SPAZI EUCLIDEI

$\mathbb{E}^n = (S, V^n, \cdot)$, $\qquad \mathbb{A}^n = (S, V^n)$ spazio affine.

$\cdot$: prodotto scalare in V^n.

- Ortogonalità: $\underline{u} \cdot \underline{v} = 0 \qquad (\underline{u} \perp \underline{v})$
- distanza tra due punti $\quad d(P_1, P_2) = \|\overrightarrow{P_1P_2}\| = \sqrt{\overrightarrow{P_1P_2} \cdot \overrightarrow{P_1P_2}}$
- angolo tra due vettori $\underline{u}$ e $\underline{v}$: $\quad \cos\theta = \dfrac{\underline{u} \cdot \underline{v}}{\|\underline{u}\| \cdot \|\underline{v}\|}$

Riferimento euclideo: $\quad \mathcal{R} = \{ O ; \underline{v}_1, \underline{v}_2, \dots, \underline{v}_n \}$

$\mathcal{B} = (\underline{v}_1, \underline{v}_2, \dots, \underline{v}_n)$ base ordinata di V^n. ORTONORMALE.

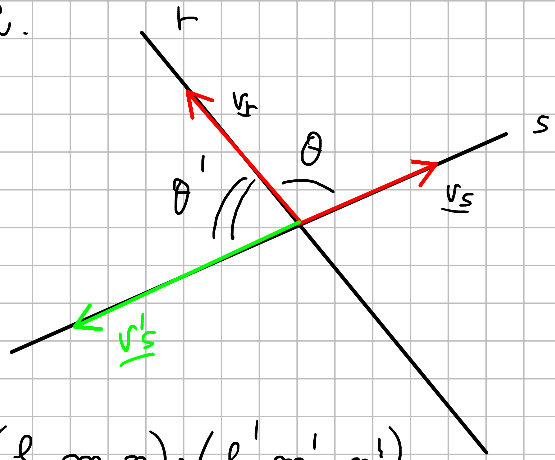
$\underline{v}_i \cdot \underline{v}_j = 0$ se $i \neq j$, $\quad \|\underline{v}_i\| = 1 \quad \forall\, i = 1, \dots, n$.

$P(x_1, x_2, \dots, x_n)_{\mathcal{R}}$: coordinate cartesiane o ORTONORMALI di P rispetto ad $\mathcal{R}$.

$\mathbb{E}^1, \mathbb{E}^2, \mathbb{E}^3$

r, s rette in $\mathbb{E}^3$

angolo θ tra r ed s, $\quad 0 \leq \theta \leq \pi$



$\underline{v}_r = (l, m, n)$

$\underline{v}_s = (l', m', n') \qquad \cos\theta = \pm \dfrac{\underline{v}_r \cdot \underline{v}_s}{\|\underline{v}_r\| \|\underline{v}_s\|} = \pm \dfrac{(l, m, n) \cdot (l', m', n')}{\sqrt{l^2 + m^2 + n^2} \cdot \sqrt{(l')^2 + (m')^2 + (n')^2}}$

(Vale anche in $\mathbb{E}^2$, $\quad \cos\theta = \pm \dfrac{(l, m) \cdot (l', m')}{\sqrt{l^2 + m^2} \cdot \sqrt{(l')^2 + (m')^2}}$)

$\mathbb{E}^3$: $\quad$ asse x: $\begin{cases} y = 0 \\ z = 0 \end{cases}$ $\qquad$ asse y: $\begin{cases} x = 0 \\ z = 0 \end{cases}$ $\qquad$ asse z: $\begin{cases} x = 0 \\ y = 0 \end{cases}$

$\underline{v}_x = \hat{i} = (1, 0, 0) \qquad \underline{v}_y = \hat{j} = (0, 1, 0) \qquad \underline{v}_z = \hat{k} = (0, 0, 1)$

θ_x = angolo che una retta r forma con l'asse x.

$\underline{v}_r = (l, m, n)$, $\quad \cos\theta_x = \pm \dfrac{(l, m, n) \cdot (1, 0, 0)}{\sqrt{l^2 + m^2 + n^2} \cdot 1} = \pm \dfrac{l}{\sqrt{l^2 + m^2 + n^2}}$

θ_y = angolo che r forma con l'asse y

$$\cos\theta_y = \pm \frac{(l,m,n)\cdot(0,1,0)}{\sqrt{l^2+m^2+n^2}} = \pm \frac{m}{\sqrt{l^2+m^2+n^2}}$$

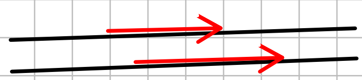
$$\cos\theta_z = \pm \frac{(l,m,n)\cdot(0,0,1)}{\sqrt{l^2+m^2+n^2}} = \frac{n}{\sqrt{l^2+m^2+n^2}}$$

$\cos\theta_x$, $\cos\theta_y$, $\cos\theta_z$ sono i COSENI DIRETTORI della retta r.

$$\cos^2\theta_x + \cos^2\theta_y + \cos^2\theta_z = \frac{l^2}{l^2+m^2+n^2} + \frac{m^2}{l^2+m^2+n^2} + \frac{n^2}{l^2+m^2+n^2}$$

$$= \frac{l^2+m^2+n^2}{l^2+m^2+n^2} = 1$$

(Idem per r in $\mathbb{E}^2$)



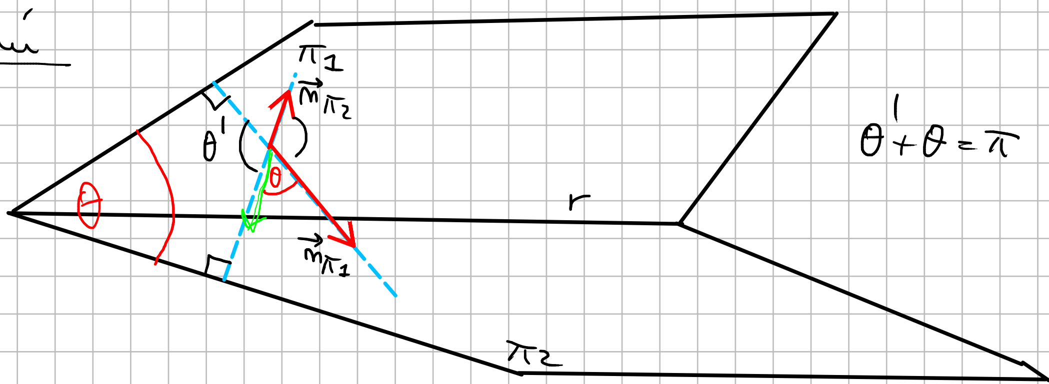
Se $r \parallel s$, $\theta_{rs} = 0$

Quando $r \perp s$? $\theta_{rs} = \frac{\pi}{2}$, $\cos\theta_{r,s} = 0$, $\frac{\underline{v}_r \cdot \underline{v}_s}{\|\underline{v}_r\| \cdot \|\underline{v}_s\|} = 0$, $\underline{v}_r \cdot \underline{v}_s = 0$

$$ll' + mm' + nn' = 0$$

Angolo tra due piani

$$\cos\theta = \pm\cos\theta' = \pm \frac{\vec{n}_{\pi_1}\cdot\vec{n}_{\pi_2}}{\|\vec{n}_{\pi_1}\|\|\vec{n}_{\pi_2}\|}$$

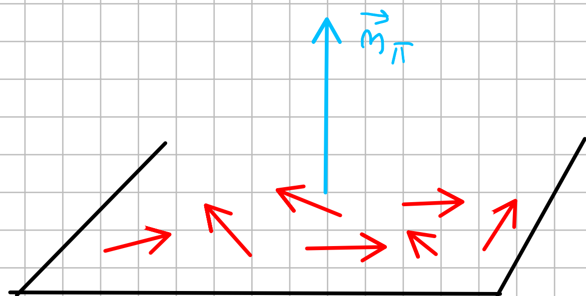


$\theta' + \theta = \pi$

$\pi \perp r$

$\pi: ax + by + cz + d = 0.$

$$W_\pi = \{ \underline{v} = (x,y,z) : ax + by + cz = 0 \}$$
$$\downarrow$$
$$(a,b,c)\cdot(x,y,z) = 0$$



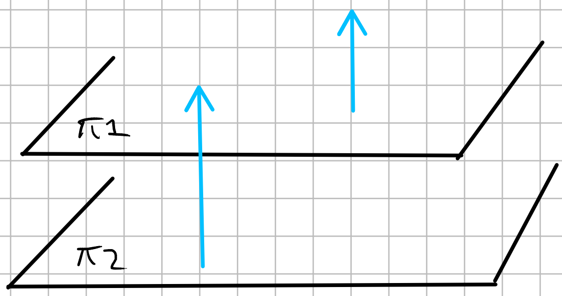
$\vec{n}_\pi = (a,b,c)$ vettore normale al piano π.

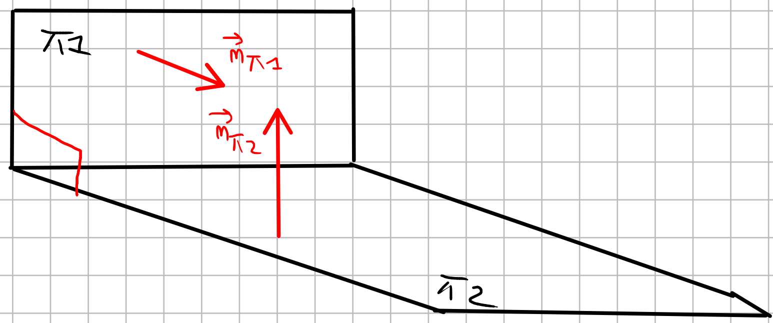
$$\hat{n}_\pi = \frac{(a,b,c)}{\|\vec{n}_\pi\|} = \frac{(a,b,c)}{\sqrt{a^2+b^2+c^2}}$$

$\pi_1 \perp \pi_2 \iff \vec{n}_{\pi_1} \cdot \vec{n}_{\pi_2} = 0 \iff aa' + bb' + cc' = 0$

$\pi_1 : ax + by + cz + d = 0$

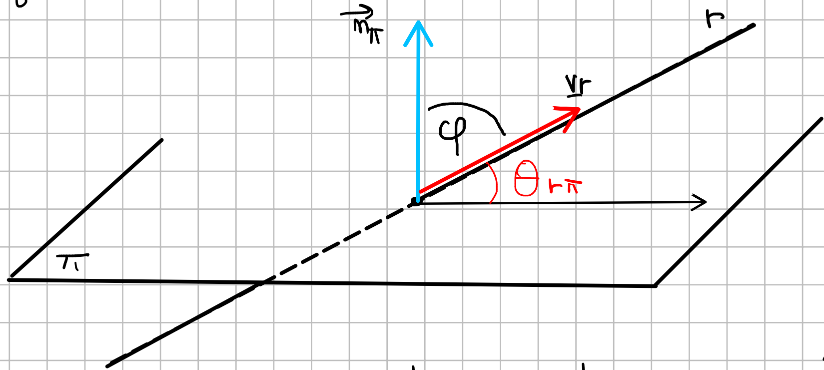
$\pi_2 : a'x + b'y + c'z + d' = 0$





$\pi_1 \parallel \pi_2 \iff \vec{n}_{\pi_1} \parallel \vec{n}_{\pi_2}$

Angolo tra una retta r e un piano π:



$0 \le \theta_{r\pi} \le \pi$

$0 \le \operatorname{sen} \theta_{r\pi} \le 1$

$|\cos \varphi| = \operatorname{sen} \theta_{r\pi}$

$\pi : ax + by + cz + d = 0$

$\underline{v_r} = (l, m, n)$

$$\operatorname{sen} \theta_{r\pi} = |\cos \varphi| = \frac{|\vec{n}_\pi \cdot \underline{v_r}|}{\|\vec{n}_\pi\| \|\underline{v_r}\|}$$

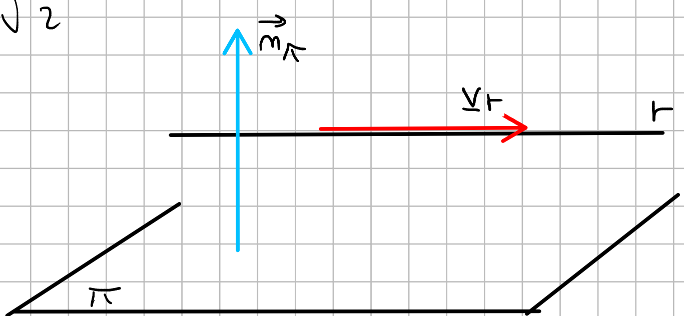
$$= \frac{|(a, b, c) \cdot (l, m, n)|}{\sqrt{a^2 + b^2 + c^2}\,\sqrt{l^2 + m^2 + n^2}} = \frac{|al + bm + cn|}{\sqrt{a^2 + b^2 + c^2}\,\sqrt{l^2 + m^2 + n^2}}$$

- $\pi : 3x + 2y - z - 1 = 0 \qquad r : \begin{cases} x = t \\ y = 5 \\ z = 1 + t \end{cases} \qquad \theta_{r\pi} = ?$

$$\operatorname{sen} \theta_{r\pi} = \frac{|(3, 2, -1) \cdot (1, 0, 1)|}{\sqrt{9 + 4 + 1} \cdot \sqrt{2}} = \frac{|3 - 1|}{\sqrt{28}} = \frac{2}{2\sqrt{7}} = \frac{1}{\sqrt{7}}$$

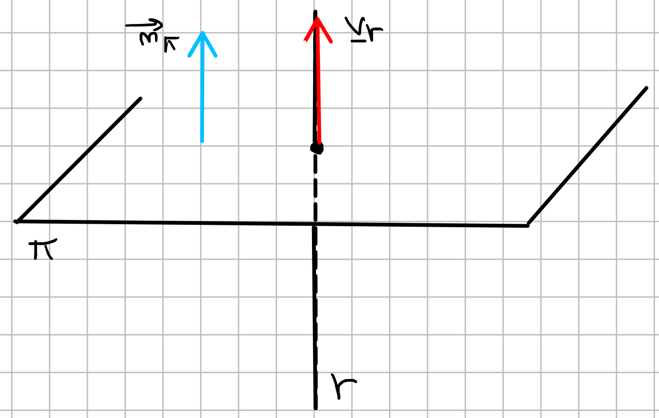
$\theta_{r\pi} = \arcsin\left(\frac{1}{\sqrt{7}}\right).$

$r \parallel \pi$



$\underline{v_r} \in W_\pi$

$\underline{v_r} \cdot \vec{n}_\pi = 0$



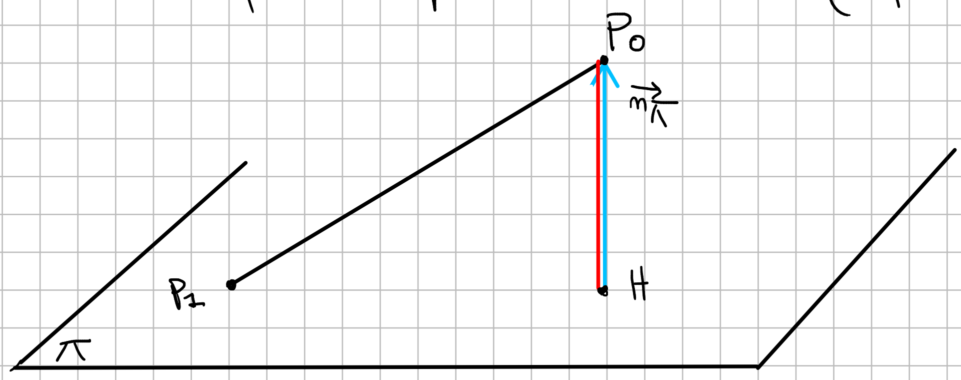
$\vec{m}_\pi \parallel \underline{v}_r \iff r \perp \pi$

DISTANZE

Distanza tra due punti: $P_1(x_1, y_1, z_1)$, $P_2(x_2, y_2, z_2)$

$$d(P_1,P_2) = \| \overrightarrow{P_1P_2} \| = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$$

Distanza punto-piano in $\mathbb{E}^3$ (punto-retta in $\mathbb{E}^2$)



$P_0(x_0, y_0, z_0)$

$$d(P_0,\pi) = \min\{d(P_0,P_1) : P_1 \in \pi\} = d(P_0,H)$$

$\pi: ax+by+cz+d=0$

$P_1(x_1,y_1,z_1) \in \pi \Rightarrow ax_1+by_1+cz_1+d=0$

$$d(P_0,\pi) = \left|C_{\vec{m}_\pi}(\overrightarrow{P_1P_0})\right| = \left|\overrightarrow{P_1P_0} \cdot \hat{m}_\pi\right| =$$

$$= \left|(x_0-x_1, y_0-y_1, z_0-z_1) \cdot \frac{(a,b,c)}{\sqrt{a^2+b^2+c^2}}\right| =$$

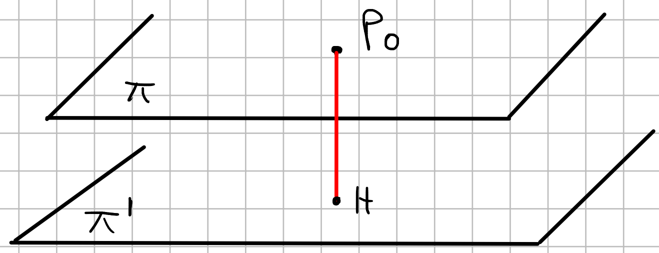
$$= \frac{|a(x_0-x_1)+b(y_0-y_1)+c(z_0-z_1)|}{\sqrt{a^2+b^2+c^2}} = \frac{|ax_0+by_0+cz_0-\overbrace{(ax_1+by_1+cz_1)}^{-d}|}{\sqrt{}}$$

$$= \frac{|ax_0+by_0+cz_0+d|}{\sqrt{a^2+b^2+c^2}}$$

Distanza tra due piani paralleli π e π':

$\pi: ax+by+cz+d=0$

$\pi': ax+by+cz+d'=0$



$$d(\pi,\pi') = \min\{ d(P_0,P_1) : P_0 \in \pi, \ P_1 \in \pi' \}$$

$$= d(P_0,\pi') \text{ con } P_0 \in \pi$$

$$= \frac{|\underbrace{ax_0+by_0+cz_0}_{-d}+d'|}{\sqrt{a^2+b^2+c^2}} = \frac{|d'-d|}{\sqrt{a^2+b^2+c^2}}$$

$P_0(x_0,y_0,z_0) \in \pi$

$ax_0+by_0+cz_0+d=0$

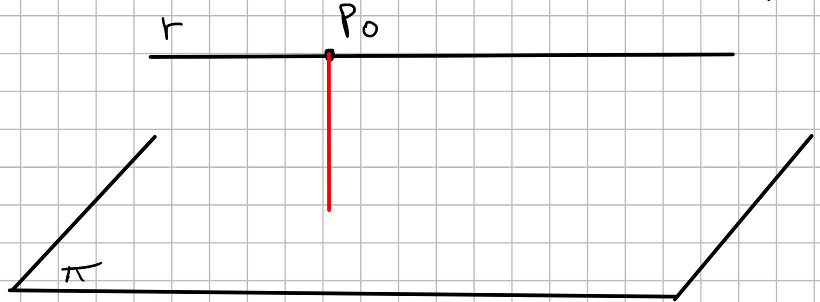
- $\pi: 8x-2y+6z+21=0$
- $\pi': 4x-y+3z-10=0$

$\frac{8}{4} = \frac{-2}{-1} = \frac{6}{3}$ $\quad \pi \parallel \pi'$

$\pi: 4x-y+3z+\frac{21}{2}=0$

$$d(\pi,\pi') = \frac{|-10-\frac{21}{2}|}{\sqrt{16+1+9}} = \frac{|-\frac{41}{2}|}{\sqrt{26}} = \frac{41}{2\sqrt{26}}$$

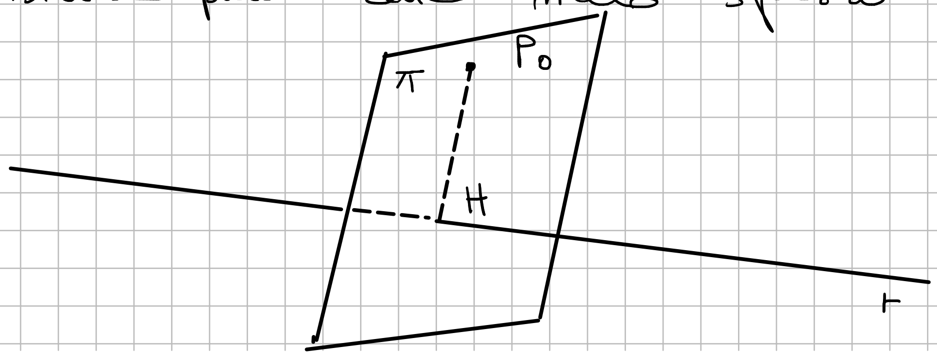
Distanza tra una retta ed un piano paralleli



$d(r,\pi) = d(P_0,\pi)$

$P_0 \in r$

Distanza punto-retta nello spazio



π: piano passante per P_0 e ortogonale ad r

$\pi \cap r = \{H\}$

$$d(P_0, r) = \min\{d(P_0, P) : P \in r\} = d(P_0, H)$$

$r: \begin{cases} x - 2y = 0 \\ y + z + 1 = 0 \end{cases}$ $\quad P_0(3, 2, -1)$ $\quad \underline{v}_r : \begin{cases} x - 2y = 0 \\ y + z = 0 \end{cases}$

$\pi \perp r \Rightarrow \underline{v}_r = \vec{n}_\pi$, $\quad \underline{v}_r = (2, 1, -1)$

$\pi: ax + by + cz + d = 0$, $\quad 2x + y - z + d = 0$

$2 \cdot 3 + 2 - (-1) + d = 0$, $\quad 9 + d = 0$, $\quad d = -9$

$\pi: 2x + y - z - 9 = 0$

$\pi \cap r : \begin{cases} 2x + y - z - 9 = 0 \\ x - 2y = 0 \\ y + z + 1 = 0 \end{cases}$ $\quad \begin{cases} x = \frac{8}{3} \\ y = \frac{4}{3} \\ z = -\frac{7}{3} \end{cases}$ $\quad d(P_0, H) = ?$

$H\left(\frac{8}{3}, \frac{4}{3}, -\frac{7}{3}\right)$ $\quad \| \quad d(P_0, r)$