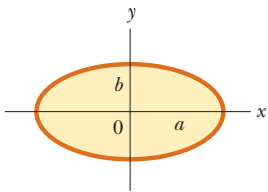


Figure B.7 An ellipse with semi-major axis a and semi-minor axis b .

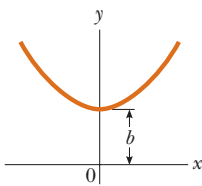


Figure B.8 A parabola with its vertex at $y = b$.

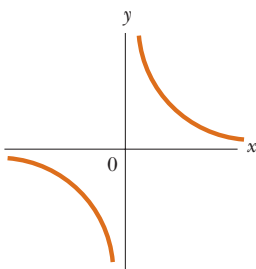
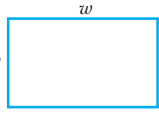
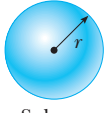

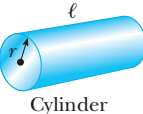
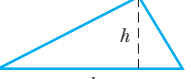
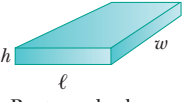


Figure B.9 A hyperbola.

TABLE B.2

Useful Information for Geometry

Shape	Area or Volume	Shape	Area or Volume
 Rectangle	Area = ℓw	 Sphere	Surface area = $4\pi r^2$ Volume = $\frac{4\pi r^3}{3}$
 Circle	Area = πr^2 Circumference = $2\pi r$	 Cylinder	Lateral surface area = $2\pi r\ell$ Volume = $\pi r^2\ell$
 Triangle	Area = $\frac{1}{2}bh$	 Rectangular box	Surface area = $2(\ell h + \ell w + hw)$ Volume = ℓwh

Two angles are equal if their sides are perpendicular, right side to right side and left side to left side. For example, the two angles marked θ in Figure B.4 are the same because of the perpendicularity of the sides of the angles. To distinguish the left and right sides of an angle, imagine standing at the angle's apex and facing into the angle.

Radian measure: The arc length s of a circular arc (Fig. B.5) is proportional to the radius r for a fixed value of θ (in radians):

$$s = r\theta \tag{B.18}$$

$$\theta = \frac{s}{r}$$

Table B.2 gives the **areas** and **volumes** for several geometric shapes used throughout this text.

The equation of a **straight line** (Fig. B.6) is

$$y = mx + b \tag{B.19}$$

where b is the y -intercept and m is the slope of the line.

The equation of a **circle** of radius R centered at the origin is

$$x^2 + y^2 = R^2 \tag{B.20}$$

The equation of an **ellipse** having the origin at its center (Fig. B.7) is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{B.21}$$

where a is the length of the semimajor axis (the longer one) and b is the length of the semi-minor axis (the shorter one).

The equation of a **parabola** the vertex of which is at $y = b$ (Fig. B.8) is

$$y = ax^2 + b \tag{B.22}$$

The equation of a **rectangular hyperbola** (Fig. B.9) is

$$xy = \text{constant} \tag{B.23}$$

B.4 Trigonometry

That portion of mathematics based on the special properties of the right triangle is called trigonometry. By definition, a right triangle is a triangle containing a 90° angle. Consider the right triangle shown in Figure B.10, where side a is opposite the angle θ , side b is adjacent to the angle θ , and side c is the hypotenuse of the triangle. The three

basic trigonometric functions defined by such a triangle are the sine (sin), cosine (cos), and tangent (tan). In terms of the angle θ , these functions are defined as follows:

$$\sin \theta = \frac{\text{side opposite } \theta}{\text{hypotenuse}} = \frac{a}{c} \tag{B.24}$$

$$\cos \theta = \frac{\text{side adjacent to } \theta}{\text{hypotenuse}} = \frac{b}{c} \tag{B.25}$$

$$\tan \theta = \frac{\text{side opposite } \theta}{\text{side adjacent to } \theta} = \frac{a}{b} \tag{B.26}$$

The Pythagorean theorem provides the following relationship among the sides of a right triangle:

$$c^2 = a^2 + b^2 \tag{B.27}$$

From the preceding definitions and the Pythagorean theorem, it follows that

$$\begin{aligned} \sin^2 \theta + \cos^2 \theta &= 1 \\ \tan \theta &= \frac{\sin \theta}{\cos \theta} \end{aligned}$$

The cosecant, secant, and cotangent functions are defined by

$$\csc \theta = \frac{1}{\sin \theta} \quad \sec \theta = \frac{1}{\cos \theta} \quad \cot \theta = \frac{1}{\tan \theta}$$

The following relationships are derived directly from the right triangle shown in Figure B.10:

$$\begin{aligned} \sin \theta &= \cos (90^\circ - \theta) \\ \cos \theta &= \sin (90^\circ - \theta) \\ \cot \theta &= \tan (90^\circ - \theta) \end{aligned}$$

Some properties of trigonometric functions are the following:

$$\begin{aligned} \sin (-\theta) &= -\sin \theta \\ \cos (-\theta) &= \cos \theta \\ \tan (-\theta) &= -\tan \theta \end{aligned}$$

The following relationships apply to *any* triangle as shown in Figure B.11:

$$\begin{aligned} \alpha + \beta + \gamma &= 180^\circ \\ \text{Law of cosines} &\begin{cases} a^2 = b^2 + c^2 - 2bc \cos \alpha \\ b^2 = a^2 + c^2 - 2ac \cos \beta \\ c^2 = a^2 + b^2 - 2ab \cos \gamma \end{cases} \\ \text{Law of sines} &\quad \frac{a}{\sin \alpha} = \frac{b}{\sin \beta} = \frac{c}{\sin \gamma} \end{aligned}$$

Table B.3 (page A-12) lists a number of useful trigonometric identities.

a = opposite side
 b = adjacent side
 c = hypotenuse

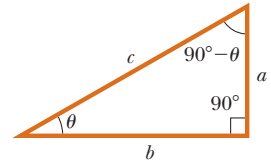


Figure B.10 A right triangle, used to define the basic functions of trigonometry.

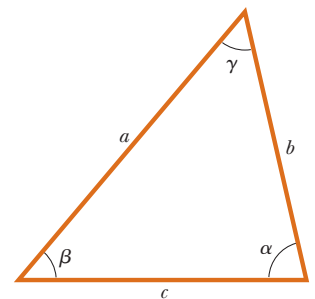


Figure B.11 An arbitrary, nonright triangle.

EXAMPLE B.3

Consider the right triangle in Figure B.12 in which $a = 2.00$, $b = 5.00$, and c is unknown. From the Pythagorean theorem, we have

$$\begin{aligned} c^2 &= a^2 + b^2 = 2.00^2 + 5.00^2 = 4.00 + 25.0 = 29.0 \\ c &= \sqrt{29.0} = 5.39 \end{aligned}$$

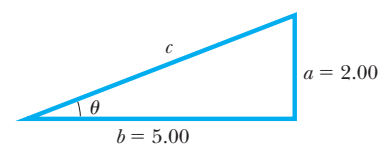


Figure B.12 (Example B.3)

To find the angle θ , note that

$$\tan \theta = \frac{a}{b} = \frac{2.00}{5.00} = 0.400$$

Using a calculator, we find that

$$\theta = \tan^{-1}(0.400) = 21.8^\circ$$

where $\tan^{-1}(0.400)$ is the notation for “angle whose tangent is 0.400,” sometimes written as $\arctan(0.400)$.

TABLE B.3

Some Trigonometric Identities

$\sin^2 \theta + \cos^2 \theta = 1$	$\csc^2 \theta = 1 + \cot^2 \theta$
$\sec^2 \theta = 1 + \tan^2 \theta$	$\sin^2 \frac{\theta}{2} = \frac{1}{2}(1 - \cos \theta)$
$\sin 2\theta = 2 \sin \theta \cos \theta$	$\cos^2 \frac{\theta}{2} = \frac{1}{2}(1 + \cos \theta)$
$\cos 2\theta = \cos^2 \theta - \sin^2 \theta$	$1 - \cos \theta = 2 \sin^2 \frac{\theta}{2}$
$\tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta}$	$\tan \frac{\theta}{2} = \sqrt{\frac{1 - \cos \theta}{1 + \cos \theta}}$
$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$	
$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$	
$\sin A \pm \sin B = 2 \sin \left[\frac{1}{2}(A \pm B) \right] \cos \left[\frac{1}{2}(A \mp B) \right]$	
$\cos A + \cos B = 2 \cos \left[\frac{1}{2}(A + B) \right] \cos \left[\frac{1}{2}(A - B) \right]$	
$\cos A - \cos B = 2 \sin \left[\frac{1}{2}(A + B) \right] \sin \left[\frac{1}{2}(B - A) \right]$	

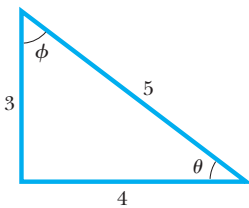


Figure B.13 (Exercise 1)

Exercises

- In Figure B.13, identify (a) the side opposite θ (b) the side adjacent to ϕ and then find (c) $\cos \theta$, (d) $\sin \phi$, and (e) $\tan \phi$.

Answers (a) 3 (b) 3 (c) $\frac{4}{5}$ (d) $\frac{4}{5}$ (e) $\frac{4}{3}$

- In a certain right triangle, the two sides that are perpendicular to each other are 5.00 m and 7.00 m long. What is the length of the third side?

Answer 8.60 m

- A right triangle has a hypotenuse of length 3.0 m, and one of its angles is 30° . (a) What is the length of the side opposite the 30° angle? (b) What is the side adjacent to the 30° angle?

Answers (a) 1.5 m (b) 2.6 m

B.5 Series Expansions

$$(a + b)^n = a^n + \frac{n}{1!} a^{n-1}b + \frac{n(n-1)}{2!} a^{n-2}b^2 + \dots$$

$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!} x^2 + \dots$$

$$\begin{aligned}
 e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \\
 \ln(1 \pm x) &= \pm x - \frac{1}{2}x^2 \pm \frac{1}{3}x^3 - \dots \\
 \sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots \\
 \cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots \\
 \tan x &= x + \frac{x^3}{3} + \frac{2x^5}{15} + \dots \quad |x| < \frac{\pi}{2}
 \end{aligned}
 \left. \vphantom{\begin{aligned} e^x \\ \ln(1 \pm x) \\ \sin x \\ \cos x \\ \tan x \end{aligned}} \right\} \text{ } x \text{ in radians}$$

For $x \ll 1$, the following approximations can be used:¹

$$\begin{aligned}
 (1 + x)^n &\approx 1 + nx & \sin x &\approx x \\
 e^x &\approx 1 + x & \cos x &\approx 1 \\
 \ln(1 \pm x) &\approx \pm x & \tan x &\approx x
 \end{aligned}$$

B.6 Differential Calculus

In various branches of science, it is sometimes necessary to use the basic tools of calculus, invented by Newton, to describe physical phenomena. The use of calculus is fundamental in the treatment of various problems in Newtonian mechanics, electricity, and magnetism. In this section, we simply state some basic properties and “rules of thumb” that should be a useful review to the student.

First, a **function** must be specified that relates one variable to another (e.g., a coordinate as a function of time). Suppose one of the variables is called y (the dependent variable), and the other x (the independent variable). We might have a function relationship such as

$$y(x) = ax^3 + bx^2 + cx + d$$

If a , b , c , and d are specified constants, y can be calculated for any value of x . We usually deal with continuous functions, that is, those for which y varies “smoothly” with x .

The **derivative** of y with respect to x is defined as the limit as Δx approaches zero of the slopes of chords drawn between two points on the y versus x curve. Mathematically, we write this definition as

$$\frac{dy}{dx} = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{y(x + \Delta x) - y(x)}{\Delta x} \tag{B.28}$$

where Δy and Δx are defined as $\Delta x = x_2 - x_1$ and $\Delta y = y_2 - y_1$ (Fig. B.14). Note that dy/dx *does not* mean dy divided by dx , but rather is simply a notation of the limiting process of the derivative as defined by Equation B.28.

A useful expression to remember when $y(x) = ax^n$, where a is a *constant* and n is *any* positive or negative number (integer or fraction), is

$$\frac{dy}{dx} = nax^{n-1} \tag{B.29}$$

If $y(x)$ is a polynomial or algebraic function of x , we apply Equation B.29 to *each* term in the polynomial and take $d[\text{constant}]/dx = 0$. In Examples B.4 through B.7, we evaluate the derivatives of several functions.

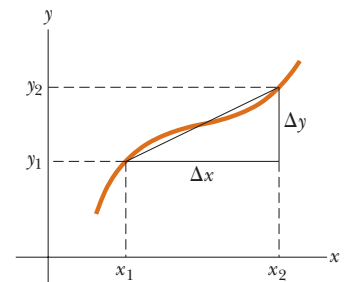


Figure B.14 The lengths Δx and Δy are used to define the derivative of this function at a point.

¹The approximations for the functions $\sin x$, $\cos x$, and $\tan x$ are for $x \leq 0.1$ rad.

TABLE B.4

Derivative for Several Functions

$$\frac{d}{dx}(a) = 0$$

$$\frac{d}{dx}(ax^n) = nax^{n-1}$$

$$\frac{d}{dx}(e^{ax}) = ae^{ax}$$

$$\frac{d}{dx}(\sin ax) = a \cos ax$$

$$\frac{d}{dx}(\cos ax) = -a \sin ax$$

$$\frac{d}{dx}(\tan ax) = a \sec^2 ax$$

$$\frac{d}{dx}(\cot ax) = -a \csc^2 ax$$

$$\frac{d}{dx}(\sec x) = \tan x \sec x$$

$$\frac{d}{dx}(\csc x) = -\cot x \csc x$$

$$\frac{d}{dx}(\ln ax) = \frac{1}{x}$$

$$\frac{d}{dx}(\sin^{-1} ax) = \frac{a}{\sqrt{1 - a^2 x^2}}$$

$$\frac{d}{dx}(\cos^{-1} ax) = \frac{-a}{\sqrt{1 - a^2 x^2}}$$

$$\frac{d}{dx}(\tan^{-1} ax) = \frac{a}{1 + a^2 x^2}$$

Note: The symbols a and n represent constants.

Special Properties of the Derivative

A. Derivative of the product of two functions If a function $f(x)$ is given by the product of two functions—say, $g(x)$ and $h(x)$ —the derivative of $f(x)$ is defined as

$$\frac{d}{dx} f(x) = \frac{d}{dx} [g(x)h(x)] = g \frac{dh}{dx} + h \frac{dg}{dx} \tag{B.30}$$

B. Derivative of the sum of two functions If a function $f(x)$ is equal to the sum of two functions, the derivative of the sum is equal to the sum of the derivatives:

$$\frac{d}{dx} f(x) = \frac{d}{dx} [g(x) + h(x)] = \frac{dg}{dx} + \frac{dh}{dx} \tag{B.31}$$

C. Chain rule of differential calculus If $y = f(x)$ and $x = g(z)$, then dy/dz can be written as the product of two derivatives:

$$\frac{dy}{dz} = \frac{dy}{dx} \frac{dx}{dz} \tag{B.32}$$

D. The second derivative The second derivative of y with respect to x is defined as the derivative of the function dy/dx (the derivative of the derivative). It is usually written as

$$\frac{d^2 y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right) \tag{B.33}$$

Some of the more commonly used derivatives of functions are listed in Table B.4.

EXAMPLE B.4

Suppose $y(x)$ (that is, y as a function of x) is given by

$$y(x) = ax^3 + bx + c$$

where a and b are constants. It follows that

$$\begin{aligned} y(x + \Delta x) &= a(x + \Delta x)^3 + b(x + \Delta x) + c \\ &= a(x^3 + 3x^2 \Delta x + 3x \Delta x^2 + \Delta x^3) + b(x + \Delta x) + c \end{aligned}$$

so

$$\Delta y = y(x + \Delta x) - y(x) = a(3x^2 \Delta x + 3x \Delta x^2 + \Delta x^3) + b \Delta x$$

Substituting this into Equation B.28 gives

$$\begin{aligned} \frac{dy}{dx} &= \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \rightarrow 0} [3ax^2 + 3ax \Delta x + a \Delta x^2] + b \\ \frac{dy}{dx} &= 3ax^2 + b \end{aligned}$$

EXAMPLE B.5

Find the derivative of

$$y(x) = 8x^5 + 4x^3 + 2x + 7$$

Solution Applying Equation B.29 to each term independently and remembering that d/dx (constant) = 0, we have

$$\frac{dy}{dx} = 8(5)x^4 + 4(3)x^2 + 2(1)x^0 + 0$$

$$\frac{dy}{dx} = 40x^4 + 12x^2 + 2$$

EXAMPLE B.6

Find the derivative of $y(x) = x^3/(x+1)^2$ with respect to x .

Solution We can rewrite this function as $y(x) = x^3(x+1)^{-2}$ and apply Equation B.30:

$$\begin{aligned} \frac{dy}{dx} &= (x+1)^{-2} \frac{d}{dx}(x^3) + x^3 \frac{d}{dx}(x+1)^{-2} \\ &= (x+1)^{-2} 3x^2 + x^3(-2)(x+1)^{-3} \end{aligned}$$

$$\frac{dy}{dx} = \frac{3x^2}{(x+1)^2} - \frac{2x^3}{(x+1)^3}$$

EXAMPLE B.7

A useful formula that follows from Equation B.30 is the derivative of the quotient of two functions. Show that

$$\frac{d}{dx} \left[\frac{g(x)}{h(x)} \right] = \frac{h \frac{dg}{dx} - g \frac{dh}{dx}}{h^2}$$

Solution We can write the quotient as gh^{-1} and then apply Equations B.29 and B.30:

$$\begin{aligned} \frac{d}{dx} \left(\frac{g}{h} \right) &= \frac{d}{dx} (gh^{-1}) = g \frac{d}{dx} (h^{-1}) + h^{-1} \frac{d}{dx} (g) \\ &= -gh^{-2} \frac{dh}{dx} + h^{-1} \frac{dg}{dx} \\ &= \frac{h \frac{dg}{dx} - g \frac{dh}{dx}}{h^2} \end{aligned}$$

B.7 Integral Calculus

We think of integration as the inverse of differentiation. As an example, consider the expression

$$f(x) = \frac{dy}{dx} = 3ax^2 + b \tag{B.34}$$

which was the result of differentiating the function

$$y(x) = ax^3 + bx + c$$

in Example B.4. We can write Equation B.34 as $dy = f(x) dx = (3ax^2 + b) dx$ and obtain $y(x)$ by “summing” over all values of x . Mathematically, we write this inverse operation as

$$y(x) = \int f(x) dx$$

For the function $f(x)$ given by Equation B.34, we have

$$y(x) = \int (3ax^2 + b) dx = ax^3 + bx + c$$

where c is a constant of the integration. This type of integral is called an *indefinite integral* because its value depends on the choice of c .

A general **indefinite integral** $I(x)$ is defined as

$$I(x) = \int f(x) dx \tag{B.35}$$

where $f(x)$ is called the *integrand* and $f(x) = dI(x)/dx$.

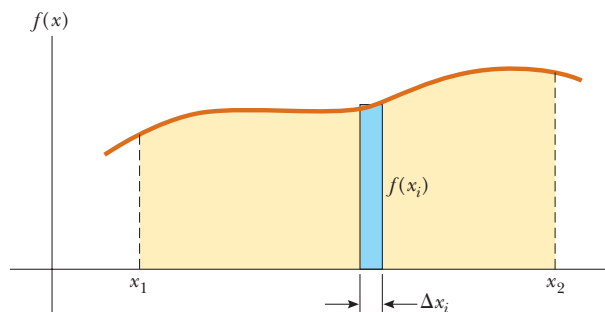
For a *general continuous* function $f(x)$, the integral can be described as the area under the curve bounded by $f(x)$ and the x axis, between two specified values of x , say, x_1 and x_2 , as in Figure B.15.

The area of the blue element in Figure B.15 is approximately $f(x_i) \Delta x_i$. If we sum all these area elements between x_1 and x_2 and take the limit of this sum as $\Delta x_i \rightarrow 0$, we obtain the *true* area under the curve bounded by $f(x)$ and the x axis, between the limits x_1 and x_2 :

$$\text{Area} = \lim_{\Delta x_i \rightarrow 0} \sum_i f(x_i) \Delta x_i = \int_{x_1}^{x_2} f(x) dx \tag{B.36}$$

Integrals of the type defined by Equation B.36 are called **definite integrals**.

Figure B.15 The definite integral of a function is the area under the curve of the function between the limits x_1 and x_2 .



One common integral that arises in practical situations has the form

$$\int x^n dx = \frac{x^{n+1}}{n+1} + c \quad (n \neq -1) \quad (\text{B.37})$$

This result is obvious, being that differentiation of the right-hand side with respect to x gives $f(x) = x^n$ directly. If the limits of the integration are known, this integral becomes a *definite integral* and is written

$$\int_{x_1}^{x_2} x^n dx = \frac{x^{n+1}}{n+1} \Big|_{x_1}^{x_2} = \frac{x_2^{n+1} - x_1^{n+1}}{n+1} \quad (n \neq -1) \quad (\text{B.38})$$

EXAMPLES

$$1. \int_0^a x^2 dx = \frac{x^3}{3} \Big|_0^a = \frac{a^3}{3}$$

$$3. \int_3^5 x dx = \frac{x^2}{2} \Big|_3^5 = \frac{5^2 - 3^2}{2} = 8$$

$$2. \int_0^b x^{3/2} dx = \frac{x^{5/2}}{5/2} \Big|_0^b = \frac{2}{5} b^{5/2}$$

Partial Integration

Sometimes it is useful to apply the method of *partial integration* (also called “integrating by parts”) to evaluate certain integrals. This method uses the property

$$\int u dv = uv - \int v du \quad (\text{B.39})$$

where u and v are *carefully* chosen so as to reduce a complex integral to a simpler one. In many cases, several reductions have to be made. Consider the function

$$I(x) = \int x^2 e^x dx$$

which can be evaluated by integrating by parts twice. First, if we choose $u = x^2$, $v = e^x$, we obtain

$$\int x^2 e^x dx = \int x^2 d(e^x) = x^2 e^x - 2 \int e^x x dx + c_1$$

Now, in the second term, choose $u = x$, $v = e^x$, which gives

$$\int x^2 e^x dx = x^2 e^x - 2x e^x + 2 \int e^x dx + c_1$$

or

$$\int x^2 e^x dx = x^2 e^x - 2x e^x + 2e^x + c_2$$

TABLE B.5

Some Indefinite Integrals (An arbitrary constant should be added to each of these integrals.)

$\int x^n dx = \frac{x^{n+1}}{n+1}$ (provided $n \neq -1$)	$\int \ln ax dx = (x \ln ax) - x$
$\int \frac{dx}{x} = \int x^{-1} dx = \ln x$	$\int xe^{ax} dx = \frac{e^{ax}}{a^2} (ax - 1)$
$\int \frac{dx}{a+bx} = \frac{1}{b} \ln(a+bx)$	$\int \frac{dx}{a+be^{cx}} = \frac{x}{a} - \frac{1}{ac} \ln(a+be^{cx})$
$\int \frac{x dx}{a+bx} = \frac{x}{b} - \frac{a}{b^2} \ln(a+bx)$	$\int \sin ax dx = \frac{1}{a} \cos ax$
$\int \frac{dx}{x(x+a)} = -\frac{1}{a} \ln \frac{x+a}{x}$	$\int \cos ax dx = \frac{1}{a} \sin ax$
$\int \frac{dx}{(a+bx)^2} = -\frac{1}{b(a+bx)}$	$\int \tan ax dx = -\frac{1}{a} \ln(\cos ax) = \frac{1}{a} \ln(\sec ax)$
$\int \frac{dx}{a^2+x^2} = \frac{1}{a} \tan^{-1} \frac{x}{a}$	$\int \cot ax dx = \frac{1}{a} \ln(\sin ax)$
$\int \frac{dx}{a^2-x^2} = \frac{1}{2a} \ln \frac{a+x}{a-x}$ ($a^2-x^2 > 0$)	$\int \sec ax dx = \frac{1}{a} \ln(\sec ax + \tan ax) = \frac{1}{a} \ln \left[\tan \left(\frac{ax}{2} + \frac{\pi}{4} \right) \right]$
$\int \frac{dx}{x^2-a^2} = \frac{1}{2a} \ln \frac{x-a}{x+a}$ ($x^2-a^2 > 0$)	$\int \csc ax dx = \frac{1}{a} \ln(\csc ax - \cot ax) = \frac{1}{a} \ln \left(\tan \frac{ax}{2} \right)$
$\int \frac{x dx}{a^2 \pm x^2} = \pm \frac{1}{2} \ln(a^2 \pm x^2)$	$\int \sin^2 ax dx = \frac{x}{2} + \frac{\sin 2ax}{4a}$
$\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1} \frac{x}{a} = -\cos^{-1} \frac{x}{a}$ ($a^2-x^2 > 0$)	$\int \cos^2 ax dx = \frac{x}{2} + \frac{\sin 2ax}{4a}$
$\int \frac{dx}{\sqrt{x^2+a^2}} = \ln(x + \sqrt{x^2+a^2})$	$\int \frac{dx}{\sin^2 ax} = -\frac{1}{a} \cot ax$
$\int \frac{x dx}{\sqrt{a^2-x^2}} = -\sqrt{a^2-x^2}$	$\int \frac{dx}{\cos^2 ax} = \frac{1}{a} \tan ax$
$\int \frac{x dx}{\sqrt{x^2 \pm a^2}} = \sqrt{x^2 \pm a^2}$	$\int \tan^2 ax dx = \frac{1}{a} (\tan ax) - x$
$\int \sqrt{a^2-x^2} dx = \frac{1}{2} \left(x\sqrt{a^2-x^2} + a^2 \sin^{-1} \frac{x}{a} \right)$	$\int \cot^2 ax dx = -\frac{1}{a} (\cot ax) - x$
$\int x\sqrt{a^2-x^2} dx = -\frac{1}{3}(a^2-x^2)^{3/2}$	$\int \sin^{-1} ax dx = x(\sin^{-1} ax) + \frac{\sqrt{1-a^2x^2}}{a}$
$\int \sqrt{x^2 \pm a^2} dx = \frac{1}{2} [x\sqrt{x^2 \pm a^2} \pm a^2 \ln(x + \sqrt{x^2 \pm a^2})]$	$\int \cos^{-1} ax dx = x(\cos^{-1} ax) - \frac{\sqrt{1-a^2x^2}}{a}$
$\int x(\sqrt{x^2 \pm a^2}) dx = \frac{1}{3}(x^2 \pm a^2)^{3/2}$	$\int \frac{dx}{(x^2+a^2)^{3/2}} = \frac{x}{a^2\sqrt{x^2+a^2}}$
$\int e^{ax} dx = \frac{1}{a} e^{ax}$	$\int \frac{x dx}{(x^2+a^2)^{3/2}} = -\frac{1}{\sqrt{x^2+a^2}}$

TABLE B.6**Gauss's Probability Integral
and Other Definite Integrals**

$$\int_0^{\infty} x^n e^{-ax} dx = \frac{n!}{a^{n+1}}$$

$$I_0 = \int_0^{\infty} e^{-ax^2} dx = \frac{1}{2} \sqrt{\frac{\pi}{a}} \quad (\text{Gauss's probability integral})$$

$$I_1 = \int_0^{\infty} x e^{-ax^2} dx = \frac{1}{2a}$$

$$I_2 = \int_0^{\infty} x^2 e^{-ax^2} dx = -\frac{dI_0}{da} = \frac{1}{4} \sqrt{\frac{\pi}{a^3}}$$

$$I_3 = \int_0^{\infty} x^3 e^{-ax^2} dx = -\frac{dI_1}{da} = \frac{1}{2a^2}$$

$$I_4 = \int_0^{\infty} x^4 e^{-ax^2} dx = \frac{d^2 I_0}{da^2} = \frac{3}{8} \sqrt{\frac{\pi}{a^5}}$$

$$I_5 = \int_0^{\infty} x^5 e^{-ax^2} dx = \frac{d^2 I_1}{da^2} = \frac{1}{a^3}$$

$$\vdots$$

$$I_{2n} = (-1)^n \frac{d^n}{da^n} I_0$$

$$I_{2n+1} = (-1)^n \frac{d^n}{da^n} I_1$$

The Perfect Differential

Another useful method to remember is that of the *perfect differential*, in which we look for a change of variable such that the differential of the function is the differential of the independent variable appearing in the integrand. For example, consider the integral

$$I(x) = \int \cos^2 x \sin x dx$$

This integral becomes easy to evaluate if we rewrite the differential as $d(\cos x) = -\sin x dx$. The integral then becomes

$$\int \cos^2 x \sin x dx = - \int \cos^2 x d(\cos x)$$

If we now change variables, letting $y = \cos x$, we obtain

$$\int \cos^2 x \sin x dx = - \int y^2 dy = -\frac{y^3}{3} + c = -\frac{\cos^3 x}{3} + c$$

Table B.5 (page A-18) lists some useful indefinite integrals. Table B.6 gives Gauss's probability integral and other definite integrals. A more complete list can be found in various handbooks, such as *The Handbook of Chemistry and Physics* (Boca Raton, FL: CRC Press, published annually).

B.8 Propagation of Uncertainty

In laboratory experiments, a common activity is to take measurements that act as raw data. These measurements are of several types—length, time interval, temperature, voltage, and so on—and are taken by a variety of instruments. Regardless of the measurement and the quality of the instrumentation, **there is always uncertainty associated with a physical measurement.** This uncertainty is a combination of that associated with the instrument and that related to the system being measured. An example of the former is the inability to exactly determine the position of a length measurement between the lines on a meterstick. An example of uncertainty related to the system being measured is the variation of temperature within a sample of water so that a single temperature for the sample is difficult to determine.

Uncertainties can be expressed in two ways. **Absolute uncertainty** refers to an uncertainty expressed in the same units as the measurement. Therefore, the length of a computer disk label might be expressed as (5.5 ± 0.1) cm. The uncertainty of ± 0.1 cm by itself is not descriptive enough for some purposes, however. This uncertainty is large if the measurement is 1.0 cm, but it is small if the measurement is 100 m. To give a more descriptive account of the uncertainty, **fractional uncertainty** or **percent uncertainty** is used. In this type of description, the uncertainty is divided by the actual measurement. Therefore, the length of the computer disk label could be expressed as

$$\ell = 5.5 \text{ cm} \pm \frac{0.1 \text{ cm}}{5.5 \text{ cm}} = 5.5 \text{ cm} \pm 0.018 \quad (\text{fractional uncertainty})$$

or as

$$\ell = 5.5 \text{ cm} \pm 1.8\% \quad (\text{percent uncertainty})$$

When combining measurements in a calculation, the percent uncertainty in the final result is generally larger than the uncertainty in the individual measurements. This is called **propagation of uncertainty** and is one of the challenges of experimental physics.

Some simple rules can provide a reasonable estimate of the uncertainty in a calculated result:

Multiplication and division: When measurements with uncertainties are multiplied or divided, add the *percent uncertainties* to obtain the percent uncertainty in the result.

Example: The Area of a Rectangular Plate

$$\begin{aligned} A = \ell w &= (5.5 \text{ cm} \pm 1.8\%) \times (6.4 \text{ cm} \pm 1.6\%) = 35 \text{ cm}^2 \pm 3.4\% \\ &= (35 \pm 1) \text{ cm}^2 \end{aligned}$$

Addition and subtraction: When measurements with uncertainties are added or subtracted, add the *absolute uncertainties* to obtain the absolute uncertainty in the result.

Example: A Change in Temperature

$$\begin{aligned} \Delta T = T_2 - T_1 &= (99.2 \pm 1.5)^\circ\text{C} - (27.6 \pm 1.5)^\circ\text{C} = (71.6 \pm 3.0)^\circ\text{C} \\ &= 71.6^\circ\text{C} \pm 4.2\% \end{aligned}$$

Powers: If a measurement is taken to a power, the percent uncertainty is multiplied by that power to obtain the percent uncertainty in the result.

Example: The Volume of a Sphere

$$\begin{aligned} V = \frac{4}{3}\pi r^3 &= \frac{4}{3}\pi(6.20 \text{ cm} \pm 2.0\%)^3 = 998 \text{ cm}^3 \pm 6.0\% \\ &= (998 \pm 60) \text{ cm}^3 \end{aligned}$$

For complicated calculations, many uncertainties are added together, which can cause the uncertainty in the final result to be undesirably large. Experiments should be designed such that calculations are as simple as possible.

Notice that uncertainties in a calculation always add. As a result, an experiment involving a subtraction should be avoided if possible, especially if the measurements being subtracted are close together. The result of such a calculation is a small difference in the measurements and uncertainties that add together. It is possible that the uncertainty in the result could be larger than the result itself!