The classical isoperimetric theorem

Nota del Socio Corrispondente Nicola Fusco *

(Adunanza del 7 maggio 2004)

Abstract. We give an introduction to some basic concepts and results of geometric measure theory. Then, this material is used to present De Giorgi's proof of the isoperimetric property of the sphere.
Sunto Diamo un'introduzione ai concetti e ai risultati di base della teoria geometrica della misura. Questo materiale introduttivo viene quindi utilizzato per presentare la dimostrazione di De Giorgi della proprietà isoperimetrica della sfera.

1 Introduction

After almost 50 years since it was published the paper by De Giorgi concerning the isoperimetric property of the sphere ([17]) still strucks the reader for its elegance, brilliant simplicity and essentiality. It is amazing that he was able to derive such a fundamental result starting from a quite general definition of perimeter. However, looking at the whole theory of sets of finite perimeter, one gradually realizes that definition (1.2) is much deeper than it appears at first sight.

These notes were written up for a six lectures course given by the author in July 2003 at a summer school in Mondello. The aim is to give a self contained proof of De Giorgi's result. In Section 2 we recall all definitions and basic properties of Hausdorff measures needed for the sequel, including area and coarea formulas. Section 3 contains the essential material on BV functions and sets of finite perimeter. Most results presented in this section are provided with proofs except Theorem 3.15 concerning the structure of sets of finite perimeter. In the first part of Section 4 we present a few results on Steiner symmetrization of sets of finite perimeter, while the isoperimetric property of the sphere is proved in the last part of the section. Finally, in the present section we give an elementary proof of the isoperimetric theorem in the plane covering the case of a domain whose boundary is an absolutely continuous curve.

*Dipartimento di Matematica e Applicazioni, Università di Napoli 'Federico II', Via Cintia, 80126 Napoli (Italy); e-mail: n.fusco@unina.it

In writing these notes my basic references for Sections 2 and 3 have been the books [3], [20] and [19] to which the reader may refer for further information and for all the statements which are not proved here. The two dimensional proof of the isoperimetric theorem is taken from the survey paper [38], while the material presented in Section 4.1 is taken from the recent paper [14]. Finally, the proof of the isoperimetric theorem contained in Section 4.2 makes use of the technical Lemma 4.12 for which I would like to thank Giovanni Alberti with whom I have discussed the subject.

Here I have made no mention of any the various results related to the classical isoperimetric theorem, usually called "isoperimetric inequalities". Good references for this subject are the books [32], [4], [9], [27], and the papers [30], [13], [28], [29], [5], [22], [31]. Further extensions of the classical isoperimetric inequality within the framework of riemannian manifolds or currents are contained respectively in [21] and in [1], [2].

1.1 The isoperimetric theorem

The classical isoperimetric theorem says that a plane curve which encloses a prescribed area and has the shortest length is a circle. Mathematicians are well acquainted with this property of the circle since more that 2500 years, but the first serious attempts to give it a rigorous proof are relatively recent. In this regard, I would like to mention the early proofs given by Steiner [37] and Edler [18] in the nineteenth century, together with later contributions (and more complete proofs) published in the first half of the twentieth century by Hurwitz [23], [24], Minkowski [26], Lebesgue [25], Carathéodory and Study [12], Blaschke [6], [7], and Bonnesen [8].

The Euclidean space counterpart of the isoperimetric theorem states that among all surfaces enclosing a prescribed volume the one with lest area is the sphere. Early proofs of this result were given by Tonelli [39], Schmidt [34], [35], [36], Radò [33]. For all these authors the relevant question was to prove the isoperimetric property of the sphere in larger classes of competing sets for which a suitable notion of surface area could be defined.

This problem was completely solved by De Giorgi who considered the largest possible class of competing sets, the Lebesgue measurable sets. Starting from some previous papers of Caccioppoli [10], [11], he showed that one can assign to each measurable set E in $\mathbb{R}^n$ a suitable measure of the boundary, i.e. a number $P(E) \in [0, \infty]$, that he called the *perimeter* of E. Of course, if E is a smooth open set its perimeter agrees with the classical measure of the boundary ∂E. By using this notion of perimeter he was able to prove that any measurable set E with finite measure satisfies the following *isoperimetric inequality*

$$[\mathcal{L}^n(E)]^{\frac{n-1}{n}} \le \frac{1}{n\omega_n^{1/n}} P(E) \tag{1.1}$$

and that the equality holds if and only if E is (equivalent to) a ball. Here and in the sequel we denote the usual Lebesgue measure in $\mathbb{R}^n$ by the symbol $\mathcal{L}^n$, while ω_n stands for the measure of the unit ball in $\mathbb{R}^n$.

In the definition of perimeter and in De Giorgi's approach to the isoperimetric inequality (1.1) a key role is played by his extension of the classical *Gauss–Green* formulas. To understand this, let us consider a bounded open set E with C^1 boundary and let us take any map $\varphi \in C^1_0(\mathbb{R}^n;\mathbb{R}^n)$. Then, the divergence theorem says that

$$\int_E \mathrm{div}\varphi \, dx = -\int_{\partial E} \langle \varphi, \nu^E \rangle \, d\sigma \, ,$$

where ν^E is the *inner normal* to the boundary of E and σ denotes the usual surface measure on ∂E. From this formula we get

$$\begin{aligned} \sigma(\partial E) &= \sup\Big\{ \int_{\partial E} \langle \varphi, \nu^E \rangle \, d\sigma : \varphi \in C^1_0(\mathbb{R}^n;\mathbb{R}^n), \ \|\varphi\|_\infty \le 1 \Big\} \\ &= \sup\Big\{ \int_E \mathrm{div}\varphi \, dx : \varphi \in C^1_0(\mathbb{R}^n;\mathbb{R}^n), \ \|\varphi\|_\infty \le 1 \Big\} . \end{aligned}$$

Notice that the last supremum in this formula makes sense for any measurable set. Thus, we may define the the perimeter $P(E)$ of a measurable set E by setting

$$P(E) = \sup\Big\{ \int_E \mathrm{div}\varphi \, dx : \varphi \in C^1_0(\mathbb{R}^n;\mathbb{R}^n), \ \|\varphi\|_\infty \le 1 \Big\} . \tag{1.2}$$

Though the definition originally given by De Giorgi in [15] is a different one, it may be easily proved that his definition is equivalent to (1.2). As we shall see in Section 3, definition (1.2) relates sets of finite perimeter to BV functions. On the other hand, the original definition of De Giorgi has the advantage of providing a very useful approximation of the characteristic function of a set of finite perimeter by means of smooth functions, a property that we shall deduce from the theory of BV functions.

1.2 The isoperimetric inequality in the plane

In this section we present a proof of the isoperimetric inequality in the plane, due to Hurwitz, which applies to any bounded domain E whose boundary is a simple, closed, absolutely continuous curve. This proof is based on the Gauss–Green formulas and on the Wirtinger inequality, which is just the one dimensional Sobolev–Poincaré inequality stated with the sharp constant. This is by no means surprising, as we shall see that also in higher dimension the isoperimetric inequality is very much related to the Sobolev imbedding theorem and to the Sobolev–Poincaré inequality. But first notice that in two dimensions inequality (1.1) can be equivalently written in the form

$$L^2 \geq 4\pi A\,,$$

where L is the length of the curve enclosing E and $A = \mathcal{L}^2(E)$ is the area of E.

Theorem 1.1 (Isoperimetric inequality in the plane) *Let E be a bounded domain in the plane, whose boundary is a simple, closed and absolutely continuous curve γ. Then,*

$$L^2 \geq 4\pi A\,. \tag{1.3}$$

Moreover, if the inequality in (1.3) holds as an equality, E is a disk.

Proof. Let us orient the boundary γ of E counterclockwise. Then, given any point (x_0, y_0) in the plane, from the Gauss–Green formulas we get that

$$A = \frac{1}{2}\int_\gamma -ydx + xdy = \int_\gamma (y_0 - y)dx + (x - x_0)dy\,.$$

Let $(x(s), y(s)) : [0, L] \mapsto \mathbb{R}^2$ be a parametrization of γ, where the parameter s is the arclength, hence $x'^2 + y'^2 = 1$. From the equality above, using the Cauchy–Schwartz inequality, we get

$$\begin{aligned} A &= \frac{1}{2}\int_0^L \big[(y_0 - y(s))x'(s) + (x(s) - x_0)y'(s)\big]\,ds \\ &\leq \frac{1}{2}\bigg(\int_0^L \big[(y_0 - y(s))^2 + (x(s) - x_0)^2\big]\,ds\bigg)^{1/2}\bigg(\int_0^L \big(x'^2 + y'^2\big)\,ds\bigg)^{1/2} \\ &= \frac{\sqrt{L}}{2}\bigg(\int_0^L \big[(y_0 - y(s))^2 + (x(s) - x_0)^2\big]\,ds\bigg)^{1/2}. \end{aligned} \tag{1.4}$$

If (x_0, y_0) is the baricenter of γ, we get

$$x(0) - x_0 = x(L) - x_0, \qquad\qquad \int_0^L (x(s) - x_0)\,ds = 0$$

and analogous relations for $y(s)$. By applying Wirtinger inequality (1.6) to the functions $x(s) - x_0$ and $y(s) - y_0$, we obtain

$$\int_0^L \big[(y_0 - y(s))^2 + (x(s) - x_0)^2\big]\,ds \leq \frac{L^2}{4\pi^2}\int_0^L \big(x'^2 + y'^2\big)\,ds = \frac{L^3}{4\pi^2}\,. \tag{1.5}$$

Then, (1.3) immediately follows from this inequality and from (1.4).
Let us now assume that equality holds in (1.3). Then, in particular, the equality also holds in (1.5). Therefore, from Proposition 1.2 we get that

$$x(s) - x_0 = a \sin \frac{2\pi s}{L} + b \cos \frac{2\pi s}{L}, \qquad y(s) - y_0 = c \sin \frac{2\pi s}{L} + d \cos \frac{2\pi s}{L}.$$

Since also (1.4) holds as an equality, we get that there exists a constant λ such that $y(s) - y_0 = \lambda x'(s) = \frac{2\pi\lambda}{L}\Big(a \cos \frac{2\pi s}{L} - b \sin \frac{2\pi s}{L}\Big)$. Therefore, setting $\mu = \sqrt{a^2 + b^2}$, $a = \mu \sin \alpha$, $b = \mu \cos \alpha$, we easily get that

$$x(s) - x_0 = \mu \cos\Big(\frac{2\pi s}{L} - \alpha\Big), \qquad y(s) - y_0 = -\frac{2\pi\lambda}{L}\mu \sin\Big(\frac{2\pi s}{L} - \alpha\Big).$$

From these equalities, recalling that $x'^2 + y'^2 = 1$, we may conclude that $\mu = |\lambda| = L/2\pi$. Therefore, since γ is oriented counterclockwise, we finally get that

$$x(s) = x_0 + \frac{L}{2\pi} \cos\Big(\frac{2\pi s}{L} - \alpha\Big), \qquad y(s) = y_0 + \frac{L}{2\pi} \sin\Big(\frac{2\pi s}{L} - \alpha\Big),$$

hence the assertion follows. □

Proposition 1.2 (Wirtinger inequality) *Let (a, b) be a bounded interval and $u \in W^{1,2}(a, b)$ such that*

$$u(a) = u(b), \qquad \int_a^b u(t)\, dt = 0.$$

Then,

$$\int_a^b u^2(t)\, dt \le \frac{(b-a)^2}{4\pi^2} \int_a^b u'^2(t)\, dt \tag{1.6}$$

and the equality holds if and only if $u = c_1 \sin \frac{2\pi(t-a)}{b-a} + c_2 \cos \frac{2\pi(t-a)}{b-a}$.

Proof. It is enough to prove the assertion in the case when the interval (a, b) coincides with $(0, 2\pi)$. The general case then follows by a simple change of variable. Let us fix $u \in W^{1,2}(0, 2\pi)$ and set for any $k \in \mathbb{Z}$

$$a_k = \frac{1}{2\pi} \int_0^{2\pi} u(t) \mathrm{e}^{-ikt}\, dt, \qquad b_k = \frac{1}{2\pi} \int_0^{2\pi} u'(t) \mathrm{e}^{-ikt}\, dt.$$

The assumption $\int_0^{2\pi} u(t)dt = 0$ yields $a_0 = 0$, while the equality $u(0) = u(2\pi)$ implies that $b_k = ika_k$ for any k. Therefore we have

$$\int_0^{2\pi} |u|^2 dt = 2\pi \sum_{k=-\infty}^{+\infty} |a_k|^2 = 2\pi \sum_{k\neq 0} |a_k|^2, \qquad \int_0^{2\pi} |u'|^2 dt = 2\pi \sum_{k=-\infty}^{+\infty} |b_k|^2 = 2\pi \sum_{k\neq 0} k^2 |a_k|^2$$

and thus (1.6) follows from the trivial inequality

$$\sum_{k\neq 0} k^2 |a_k|^2 \geq \sum_{k\neq 0} |a_k|^2 .$$

Notice that this last inequality implies also that the equality in (1.6) holds if and only if $a_k = 0$ for any $k \neq 1, -1$, i.e. if and only if u is a linear combination of e^{it} and e^{-it}. □

2 Hausdorff measures

2.1 Definition and basic properties

Before giving the definition and the main properties of Hausdorff measures in $\mathbb{R}^n$, let us fix some notation and recall some elementary facts of measure theory needed in the sequel.

Let $\mu : \mathcal{P}(\mathbb{R}^n) \to [0, \infty]$ be a set function defined on all subsets of $\mathbb{R}^n$. We say that μ is an *outer measure* if

(i) $\mu(\emptyset) = 0$;

(ii) for any sequence E_h of subsets of $\mathbb{R}^n$, then $\mu\Big(\bigcup_{h\in\mathbb{N}} E_h\Big) \leq \sum_{h\in\mathbb{N}} \mu(E_h)$.

In the sequel an outer measure will be also called a *'measure'*. Starting from a measure μ one can give the notion of *μ-measurable* set. A set E is said to be μ-measurable if

$$\mu(F) = \mu(F \cap E) + \mu(F \setminus E) \qquad \text{for any subset } F \text{ of } \mathbb{R}^n . \tag{2.1}$$

Notice that to prove that a set E is μ-measurable it is enough to show that for any F the left hand side in (2.1) is greater than or equal to the right hand side, since the reverse inequality follows from the subadditivity of μ.

Outer measure behave nicely on measurable sets, as it is shown by the next result (see [19, Theorem 1, Section 1.1]).

Theorem 2.1 *Let μ be an outer measure and $\mathcal{M}_\mu$ the family of all μ-measurable sets. Then $\mathcal{M}_\mu$ is a σ-algebra and μ is countably additive on $\mathcal{M}_\mu$, i.e.*

$$E_h \textit{ is a sequence of pairwise disjoint sets in } \mathcal{M}_\mu \quad \Longrightarrow \quad \mu\Big(\bigcup_{h\in\mathbb{N}} E_h\Big) = \sum_{h\in\mathbb{N}} \mu(E_h)\,.$$

When the σ-algebra $\mathcal{M}_\mu$ of μ-measurable sets contains the σ-algebra $\mathcal{B}(\mathbb{R}^n)$ of Borel subsets of $\mathbb{R}^n$ the measure μ is called a *Borel measure.* If μ is also finite on compact sets, then we say that μ is a (positive) *Radon measure.* The next result, due to Carathéodory, provides a useful criterion ensuring that an outer measure is a Borel measure.

Theorem 2.2 *Let μ be an outer measure with the property that*

$$\mu(E\cup F) = \mu(E)+\mu(F) \quad \textit{for any } E, F \subset \mathbb{R}^n \textit{ such that } \operatorname{dist}(E,F) > 0\,. \tag{2.2}$$

Then μ is a Borel measure.

Proof. To prove the assertion it is enough to show that if C is a closed subset of $\mathbb{R}^n$, then

$$\mu(F) \geq \mu(F\cap C) + \mu(F\setminus C) \quad \text{for any subset } F \text{ of } \mathbb{R}^n \text{ with } \mu(F) < \infty\,. \tag{2.3}$$

In fact, as we have observed above, from this inequality it follows that C is μ-measurable, hence $\mathcal{M}_\mu$ contains the closed sets. Therefore, $\mathcal{M}_\mu$ contains also the smallest σ-algebra containing the closed sets, i.e. the σ-algebra of Borel sets.
Let us fix a set F of finite μ-measure and set $F_0 = \{x \in F : \operatorname{dist}(x, C) \geq 1\}$ and

$$F_i = \Big\{x \in F : \frac{1}{i+1} \leq \operatorname{dist}(x,C) < \frac{1}{i}\Big\} \qquad \text{for } i \geq 1\,.$$

From assumption (2.2) it follows that for any h

$$\sum_{i=0}^{h} \mu(F_{2i}) = \mu\Big(\bigcup_{i=0}^{h} F_{2i}\Big) \leq \mu(F), \qquad \sum_{i=0}^{h} \mu(F_{2i+1}) = \mu\Big(\bigcup_{i=0}^{h} F_{2i+1}\Big) \leq \mu(F)\,,$$

hence the series $\sum_i \mu(F_i)$ is convergent. Since $F\setminus C = \bigcup_{i=0}^{\infty} F_i$, by recalling (2.2) and using the countable subadditivity of μ, we get that for any h

$$\begin{aligned}\mu(F\cap C) + \mu(F\setminus C) \;&\leq\; \mu(F\cap C) + \mu\Big(\bigcup_{i=0}^{h} F_h\Big) + \sum_{i=h}^{\infty} \mu(F_i) \\ &= \mu\Big((F\cap C) \cup \bigcup_{i=0}^{h} F_i\Big) + \sum_{i=h}^{\infty} \mu(F_i) \leq \mu(F) + \sum_{i=h}^{\infty} \mu(F_i)\,.\end{aligned}$$

Then (2.3) follows from the previous inequality, letting $h \to \infty$. $\square$

If μ is an outer measure and B is a Borel subset of $\mathbb{R}$, we shall denote by $\mu \llcorner B$ the outer measure defined for any $E \subset \mathbb{R}^n$ by

$$\mu \llcorner B(E) = \mu(B \cap E)\,.$$

If μ is a Borel measure, $\mu \llcorner B$ is Borel too.

Example 2.3 The best known example of outer measure is Lebesgue measure on $\mathbb{R}^n$, which is defined by setting, for any $E \subset \mathbb{R}^n$,

$$\mathcal{L}^n(E) = \inf\left\{\sum_{h=1}^{\infty} |Q_h| :\, E \subset \bigcup_{h=1}^{\infty} Q_h\right\},$$

where by Q we denote an open cube with sides parallel to the coordinate axes and by $|Q|$ its elementary measure. From this definition, using Theorem 2.2, it can be easily checked that $\mathcal{L}^n$ is a positive Radon measure.

In the sequel the unit ball in $\mathbb{R}^n$ will be denoted by B_1. We recall that the Lebesgue measure of B_1, denoted by ω_n, is given by

$$\omega_n = \mathcal{L}^n(B_1) = \frac{\pi^{n/2}}{\Gamma(1+n/2)}\,,$$

where $\Gamma(t) = \displaystyle\int_0^{+\infty} s^{t-1} \mathrm{e}^{-s} ds$, for $t > 0$, is the Euler Γ-function. Similarly, if s is a nonnegative real number, we shall denote by ω_s the quantity

$$\omega_s = \frac{\pi^{s/2}}{\Gamma(1+s/2)}\,.$$

Definition 2.4 Let $s \geq 0$, $\delta \in (0, \infty]$. The *s-dimensional Hausdorff pre-measure* $\mathcal{H}^s_\delta$ is defined by setting for any subset E of $\mathbb{R}^n$

$$\mathcal{H}^s_\delta(E) = \inf\left\{\frac{\omega_s}{2^s}\sum_{h=1}^{\infty} (\operatorname{diam} C_h)^s :\, E \subset \bigcup_{h=1}^{\infty} C_h,\ \operatorname{diam} C_h < \delta\right\}.$$

Starting from the measures $\mathcal{H}^s_\delta$, we may define the *s-dimensional Hausdorff measure* of E setting

$$\mathcal{H}^s(E) = \sup_{\delta>0} \mathcal{H}^s_\delta(E) = \lim_{\delta \to 0^+} \mathcal{H}^s_\delta(E)\,. \tag{2.4}$$

Notice that since $\mathcal{H}^s_{\delta_1}(E)$ is a decreasing function of δ the limit appearing in (2.4) always exists. The following properties of Hausdorff measures are easily deduced from Definition 2.4.

Proposition 2.5 (Elementary properties of Hausdorff measures) *The measures H^s in $\mathbb{R}^n$ enjoy the the following properties.*
(i) $\mathcal{H}^s$ *is a Borel measure;*
(ii) for any $\delta > 0$, $\mathcal{H}^0_\delta(E) = \#(E)$;
(iii) if $s > n$, *then* $\mathcal{H}^s \equiv 0$;
(iv) $\mathcal{H}^s(x + E) = \mathcal{H}^s(E)$ *for any* $x \in \mathbb{R}^n$ *and* $\mathcal{H}^s(\lambda E) = \lambda^s \mathcal{H}^s(E)$ *for any* $\lambda > 0$;
(v) if $s > s' \geq 0$ *and* $\mathcal{H}^s(E) > 0$, *then* $\mathcal{H}^{s'}(E) = \infty$;
(vi) if $\Phi : \mathbb{R}^n \to \mathbb{R}^m$ *is a Lipschitz map, then*

$$\mathcal{H}^s(\Phi(E)) \leq [\mathrm{Lip}\Phi]^s \, \mathcal{H}^s(E) ,$$

where $\mathrm{Lip}\Phi$ *is the Lipschitz constant of* Φ.

Proof. The fact that $\mathcal{H}^s$ is an outer measure and that properties (ii), (iv) and (vi) hold is a straightforward consequence of Definition 2.4.
(i) To prove that $\mathcal{H}^s$ is a Borel measure we use Theorem 2.2. To this aim, let us fix two subsets E and F of $\mathbb{R}^n$ such that $\mathrm{dist}(E, F) > 0$. In order to prove (2.2) it is enough to show that $\mathcal{H}^s(E) + \mathcal{H}^s(F) \leq \mathcal{H}^s(E \cup F)$ whenever $\mathcal{H}^s(E \cup F) < \infty$. Let us fix a positive number $\delta < \frac{1}{2}\mathrm{dist}(E, F)$ and let us denote by C_h a countable covering of $E \cup F$ such that $\mathrm{diam}\, C_h < \delta$ for any h and

$$\frac{\omega_s}{2^s} \sum_{h=1}^{\infty} (\mathrm{diam}\, C_h)^s < \mathcal{H}^s_\delta(E \cup F) + \delta .$$

Notice that, since the diameter of each C_h is less than δ, any of the C_h's may intersect at most one of the two sets E and F. Therefore, we get

$$\begin{aligned} \mathcal{H}^s_\delta(E) + \mathcal{H}^s_\delta(F) &\leq \frac{\omega_s}{2^s} \sum_{\{h : C_h \cap E \neq \emptyset\}} (\mathrm{diam}\, C_h)^s + \frac{\omega_s}{2^s} \sum_{\{h : C_h \cap F \neq \emptyset\}} (\mathrm{diam}\, C_h)^s \\ &\leq \frac{\omega_s}{2^s} \sum_{h=1}^{\infty} (\mathrm{diam}\, C_h)^s < \mathcal{H}^s_\delta(E \cup F) + \delta \end{aligned}$$

and the assertion follows by letting δ go to zero.
(iii) Let us fix $\delta > 0$, and $h \in \mathbb{N}$ such that $\sqrt{n}/h < \delta$. Let us cover the cube $Q = [0, 1]^n$ with h^n closed cubes Q_i of side length $1/h$. Then, we have

$$\mathcal{H}^s_\delta(Q) \leq \frac{\omega_s}{2^s} \sum_{i=1}^{h^n} (\mathrm{diam}\, Q_i)^s = \frac{\omega_s}{2^s} h^{n-s} n^{s/2} \to 0 \qquad \text{as } h \to \infty .$$

Therefore $\mathcal{H}^s_\delta(Q) = 0 = \mathcal{H}^s(Q)$, hence $\mathcal{H}^s \equiv 0$.

(v) We argue by contradiction, assuming that $\mathcal{H}^s(E) > 0$ and $\mathcal{H}^{s'}(E) < \infty$, with $s > s' \geq 0$. Then, for any $\delta > 0$ there exists a countable covering C_h of E such that $\operatorname{diam} C_h < \delta$ for any h and

$$\frac{\omega_{s'}}{2^{s'}} \sum_{h=1}^{\infty} (\operatorname{diam} C_h)^{s'} < \mathcal{H}_\delta^{s'}(E) + 1 \leq \mathcal{H}^{s'}(E) + 1 \,.$$

Therefore we have also

$$\mathcal{H}_\delta^s(E) \leq \frac{\omega_s}{2^s} \sum_{h=1}^{\infty} (\operatorname{diam} C_h)^s \leq \delta^{s-s'} \frac{\omega_s}{2^s} \sum_{h=1}^{\infty} (\operatorname{diam} C_h)^{s'} < \delta^{s-s'} \frac{2^{s'} \omega_s}{2^s \omega_{s'}} \left[\mathcal{H}^{s'}(E) + 1 \right] \,,$$

hence, letting δ go to zero, we get that $\mathcal{H}^s(E) = 0$, a contradiction to the assumption $\mathcal{H}^s(E) > 0$. □

The *Hausdorff dimension* of a subset E is given by

$$\mathcal{H}\text{-}\dim(E) = \inf \{ s \geq 0 : \, \mathcal{H}^s(E) = 0 \} \,.$$

Notice that if $k = \mathcal{H}\text{-}\dim(E)$, then $\mathcal{H}^s(E) = \infty$ for any $s \in [0, k)$. In fact if we had $\mathcal{H}^{s'}(E) < \infty$ for some $s' < k$, then from Proposition 2.5 (v) we would also get that $\mathcal{H}^s(E) = 0$ for any $s' < s < k$ and this would contradict the fact that k is the Hausdorff dimension of E. Notice that Proposition 2.5 (v) implies also that $\mathcal{H}^s(E) = 0$ for any $s > k$. However, if k is the Hausdorff dimension of E, then $\mathcal{H}^k(E)$ can be any number in $[0, \infty]$.

Examples 2.6 (i) If $E = \{x_h\}_{h \in \mathbb{N}}$, then $\mathcal{H}\text{-}\dim(E) = 0$.
(ii) If E is a k-dimensional smooth manifold in $\mathbb{R}^n$, then $\mathcal{H}\text{-}\dim(E) = k$.
(iii) If E is the Cantor set, then it can be easily checked that $\mathcal{H}\text{-}\dim(E) = \dfrac{\log 2}{\log 3}$.

Let us now introduce the Steiner symmetral of a set $E \subset \mathbb{R}^n$ with respect to a fixed direction $\nu \in \mathbb{S}^{n-1}$.

Definition 2.7 Let E be any subset in $\mathbb{R}^n$ and $\nu \in \mathbb{S}^{n-1}$. Let us denote by π_ν the $(n-1)$-dimensional hyperplane passing through the origin, orthogonal to ν. The *Steiner symmetral* of E in the direction ν is the set

$$E_\nu^s = \left\{ x \in \mathbb{R}^n : \, x = z + t\nu, \; z \in \pi_\nu, \; |t| < \frac{1}{2} \mathcal{H}^1(E_{z,\nu}) \right\} , \tag{2.5}$$

where $E_{z,\nu} = \{ x \in E : x = z + s\nu, \; s \in \mathbb{R} \}$ is the intersection of E with the line through z, parallel to ν.

Clearly, the set E^s_ν is symmetric with respect to the hyperplane π_ν. Moreover, the following properties are an easy consequence of Definition 2.7.

Proposition 2.8 *Let E be any subset of $\mathbb{R}^n$ and $\nu \in \mathbb{S}^{n-1}$ any direction. Then,* $\operatorname{diam} E^s_\nu \le \operatorname{diam} E$. *Moreover, if E is measurable, $\mathcal{L}^n(E^s_\nu) = \mathcal{L}^n(E)$.*

Proof. The equality $\mathcal{L}^n(E^s_\nu) = \mathcal{L}^n(E)$ is a straightforward consequence of Fubini's theorem.
To prove that the diameter decreases under Steiner symmetrization we fix two points $x_1, x_2 \in E^s$, $x_i = z_i + t_i\nu$ for $i = 1, 2$, where $z_i \in \pi_\nu$ and $|t_i| < \frac{1}{2}\mathcal{H}^1(E_{z_i,\nu})$. Let us now set for $i = 1, 2$

$$\alpha_i = \inf\{t : \, z_i + t\nu \in E\}, \qquad \beta_i = \sup\{t : \, z_i + t\nu \in E\}$$

and assume, without loss of generality, that $\beta_2 - \beta_1 \ge \alpha_1 - \alpha_2$. Thus, we have

$$\begin{aligned}\beta_2 - \alpha_1 &\ge \frac{1}{2}(\beta_2 - \alpha_1) + \frac{1}{2}(\beta_1 - \alpha_2) = \frac{1}{2}(\beta_2 - \alpha_2) + \frac{1}{2}(\beta_1 - \alpha_1) \\ &\ge \frac{1}{2}\mathcal{H}^1(E_{z_1,\nu}) + \frac{1}{2}\mathcal{H}^1(E_{z_2,\nu}) \ge |t_1| + |t_2| \ge |t_2 - t_1| \, .\end{aligned}$$

Therefore, we get that

$$|x_2 - x_1|^2 = |z_2 - z_1|^2 + |t_2 - t_1|^2 \le |z_2 - z_1|^2 + (\beta_2 - \alpha_1)^2 = |z_2 + \beta_2\nu - z_1 - \alpha_1\nu|^2 \le (\operatorname{diam} E)^2$$

and from this inequality the assertion follows. □

The inequality stated in the following theorem is called *isodiametral inequality*. It says that the Lebesgue measure of any set E is always less than or equal to the measure of the ball having the same diameter of E. It follows from Proposition 2.8, but it cannot be proved by taking the smallest ball containing E since the diameter of this ball can be larger than the diameter of E (think for instance of E being an equilateral triangle).

Theorem 2.9 *Let E be any subset of $\mathbb{R}^n$. Then*

$$\mathcal{L}^n(E) \le \omega_n \Big(\frac{\operatorname{diam} E}{2}\Big)^n . \tag{2.6}$$

Proof. Let us assume that $\operatorname{diam} E < \infty$, otherwise (2.6) is trivial. If E is measurable, we set $E_1 = E^s_{e_1}$, $E_2 = (E_1)^s_{e_2}, \ldots, E^\star = (E_{n-1})^s_{e_n}$, where $\{e_1, \ldots, e_n\}$ is the standard base in $\mathbb{R}^n$. By construction, $E^\star$ is symmetric with respect to all the coordinates planes, hence $x \in E^\star$ if and only if $-x \in E^\star$ and thus $E^\star$ is contained in the closed ball centered at the origin and having the same diameter of $E^\star$. Therefore, we have

$$\mathcal{L}^n(E^\star) \le \omega_n \Big(\frac{\operatorname{diam} E^\star}{2} \Big)^n .$$

From this inequality (2.6) follows immediately, since by Proposition 2.8 $\mathcal{L}^n(E) = \mathcal{L}^n(E^\star)$ and $\operatorname{diam} E^\star \le \operatorname{diam} E$.
If E is any set (not necessarily measurable) (2.6) holds for $\overline{E}$, which is a measurable set, hence it holds also for E, since $\mathcal{L}^n(E) \le \mathcal{L}^n(\overline{E})$ and $\operatorname{diam} E = \operatorname{diam} \overline{E}$. □

Notice that from Theorem 2.9 it follows that, for any $\delta > 0$, the Lebesgue measure $\mathcal{L}^n$ is smaller than or equal to the measure $\mathcal{H}^n_\delta$. In fact, if E is any subset of $\mathbb{R}^n$ and C_h is a sequence of sets with diameter less than δ covering E, from (2.6) we have that

$$\mathcal{L}^n(E) \le \sum_{h=1}^{\infty} \mathcal{L}^n(C_h) \le \frac{\omega_n}{2^n} \sum_{h=1}^{\infty} (\operatorname{diam} C_h)^n ,$$

hence, we get that $\mathcal{L}^n(E) \le \mathcal{H}^n_\delta(E)$. The following theorem (see [3, Theorem 2.53]) says that also the opposite inequality is true.

Theorem 2.10 *Let B a Borel set in $\mathbb{R}^n$. For any $\delta \in (0, +\infty]$ we have*

$$\mathcal{L}^n(B) = \mathcal{H}^n_\delta(B) = \mathcal{H}^n(B) .$$

2.2 Area and coarea formulas

Last theorem of the previous section states that the Hausdorff measure $\mathcal{H}^n$ on $\mathbb{R}^n$ coincides with the usual Lebesgue measure. On the other hand, as a consequence of the area formula, we shall see in this section that if $1 \le k < n$ is an integer, then the restriction of the Hausdorff measure $\mathcal{H}^k$ to a k-dimensional smooth manifold coincides with the classical measure on the manifold. But before going in further details, we need to recall some useful facts of linear algebra.

Let $L : \mathbb{R}^n \mapsto \mathbb{R}^m$, where $m \ge n$, be a linear operator. In the sequel we shall always identify the linear operator L with the $m \times n$ matrix representing it with respect to the canonical bases of $\mathbb{R}^n$ and $\mathbb{R}^m$. The *n-dimensional jacobian* of L is defined by

$$\mathbf{J}_n L = \sqrt{\det(L^* \circ L)} \tag{2.7}$$

where $L^* : \mathbb{R}^m \mapsto \mathbb{R}^n$ is the adjoint of L. If $m = n$, from (2.7) we get that $\mathbf{J}_n L = |\det L|$. In the general case, denoting by $X_i(L)$, for $i = 1 \dots , \binom{m}{n}$, the minors of order n of L, the *Cauchy–Binet formula* ([3, Proposition 2.69]) states that

$$\mathbf{J}_n L = \sqrt{\sum_{i=1}^{\binom{m}{n}} |X_i(L)|^2}\,. \tag{2.8}$$

Notice that as a consequence of (2.8) we have that $\mathbf{J}_n L > 0$ if and only if the rank of L is n.

Let us consider now a Lipschitz map $\Phi : \mathbb{R}^n \mapsto \mathbb{R}^m$. From Rademacher theorem ([3, Theorem 2.14]) we know that Φ is differentiable (in classical sense) for $\mathcal{L}^n$-a.e. x in $\mathbb{R}^n$. Then, if x is a point where Φ is differentiable, we denote by $d\Phi_x$ its differential at x, which is a linear operator from $\mathbb{R}^n$ to $\mathbb{R}^m$. Having fixed these notations, we may now state the area formula for Lipschitz maps (see [3, Theorem 2.71]).

Theorem 2.11 (Area formula) *Let $\Phi : \mathbb{R}^n \mapsto \mathbb{R}^m$ be a Lipschitz continuous map with $m \geq n$. Then, for any Borel subset E of $\mathbb{R}^n$, the multiplicity function $y \in \mathbb{R}^m \to \mathcal{H}^0(E \cap \Phi^{-1}(y))$ is $\mathcal{H}^n$-measurable and*

$$\int_{\mathbb{R}^m} \mathcal{H}^0(E \cap \Phi^{-1}(y))\, d\mathcal{H}^n(y) = \int_E \mathbf{J}_n d\Phi_x \, dx\,. \tag{2.9}$$

As a first consequence of formula (2.9) we have that if E_0 is the set of points $x \in \mathbb{R}^n$ such that Φ is not differentiable at x or the rank of $d\Phi_x$ is less than n, then $\mathcal{H}^n(\Phi(E_0)) = 0$.

Next examples show some interesting applications of the area formula.

Examples 2.12 (i) If Φ is one-to-one, from (2.9) we get that the n-dimensional measure of $\Phi(E)$ is given by

$$\mathcal{H}^n(\Phi(E)) = \int_E \mathbf{J}_n d\Phi_x \, dx\,.$$

(ii) If $\Phi : \mathbb{R} \mapsto \mathbb{R}^n$, (2.8) implies that $\mathbf{J}_1 d\Phi_t = |\Phi'(t)|$. Therefore, if Φ is one-to-one, (2.9) agrees with the classical formula for the length of the parametrized curve $\Phi(t)$,

$$\mathcal{H}^1(\Phi(E)) = \int_E |\Phi'(t)| \, dt\,.$$

(iii) Let $\gamma : \mathbb{R}^n \to \mathbb{R}$ be a Lipschitz function. Let us denote by $\Phi(x) = (x, \gamma(x))$ the vector-valued function mapping $\mathbb{R}^n$ onto the graph $G_\gamma \subset \mathbb{R}^{n+1}$ of γ. Clearly, Φ is one-to-one and for $\mathcal{L}^n$-a.e. $x \in \mathrm{I\!R}^n$ we have $\mathbf{J}_n d\Phi_x = \sqrt{1+|\nabla\gamma(x)|^2}$. Therefore, if E is a Borel subset of $\mathrm{I\!R}^n$, we get from (2.9) that

$$\mathcal{H}^n(G_\gamma \cap (E \times \mathrm{I\!R})) = \int_E \sqrt{1+|\nabla\gamma(x)|^2}\, dx\,. \tag{2.10}$$

Generalizing this example, let us consider a Lipschitz map $\gamma : \mathrm{I\!R}^n \mapsto \mathrm{I\!R}^k$, $k > 1$, and let us set, for any $x \in \mathrm{I\!R}^n$, $\Phi(x) = (x, \gamma(x)) \in \mathrm{I\!R}^{n+k}$. Then we may easily check that

$$\mathbf{J}_n d\Phi_x = \sqrt{1 + \sum_{j=1}^{\min\{k,n\}} |M_j(d\gamma_x)|^2}\,,$$

where, for any j, $M_j(d\gamma_x)$ denotes the vector whose components are all the $\binom{k}{j}\binom{n}{j}$ minors of order j of the $k \times n$ matrix $d\gamma_x$.
(iv) Let Ω be an open set in $\mathrm{I\!R}^2$ and $\Phi : \Omega \mapsto \mathrm{I\!R}^3$ a parametrization of a C^1 surface $M = \Phi(\Omega)$. Then, (2.9) gives the classical formula for the area of the parametrized surface, since

$$\mathbf{J}_2 d\Phi_{(s,t)} = \sqrt{\left|\frac{\partial(\Phi_2,\Phi_3)}{\partial(s,t)}\right|^2 + \left|\frac{\partial(\Phi_3,\Phi_1)}{\partial(s,t)}\right|^2 + \left|\frac{\partial(\Phi_1,\Phi_2)}{\partial(s,t)}\right|^2}\,,$$

where Φ_1, Φ_2, Φ_3 are the components of the map Φ.

From the area formula one can easily obtain the following change of variables formula.

Theorem 2.13 (Change of variables formula) *Let $\Phi : \mathrm{I\!R}^n \mapsto \mathrm{I\!R}^m$ be a Lipschitz map, with $m \geq n$, and let $f : \mathrm{I\!R}^n \to [0, +\infty]$ be a Borel function. Then,*

$$y \in \mathrm{I\!R}^m \to \int_{\Phi^{-1}(y)} f(x)\, d\mathcal{H}^0(x)$$

is a $\mathcal{H}^n$-measurable function and

$$\int_{\mathrm{I\!R}^m} \left(\int_{\Phi^{-1}(y)} f(x)\, d\mathcal{H}^0(x) \right) d\mathcal{H}^n(y) = \int_{\mathrm{I\!R}^n} f(x) \mathbf{J}_n d\Phi_x\, dx\,. \tag{2.11}$$

In particular, if Φ is one-to-one, we have

$$\int_{\Phi(\mathrm{I\!R}^n)} f(\Phi^{-1}(y))\, d\mathcal{H}^n(y) = \int_{\mathrm{I\!R}^n} f(x) \mathbf{J}_n d\Phi_x\, dx\,.$$

Proof. Equation (2.11) follows immediately from (2.9) and from next lemma, by a simple application of B.Levi monotone convergence theorem. □

Lemma 2.14 *Let $f : \mathbb{R}^n \to [0, +\infty]$ be a Borel function. Then, there exists a sequence of Borel sets E_h of $\mathbb{R}^n$ such that for any $x \in \mathbb{R}^n$*

$$f(x) = \sum_{h=1}^{\infty} \frac{1}{h} \chi_{E_h}(x) .$$

Proof. Let us set for any $h > 1$

$$E_1 = \{x \in \mathbb{R}^n : f(x) \geq 1\}, \qquad E_h = \Big\{x \in \mathbb{R}^n : f(x) \geq \frac{1}{h} + \sum_{i=1}^{h-1} \frac{1}{i} \chi_{E_i}(x)\Big\} .$$

Notice that from the definition of E_h we get easily that for any $x \in \mathbb{R}^n$

$$f(x) \geq \sum_{h=1}^{\infty} \frac{1}{h} \chi_{E_h(x)} . \tag{2.12}$$

The opposite inequality is trivially satisfied if $f(x) = 0$. If $0 < f(x) < \infty$, then from (2.12) we get that $x \notin E_h$ for infinitely many integers h. Thus, if $x \notin E_{h_i}$ for some h_i, we have that

$$f(x) < \frac{1}{h_i} + \sum_{j=1}^{h_i-1} \frac{1}{j} \chi_{E_j}(x)$$

hence, letting h_i go to ∞, we get that the opposite inequality to (2.12) holds. Finally, if $f(x) = \infty$, we have that $x \in E_h$ for any h, hence the assertion is trivially satisfied. $\square$

Let us now introduce the definition of rectifiable set.

Definition 2.15 (Rectifiable sets) Let $S \subset \mathbb{R}^n$ be a $\mathcal{H}^k$-measurable subset of $\mathbb{R}^n$, where $1 \leq k \leq n$ is an integer. We say that S is a *k-rectifiable* set if there exists a sequence of Lipschitz maps $\Phi_i : \mathbb{R}^k \mapsto \mathbb{R}^n$ such that

$$\mathcal{H}^k\Big(S \setminus \bigcup_{i=1}^{\infty} \Phi_i(\mathbb{R}^k)\Big) = 0 .$$

Obviously, an n-rectifiable set in $\mathbb{R}^n$ is simply a Lebesgue measurable set. On the other hand, it can be proved (see [20, Theorem 3.1.16]) that if $k < n$ then S is k-rectifiable if and only if there exists a sequence K_i of pairwise disjoint compact sets such that each K_i is a subset of a C^1 manifold M_i and

$$\mathcal{H}^k\Big(S \setminus \bigcup_{i=1}^{\infty} K_i\Big) = 0 .$$

An interesting fact about rectifiable sets is that a suitable notion of tangent plane can be given so that the tangent plane exists almost everywhere (see [3, Section 2.11]).

Theorem 2.16 *Let S be a k-rectifiable set. Then, for $\mathcal{H}^k$-a.e. $x \in S$ there exists a k-dimensional hyperplane π_x^S such that for any $\varphi \in C_0(\mathbb{R}^n)$*

$$\lim_{\varepsilon\to 0^+} \int_{\frac{S-x}{\varepsilon}} \varphi(y)\, d\mathcal{H}^k(y) = \int_{\pi_x^S} \varphi(y)\, d\mathcal{H}^k(y)\,. \tag{2.13}$$

The k-plane π_x^S such that (2.13) holds is called the *approximate tangent plane* to S at x.

Example 2.17 (Lipschitz graphs) Let $\gamma : \mathbb{R}^{n-1} \to \mathbb{R}$ be a Lipschitz function. By Definition 2.15 the graph G_γ of γ is a $(n-1)$-rectifiable set in $\mathbb{R}^n$. Let us check that for $\mathcal{H}^{n-1}$-a.e. $x = (x', \gamma(x')) \in G_\gamma$ the approximate tangent plane $\pi_x^{G_\gamma}$ is given by the hyperplane orthogonal to the vector

$$\nu_\gamma(x) = \Big(\frac{\nabla_1\gamma(x')}{\sqrt{1+|\nabla\gamma(x')|^2}}, \dots, \frac{\nabla_{n-1}\gamma(x')}{\sqrt{1+|\nabla\gamma(x')|^2}}, \frac{-1}{\sqrt{1+|\nabla\gamma(x')|^2}}\Big)\,. \tag{2.14}$$

More precisely, let us denote by Z_0 the complement of the set of the points $x' \in \mathbb{R}^{n-1}$ such that γ is differentiable in x' and x' is a Lebesgue point for $\nabla\gamma$ and set $G_0 = \{(x', \gamma(x')) \in G_\gamma : x' \in Z_0\}$. By Rademacher theorem $\mathcal{H}^{n-1}(Z_0) = 0$ and thus from the area formula (2.10) we have also that $\mathcal{H}^{n-1}(G_0) = 0$. Let us fix $x_0 = (x'_0, \gamma(x'_0)) \in G_\gamma \backslash G_0$ and let us prove that the approximate tangent plane to G_γ at x_0 is orthogonal to the vector $\nu_\gamma(x_0)$ defined in (2.14). To this aim, let us fix a function $\varphi \in C_0(\mathbb{R}^n)$ and $\varepsilon > 0$ and apply the change of variable formula (2.11) to $f(x') = \varphi(\Phi(x'))$, where $\Phi : \mathbb{R}^{n-1} \to \mathbb{R}^n$ is given by $\Phi(x') = \Big(\frac{x'-x'_0}{\varepsilon}, \frac{\gamma(x')-\gamma(x'_0)}{\varepsilon}\Big)$. Thus, we get

$$\begin{aligned}\int_{\frac{G_\gamma - x_0}{\varepsilon}} \varphi(x) d\mathcal{H}^{n-1} &= \varepsilon^{1-n} \int_{\mathbb{R}^{n-1}} \varphi\Big(\frac{x'-x'_0}{\varepsilon}, \frac{\gamma(x')-\gamma(x'_0)}{\varepsilon}\Big)\sqrt{1+|\nabla\gamma(x')|^2}dx' \\ &= \varepsilon^{1-n} \int_{\mathbb{R}^{n-1}} \varphi\Big(\frac{x'-x'_0}{\varepsilon}, \frac{\gamma(x')-\gamma(x'_0)}{\varepsilon}\Big)\sqrt{1+|\nabla\gamma(x'_0)|^2}dx' + I_\varepsilon\,.\end{aligned} \tag{2.15}$$

Denoting by B_R a ball centered in 0 containing the support of φ, the term I_ε in (2.15) can be estimated by

$$\begin{aligned}&\varepsilon^{1-n}\Big|\int_{\mathbb{R}^{n-1}} \varphi\Big(\frac{x'-x'_0}{\varepsilon}, \frac{\gamma(x')-\gamma(x'_0)}{\varepsilon}\Big)\Big(\sqrt{1+|\nabla\gamma(x')|^2} - \sqrt{1+|\nabla\gamma(x'_0)|^2}\Big)dx'\Big| \\ &\le \varepsilon^{1-n}\|\varphi\|_\infty \int_{B^{n-1}_{\varepsilon R}(x'_0)} |\nabla\gamma(x') - \nabla\gamma(x'_0)|\, dx'\,,\end{aligned} \tag{2.16}$$

where $B_r^{n-1}(x_0')$ is the $(n-1)$-ball centered at x_0' of radius r. Recalling that x_0' is a Lebesgue point for $\nabla\gamma$, from (2.16) we get immediately that $\lim\limits_{\varepsilon\to 0} I_\varepsilon = 0$. Therefore from (2.15), by changing variables and recalling that γ is differentiable in x_0', we have

$$\begin{aligned}\lim_{\varepsilon\to 0^+}\int_{\frac{G_\gamma - x_0}{\varepsilon}}\varphi(x)d\mathcal{H}^{n-1} &= \sqrt{1+|\nabla\gamma(x_0')|^2}\lim_{\varepsilon\to 0^+}\int_{\mathbb{R}^{n-1}}\varphi\Big(y',\frac{\gamma(x_0'+\varepsilon y')-\gamma(x_0')}{\varepsilon}\Big)dy' \\ &= \sqrt{1+|\nabla\gamma(x_0')|^2}\int_{\mathbb{R}^{n-1}}\varphi(y',\langle\nabla\gamma(x_0'),y'\rangle)\,dy' = \int_{\pi_0}\varphi(y)d\mathcal{H}^{n-1}\,,\end{aligned}$$

where π_0 is the $(n-1)$-plane orthogonal to $\nu_\gamma(x_0')$. By (2.13), π_0 is the approximate tangent plane to G_γ at x_0.

It can be easily checked that whenever the approximate tangent plane at $x \in S$ exists, it is also unique. In fact, it is possible to prove even more ([3, Remark 2.87]), namely that if S_1 and S_2 are k-rectifiable sets, then

$$\pi_x^{S_1} = \pi_x^{S_2} \qquad\qquad \text{for } \mathcal{H}^k\text{-a.e. } x\in S_1\cap S_2\,.$$

Given a function $f:\mathbb{R}^n\to\mathbb{R}$ and a k-rectifiable set S, we say that f *is tangentially differentiable at* x if in x there exists the approximate tangent plane π_x^S and the restriction of f to the affine space $x+\pi_x^S$ is differentiable at x. The tangential differential of f at the point $x\in S$ will be denoted by $d^S f_x$. Notice that if f is a C^1 function, Theorem 2.16 yields that f is tangentially differentiable at $\mathcal{H}^k$-a.e. x in S and that, for any $h\in\pi_x^S$, $d^S f_x(h) = \langle\nabla f(x),h\rangle$. On the other hand, if f is Lipschitz map, hence $\mathcal{L}^n$-a.e. differentiable, it may happen that f is not differentiable at any point of S (which is a set of zero Lebesgue measure). However it is still true that f is tangentially differentiable at $\mathcal{H}^k$-a.e. point x in S ([3, Theorem 2.90]).
Let us now introduce the coarea factor of a linear map. To this aim, let $L:\mathbb{R}^k\mapsto\mathbb{R}^m$ be a linear map, with $k\geq m$. Then, *the m-dimensional coarea factor* of L is defined by

$$\mathbf{C}_m L = \sqrt{\det L\circ L^*}\,.$$

As in the case of jacobian, the Cauchy–Binet formula easily implies that

$$\mathbf{C}_m L = \sqrt{\sum_{i=1}^{\binom{k}{m}}|X_i(L)|^2}\,,$$

where $X_i(L)$, $i=1,\dots,\binom{k}{m}$, is the vector whose components are the minors of order m of L.

Theorem 2.18 *Let $\Phi : \mathbb{R}^n \to \mathbb{R}^m$ be a Lipschitz map and $S \subset \mathbb{R}^n$ a k-rectifiable set, with $m \leq k \leq n$. Then, the function $y \in \mathbb{R}^m \to \mathcal{H}^{k-m}(S \cap \Phi^{-1}(y))$ is $\mathcal{L}^m$-measurable and $S \cap \Phi^{-1}(y)$ is $(k-m)$-rectifiable for $\mathcal{L}^m$-a.e. $y \in \mathbb{R}^m$. Moreover, we have*

$$\int_S \mathbf{C}_m d^S \Phi_x \, d\mathcal{H}^k(x) = \int_{\mathbb{R}^m} \mathcal{H}^{k-m}(S \cap \Phi^{-1}(y)) \, dy \,. \tag{2.17}$$

For a proof of Theorem 2.18 see [3, Section 2.12]. Here, we limit ourselves to observe that, as for the area formula, also (2.17) can be easily extended to the case where on the left hand side we have the integral over S of a nonnegative Borel function f. In fact, from Lemma 2.14 we get that under the same assumptions of Theorem 2.18 and if $f : \mathbb{R}^n \to [0, +\infty]$ is a Borel function, then the following *generalized coarea formula* holds

$$\int_S f(x) \mathbf{C}_m d^S \Phi_x \, d\mathcal{H}^k(x) = \int_{\mathbb{R}^m} \left(\int_{S \cap \Phi^{-1}(y)} f(x) \, d\mathcal{H}^{k-m}(x) \right) dy \,. \tag{2.18}$$

Notice that if $E \subset S$ is a set of $\mathcal{H}^k$ zero measure, then from (2.18) we get that, for $\mathcal{L}^m$-a.e. $y \in \mathbb{R}^m$, $\mathcal{H}^{k-m}(E \cap \Phi^{-1}(y)) = 0$. Similarly, if we set $E_0 = \{x \in S : \text{rank of } d^S\Phi_x < m\}$, we have $\mathcal{H}^{k-m}(E_0 \cap \Phi^{-1}(y)) = 0$ for $\mathcal{L}^m$-a.e. $y \in \mathbb{R}^m$.
Notice also that the property that for $\mathcal{L}^m$-a.e. $y \in \mathbb{R}^m$ the set $S \cap \Phi^{-1}(y)$ is $(k-m)$-rectifiable can be viewed as a weak version of Sard's theorem (see [20, Theorem 3.4.3]), which states that if $\Phi : \mathbb{R}^n \to \mathbb{R}^m$ is of class C^h and $h > n+1-m$, then the set $\Phi^{-1}(y)$ is a $(n-m)$-dimensional manifold of class C^h for $\mathcal{L}^m$-a.e. $y \in \mathbb{R}^m$.

Examples 2.19 (i) Let $n = k > m \geq 1$ and let $\Phi : \mathbb{R}^n \to \mathbb{R}^m$ be the projection over the first m factors. Choosing $S = \mathbb{R}^n$ and noticing that $\mathbf{C}_m d\Phi_x = 1$, we obtain the classical Fubini's theorem as a special case of (2.18). And in fact the generalized coarea formula can be seen as the natural extension of Fubini's theorem to manifolds and rectifiable sets.
(ii) Let $n = k$ and let Φ be a Lipschitz function from $\mathbb{R}^n$ into $\mathbb{R}$. Then $\mathbf{C}_1 d\Phi_x = |\nabla\Phi(x)|$, hence in this case (2.18) reduces to the classical coarea formula

$$\int_{\mathbb{R}^n} f(x) |\nabla \Phi(x)| \, dx = \int_{-\infty}^{+\infty} \left(\int_{\{\Phi(x)=t\}} f(x) \, d\mathcal{H}^{n-1}(x) \right) dy \,. \tag{2.19}$$

(iii) Another special case of formula (2.18) that will be useful in the sequel is when S is an $(n-1)$-rectifiable set and $\Phi : \mathbb{R}^n \to \mathbb{R}^{n-1}$ is the projection over the first $n-1$ factors. In this case we may easily check that $\mathbf{C}_{n-1} d^S \Phi_x = |\nu_n^S(x)|$, where $\nu_n^S(x)$ is the n-th component of the normal $\nu^S(x)$ to the approximate tangent plane

π_x^S. Thus, denoting the points in $\mathbb{R}^n$ by (x', y), $x' \in \mathbb{R}^{n-1}$, $y \in \mathbb{R}$, (2.18) yields

$$\int_S f(x)|\nu_n^S(x)|\, d\mathcal{H}^{n-1}(x) = \int_{\mathbb{R}^{n-1}} \left(\int_{S_{x'}} f(x', y)\, \mathcal{H}^0(y) \right) dx' , \tag{2.20}$$

where $S_{x'} = \{y \in \mathbb{R} : (x', y) \in S\}$. An interesting consequence of (2.20) can be obtained by choosing $f = \chi_V$, where $V = \{x \in S : \nu_n^S(x) = 0\}$ is the 'vertical part' of S. In this case the left hand side of (2.20) is equal to zero, and thus we get that, for $\mathcal{L}^{n-1}$-a.e. $x' \in \mathbb{R}^{n-1}$, $V_{x'} = \emptyset$ (compare this example with part (iii) of Theorem 3.21).

3 BV functions and sets of finite perimeter

3.1 Vector-valued Radon measures

Let us recall some basic definitions and results of measure theory.

Let $\Omega \subset \mathbb{R}^n$ be an open set. We denote by $\mathcal{B}(\Omega)$ the σ-algebra of all Borel subsets of Ω. We say that a function $\nu : \mathcal{B}(\Omega) \mapsto \mathbb{R}^k$ is a *vector-valued Radon measure* (or a *real Radon measure*, if $k = 1$) on Ω if $\nu(\emptyset) = 0$ and ν is countably additive on $\mathcal{B}(\Omega)$. We denote by $\mathcal{M}(\Omega; \mathbb{R}^k)$ the space of all Radon measures on Ω with values in $\mathbb{R}^k$. Recall that by Riesz' theorem this space can be identified with the dual of the Banach space $C_c(\Omega, \mathbb{R}^k)$, which is defined as the closure of the space $C_0(\Omega, \mathbb{R}^k)$ of continuous functions with compact support with respect to the sup norm. In fact we may identify each Radon measure ν with the linear continuous functional on $C_c(\Omega, \mathbb{R}^k)$ defined by

$$L_\nu(\varphi) = \sum_{i=1}^k \int_\Omega \varphi_i(x)\, d\nu_i(x) \qquad \text{for any } \varphi \in C_c(\Omega, \mathbb{R}^k) ,$$

where ν_i, $i = 1, \ldots, k$, is the i-th component of the vector measure ν. From this identification it follows that the *total variation* of ν in Ω, which is defined by

$$|\nu|(\Omega) = \sup\Big\{ \sum_{i=1}^k \int_\Omega \varphi_(x)\, d\nu_i(x) : \varphi \in C_0(\Omega, \mathbb{R}^k),\ \|\varphi\|_\infty \le 1 \Big\} ,$$

coincides with the norm of the functional L_ν in the dual space of $C_c(\Omega, \mathbb{R}^k)$. We recall that the total variation of ν in an open subset A of Ω is defined similarly and is extended to any subset E of Ω by setting

$$|\nu|(E) = \inf\{|\nu|(A) : E \subset A,\ A \text{ open subset of } \Omega\} .$$

It can be easily checked that in this way $|\nu|$ becomes a Borel measure on Ω.

We recall that a sequence ν_h of Radon measures in $\mathcal{M}(\Omega;\mathbb{R}^k)$ is said to converge weakly* to a Radon measure ν if $\lim_{h\to\infty}\sum_{i=1}^k\int_\Omega \varphi_i d(\nu_h)_i = \sum_{i=1}^k\int_\Omega \varphi_i d\nu_i$ for any $\varphi\in C_c(\Omega,\mathbb{R}^k)$. Notice that the weak* convergence of ν_h is nothing else than the weak* convergence of the associated linear functionals L_{ν_h} in the dual space of $C_c(\Omega,\mathbb{R}^k)$. Therefore, from the classical Banach–Alaoglu–Bourbaki theorem on the weak* compactness of the unit ball of the dual of a Banach space we deduce at once the following compactness result.

Theorem 3.1 (De La Vallée–Poussin theorem) *Let ν_h be a sequence of Radon measures in $\mathcal{M}(\Omega;\mathbb{R}^k)$, such that $\sup_h|\nu_h|(\Omega)<\infty$. Then, there exists a subsequence ν_{h_r} weakly* converging to a Radon measure ν. Moreover, $|\nu|(\Omega)\le\liminf_{r\to\infty}|\nu_{h_r}|(\Omega)$.*

Let ν be a vector-valued Radon measure and μ a Borel measure. We say that ν is *absolutely continuous* with respect to μ (in symbols, $\nu << \mu$) if $|\nu|(B)=0$ for any Borel set B such that $\mu(B)=0$. We say instead that ν is *singular* with respect to μ (and write $\nu\perp\mu$) if there exists a Borel set $B_0\subset\Omega$ such that $\mu(B_0)=0$ and $|\nu|(\Omega\setminus B_0)=0$.

In the sequel we denote by $B_r(x)$ the open ball with center at x and radius r; if the ball is centered at the origin we simply write B_r in place of $B_r(x)$. If μ is a Borel measure in Ω and $f:\Omega\to\mathbb{R}$ is a μ-summable function, we denote by $f\mu$ the real Radon measure defined by $f\mu(B)=\int_B f(x)d\mu$, where B is a Borel subset of Ω. From this definition it follows that if $\varphi:\Omega\to\mathbb{R}$ is any bounded Borel function, then

$$\int_\Omega \varphi(x)\,d(f\mu)=\int_\Omega\varphi(x)f(x)\,d\mu\,.$$

If f is a μ-summable function with values into $\mathbb{R}^k$, the measure $f\mu$ is defined similarly and it turns out to be a Radon measure with values into $\mathbb{R}^k$. Clearly, $f\mu$ is absolutely continuous with respect to μ.

Theorem 3.2 (Besicovitch derivation theorem) *Let μ be a positive Radon measure in the open subset Ω of $\mathbb{R}^n$ and ν a Radon measure with values in $\mathbb{R}^k$. Then, for μ-a.e. $x\in\Omega$, there exists the limit*

$$\sigma(x)=\lim_{r\to0}\frac{\nu(B_r(x))}{\mu(B_r(x))}. \tag{3.1}$$

Moreover, $\nu=\sigma\mu+\nu^s$, where $\nu^s=\nu\llcorner E$ and $E\subset\Omega$ is a Borel set such that $\mu(E)=0$.

Thus, the above theorem (see [3, Theorem 2.22]) states that ν can be decomposed into the sum of the absolutely continuous (with respect to μ) measure $\sigma\mu$ and the singular measure ν^s. This decomposition, known as *Radon–Nikodým decomposition* of ν is unique. The function σ, defined by (3.1) is called the *derivative of ν with respect to μ* and is often denoted with the symbol $\dfrac{d\nu}{d\mu}$. Notice that if ν is absolutely continuous with respect to μ then $\nu = \sigma\mu$. Moreover, if $\sigma = \dfrac{d\nu}{d|\nu|}$, it can be easily checked that $|\sigma(x)| = 1$ for $|\nu|$-a.e. $x \in \Omega$.

3.2 BV functions

Definition 3.3 (BV functions) Let $\Omega \subset \mathbb{R}^n$ be an open set and $u \in L^1(\Omega)$. We say that u is a *function with bounded variation* in Ω, shortly, a *BV function*, if there exists a Radon measure λ with values in $\mathbb{R}^n$ such that for any $i = 1, \ldots, n$ and any $\varphi \in C_0^1(\Omega)$

$$\int_\Omega u \frac{\partial \varphi}{\partial x_i}\, dx = -\int_\Omega \varphi\, d\lambda_i\,. \tag{3.2}$$

The measure λ is also called the *measure derivative* of u and is denoted by the symbol Du. By $BV(\Omega)$ we denote the vector space of functions with bounded variation in Ω. This space can be endowed with the norm $\|u\|_{BV(\Omega)} = \|u\|_{L^1(\Omega)} + |Du|(\Omega)$, thus becoming a Banach space.

Proposition 3.4 (Total variation of BV functions) *Let $u \in L^1(\Omega)$. Then, $u \in BV(\Omega)$ if and only if*

$$V = \sup\Big\{\int_\Omega u(x)\mathrm{div}\varphi(x)\, dx \,:\, \varphi \in C_0^1(\Omega;\mathbb{R}^n),\ \|\varphi\|_\infty \le 1\Big\} < \infty\,. \tag{3.3}$$

Moreover the supremum V in (3.3) is equal to the total variation $|Du|(\Omega)$ of Du in Ω.

Proof. Let u be a function from $BV(\Omega)$. From (3.2) and from the definition of $|Du|(\Omega)$ we get that, for any $\varphi \in C_0^1(\Omega;\mathbb{R}^n)$ with $\|\varphi\|_\infty \le 1$,

$$\int_\Omega u(x)\mathrm{div}\varphi(x)\, dx = -\sum_{i=1}^n \int_\Omega \varphi_i(x)\, dD_i u = -\sum_{i=1}^n \int_\Omega \varphi_i(x)\sigma_i(x)\, d|Du| \le |Du|(\Omega)$$

and from this inequality it follows that $V \le |Du|(\Omega) < \infty$.
Conversely, if $V < \infty$, the linear functional $L : C_0^1(\Omega;\mathbb{R}^n) \to \mathbb{R}$ defined by $L(\varphi) = \displaystyle\int_\Omega \mathrm{div}\varphi dx$ is continuous with respect to the sup norm. Since $C_0^1(\Omega;\mathbb{R}^n)$ is a

dense subspace of $C_c(\Omega, \mathbb{R}^n)$, L can be extended in a unique way to a linear continuous functional $\overline{L}$ defined over the whole space. Moreover, $\|\overline{L}\|_{(C_c(\Omega,\mathbb{R}^n))'} = \|L\| = V$. Therefore, by the Riesz' theorem it follows that there exists a vector-valued Radon measure ν representing $\overline{L}$ and, in particular, L. Thus, we have that for any $\varphi \in C_0^1(\Omega, \mathbb{R}^n)$

$$\sum_{i=1}^{n} \int_\Omega \varphi_i \, d\nu_i = L(\varphi) = \int_\Omega \mathrm{div}\varphi \, dx$$

and from Definition 3.3 it follows that u is in $BV(\Omega)$, that $Du = -\nu$ and that $|Du|(\Omega) = \|\overline{L}\| = \|L\| = V$. □

Examples 3.5 (i) If u is a function from the Sobolev space $W^{1,1}(\Omega)$, denoting by ∇u the weak gradient of u, we have that $Du = \nabla u \mathcal{L}^n$, i.e. $D_i u(E) = \displaystyle\int_E \frac{\partial u}{\partial x_i} dx$ for any Borel set $E \subset \Omega$.

(ii) Let $u(t) = \dfrac{t}{|t|}$, with $t \in (-1, 1)$. For any $\varphi \in C_0^1(-1, 1)$ we have

$$\int_{-1}^{1} u\varphi' \, dt = \int_0^1 \varphi' \, dt - \int_{-1}^{0} \varphi' \, dt = -2\varphi(0) = -2 \int_{-1}^{1} \varphi \, d\delta_0 \, ,$$

where δ_0 is the Dirac measure concentrated at zero. In this case, the measure derivative $Du = 2\delta_0$ is singular with respect to the Lebesgue measure $\mathcal{L}^1$ and measures the 'jump' of u across the origin.

(iii) Let Ω be a bounded open subset of $\mathbb{R}^n$, $n > 1$, and let Ω_i, $i = 1, \ldots, N$, be pairwise disjoint open subsets of Ω such that $\overline{\Omega} = \cup_{i=1}^N \overline{\Omega}_i$ and, for any i, $\partial\Omega_i \cap \Omega$ is a hypersurface of class C^1. For any i, let u_i be a function of class $C^1(\overline{\Omega}_i)$. Let us define $u : \Omega \to \mathbb{R}$, by setting $u(x) = u_i(x)$ if $x \in \Omega_i$. If φ is a function from $C_0^1(\Omega)$, from the classical Gauss–Green formulas we obtain that for any $s = 1, \ldots, n$,

$$\int_\Omega u \frac{\partial \varphi}{\partial x_s} \, dx = \sum_{i=1}^{N} \int_{\Omega_i} u_i \frac{\partial \varphi}{\partial x_s} \, dx = -\sum_{i=1}^{N} \int_{\Omega_i} \varphi \frac{\partial u_i}{\partial x_s} \, dx - \sum_{i=1}^{N} \int_{\partial\Omega_i \cap \Omega} u_i \varphi \nu_s^{\Omega_i} \, d\mathcal{H}^{n-1} \, ,$$

where ν^{Ω_i} is the inner normal to the boundary of Ω_i. Thus, from this equality it follows that u is in $BV(\Omega)$ and that

$$Du = \Big(\sum_{i=1}^{N} \nabla u_i \, \chi_{\Omega_i} \Big) \mathcal{L}^n + \sum_{i=1}^{N} \nu^{\Omega_i} \mathcal{H}^{n-1} \llcorner \partial\Omega_i \, .$$

Notice that in this case $\dfrac{dDu}{d\mathcal{L}^n} = \displaystyle\sum_{i=1}^{N} \nabla u_i \, \chi_{\Omega_i}$, while $\displaystyle\sum_{i=1}^{N} \nu^{\Omega_i} \mathcal{H}^{n-1} \llcorner \partial\Omega_i$ is the singular part of Du with respect to the Lebesgue measure.

(iv) Let us recall the construction of the *Cantor–Vitali* function $f : [0,1] \to [0,1]$. Denoting by C the Cantor set, then $C = \bigcap_{h=0}^{\infty} C_h$, where $C_0 = [0,1]$ and any other set C_{h+1} is obtained by C_h by splitting each interval of C_h in three closed intervals of equal length and removing the interior of the middle one. Thus, each set C_h consists of 2^h pairwise disjoint closed intervals of size $1/3^h$. Let us now define a sequence of nondecreasing, piecewise affine functions $f_h : [0,1] \to [0,1]$ by setting, for any $x \in [0,1]$,

$$f_h(x) = \frac{3^h}{2^h} \int_0^x \chi_{C_h}(t)\, dt\,.$$

Then, one can easily show that f_h is a Cauchy sequence in $C([0,1])$, hence it converges to a continuous, monotone function f. Moreover, since

$$\int_0^1 |f_h'(t)|\, dt = \int_0^1 f_h'(t)\, dt = 1\,,$$

Theorem 3.1 implies that the sequence of measures $f_h'\mathcal{L}^n$ has a subsequence $f_{h_r}'\mathcal{L}^n$ converging weakly* to a Radon measure λ and that λ is the distributional derivative of the Cantor–Vitali function f. In fact we have, for any $\varphi \in C_0^1(0,1)$,

$$\int_0^1 f(t)\varphi'(t)\, dt = \lim_{r\to\infty} \int_0^1 f_{h_r}(t)\varphi'(t)\, dt = -\lim_{r\to\infty} \int_0^1 f_{h_r}'(t)\varphi(t)\, dt = -\int_0^1 \varphi(t)\, d\lambda\,.$$

Hence, f is in $BV(0,1)$ and $Df = \lambda$. Moreover, a simple compactness argument shows that indeed the whole sequence $f_h'\mathcal{L}^n$ converges weakly* to Df. The reader may also check that Df is equal to $c\mathcal{H}^s \llcorner C$, where $s = \log 2/\log 3$ and that the constant c is equal to $1/\mathcal{H}^s(C)$.

Arguing exactly as we just did in the Example 3.5 (iv) one can easily prove the following useful fact.

Proposition 3.6 *Let u_h be a sequence of functions from $BV(\Omega)$, converging in $L^1(\Omega)$ to a function u and such that $\sup_h |Du_h|(\Omega) < \infty$. Then, $u \in BV(\Omega)$ and the sequence Du_h converges weakly* in the sense of measures to Du.*

We shall refer to the convergence considered in Proposition 3.6 as to *weak* convergence in BV*. Namely, we say that a sequence u_h in $BV(\Omega)$ converges weakly* to a $BV(\Omega)$ function u if $u_h \to u$ in $L^1(\Omega)$ and Du_h converges to Du weakly* in Ω in the sense of measures. In the sequel we shall use only weak* convergence since norm convergence in BV is too strong for most purposes.

The following approximation theorem states that although we cannot approximate BV functions by smooth functions in the norm sense (otherwise we would fall within the class of $W^{1,1}$ functions!), such an approximation holds in the weak* convergence sense.

Theorem 3.7 (Approximation of BV functions) *Let Ω be an open set in $\mathbb{R}^n$ and u a function from $BV(\Omega)$. Then, there exists a sequence of functions u_h from $C^\infty(\Omega) \cap BV(\Omega)$ such that*

$$u_h \to u \quad \text{weakly* in } BV(\Omega), \qquad \int_\Omega |\nabla u_h|\, dx \to |Du|(\Omega)\,.$$

Proof. We shall prove the result in the model case $\Omega = \mathbb{R}^n$, since the general case presents only some extra technical complications which can be overcome arguing as in the proof of the classical result $H = W$ (see the proof of Theorem 3.9 in [3]). Let us denote by ϱ a positive, radially symmetric function with compact support in B_1, such that $\int_{\mathbb{R}^n} \varrho\, dx = 1$. For any $\varepsilon > 0$, we set $\varrho_\varepsilon(x) = \varepsilon^{-n}\varrho(x/\varepsilon)$ and

$$u_\varepsilon(x) = (u * \varrho_\varepsilon)(x) = \int_{\mathbb{R}^n} \varrho_\varepsilon(x-y)u(y)\, dy\,.$$

Then, $u_\varepsilon \to u$ in $L^1(\mathbb{R}^n)$ and for any $x \in \mathbb{R}^n$

$$\frac{\partial u_\varepsilon}{\partial x_i}(x) = \int_{\mathbb{R}^n} u(y)\frac{\partial}{\partial x_i}\varrho_\varepsilon(x-y)dy = \int_{\mathbb{R}^n} u(y)\frac{\partial}{\partial y_i}\varrho_\varepsilon(x-y)dy = -\int_{\mathbb{R}^n} \varrho_\varepsilon(x-y)dD_i u(y),$$

for $i = 1, \ldots, n$. Let us now fix $\varphi \in C_0(\mathbb{R}^n)$, with $\|\varphi\|_\infty \le 1$. From the previous equality we get

$$\begin{aligned}\int_{\mathbb{R}^n} \langle \varphi, \nabla u_\varepsilon\rangle\, dx \;&=\; -\sum_{i=1}^n \int_{\mathbb{R}^n} \varphi_i(x)\, dx \int_{\mathbb{R}^n} \varrho_\varepsilon(x-y)\, dD_i u(y) \\ &= -\sum_{i=1}^n \int_{\mathbb{R}^n} dD_i u(y) \int_{\mathbb{R}^n} \varrho_\varepsilon(x-y)\varphi_i(x)\, dx = -\sum_{i=1}^n \int_{\mathbb{R}^n} (\varphi_i * \varrho_\varepsilon)(y)\, dD_i u(y)\end{aligned}$$

and thus, since $\|\varphi * \varrho_\varepsilon\|_\infty \le 1$, taking the supremum over all such functions φ, we get

$$\int_{\mathbb{R}^n} |\nabla u_\varepsilon|\, dx \le |Du|(\mathbb{R}^n)\,. \tag{3.4}$$

Therefore from Proposition 3.6 we have that the measures $\nabla u_\varepsilon \mathcal{L}^n$ converge weakly* to the measure Du and thus, by the lower semicontinuity of the total variation (see

Theorem 3.1), we have also

$$|Du|(\mathbb{R}^n) \leq \liminf_{\varepsilon \to 0} \int_{\mathbb{R}^n} |\nabla u_\varepsilon| \, dx \, .$$

This inequality, together with (3.4), concludes the proof. □

Remark 3.8 Notice that if $u \in L^1_{\text{loc}}(\mathbb{R}^n)$ is a function such that its distributional gradient Du is a Radon measure, the same argument used in the proof above shows that there exists a sequence u_h of functions from $C^\infty(\mathbb{R}^n)$ such that $u_h \to u$ in $L^1_{\text{loc}}(\mathbb{R}^n)$ and

$$|Du|(\mathbb{R}^n) = \lim_{h \to \infty} \int_{\mathbb{R}^n} |\nabla u_h| dx, \quad |D_i u|(\mathbb{R}^n) = \lim_{h \to \infty} \int_{\mathbb{R}^n} \left| \frac{\partial u_h}{\partial x_i} \right| dx \quad \text{for all } i = 1, \ldots, n.$$

A similar remark applies to a function $u \in L^1_{\text{loc}}(\Omega)$ whose distributional gradient is a Radon measure in Ω.

By using the approximation result Theorem 3.7 we can immediately generalize to BV the extension and compactness properties of the Sobolev space $W^{1,1}$.

Theorem 3.9 (Extension of BV functions) *Let Ω be a bounded open set in $\mathbb{R}^n$ with Lipschitz boundary and let Ω_0 be an open set such that $\overline{\Omega} \subset \Omega_0$. Then, there exists a linear continuous operator $T : BV(\Omega) \to BV(\Omega_0)$ such that, for any u in $BV(\Omega)$, Tu has compact support in Ω_0 and $Tu(x) = u(x)$ for all $x \in \Omega$.*

Theorem 3.10 (Compactness) *Let Ω be a bounded open set in $\mathbb{R}^n$ with Lipschitz boundary, $n > 1$. Then, $BV(\Omega)$ is continuously imbedded in $L^p(\Omega)$ for any $1 \leq p \leq n/(n-1)$. Moreover, this imbedding is compact if $p < n/(n-1)$. In particular, if u_h is a bounded sequence in $BV(\Omega)$, there exists a subsequence u_{h_k} weakly* converging to a function $u \in BV(\Omega)$.*

3.3 Sets of finite perimeter

Definition 3.11 (Perimeter) Let E be a measurable subset of $\mathbb{R}^n$ and Ω an open set. The *perimeter* of E in Ω is defined by the quantity

$$P(E;\Omega) = \sup \Big\{ \int_E \text{div}\varphi \, dx : \ \varphi \in C^1_0(\Omega;\mathbb{R}^n), \ \|\varphi\|_\infty \Big\} \, .$$

If $P(E;\Omega) < \infty$, we say that E is a *set of finite perimeter* in Ω.

Thus, if E has finite perimeter in Ω, its characteristic function χ_E has a distributional derivative $D\chi_E$ which is a finite Radon measure with values in $\mathbb{R}^n$ and $P(E;\Omega) = |D\chi_E|(\Omega)$. We shall write simply $P(E)$ to denote the perimeter of E in $\mathbb{R}^n$ and if B is any Borel set we define the perimeter of E in B by setting

$$P(E;B) = |D\chi_E|(B)\,.$$

Notice that the measure $D\chi_E$ is concentrated on the topological boundary ∂E of E, since by the definition we get immediately that $|D\chi_E|(\Omega\setminus\partial E) = P(E;\Omega\setminus\partial E) = 0$. Moreover, if E has finite perimeter in Ω, then for any $\varphi \in C_0^1(\Omega;\mathbb{R}^n)$

$$\int_E \mathrm{div}\varphi\, dx = \int_\Omega \chi_E \mathrm{div}\varphi\, dx = -\sum_{i=1}^n \int_\Omega \varphi_i(x)\, dD_i\chi_E(x)\,. \tag{3.5}$$

Setting

$$\nu^E(x) = \frac{dD\chi_E}{d|D\chi_E|}(x)\,,$$

whenever this derivative exists, (3.5) becomes

$$\int_E \mathrm{div}\varphi\, dx = -\int_\Omega \langle\varphi,\nu^E\rangle\, d|D\chi_E| = -\int_{\partial E\cap\Omega} \langle\varphi,\nu^E\rangle\, d|D\chi_E|\,. \tag{3.6}$$

The vector $\nu^E(x)$ exists and has norm equal to 1 for $|D\chi_E|$-a.e. $x\in\Omega$; $\nu^E(x)$ is called the *generalized inner normal* to E at x, a name which is justified by the integration by parts formula (3.6).

An important feature of the definition of perimeter is that it does not change if we modify E by a set of zero Lebesgue measure. Therefore if E is equivalent to E', i.e. $\mathcal{L}^n(E\triangle E') = 0$, then $P(E;B) = P(E';B)$ for any Borel set B.

Examples 3.12 (i) Let $U \subset \mathbb{R}^{n-1}$ be a bounded open set and $u : U \to \mathbb{R}$ a Lipschitz function. Let us set $\mathcal{S}_u = \{(x',y)\in\mathbb{R}^n : x'\in U,\, y < u(x')\}$. Then, $\varphi \in C_0^1(U\times\mathbb{R};\mathbb{R}^n)$

$$\int_{\mathcal{S}_u} \mathrm{div}\varphi(x)\, dx = -\int_{G_u} \langle\varphi,\nu_u\rangle\, d\mathcal{H}^{n-1}\,, \tag{3.7}$$

where G_u is the graph of u and ν_u is defined as in (2.14). In fact, from Fubini's theorem and the area formula (2.11), we have

$$\begin{aligned}\int_{\mathcal{S}_u} \mathrm{div}\varphi(x)dx &= \sum_{i=1}^{n-1}\int_U dx'\int_{-\infty}^{u(x')}\frac{\partial\varphi_i}{\partial x_i}(x',y)dy + \int_U dx'\int_{-\infty}^{u(x')}\frac{\partial\varphi_n}{\partial y}(x',y)dy \\ &= \sum_{i=1}^{n-1}\int_U \frac{\partial}{\partial x_i}\left(\int_{-\infty}^{u(x')}\varphi_i(x',y)dy\right)dx'\end{aligned} \tag{3.8}$$

$$
-\sum_{i=1}^{n-1}\int_U \varphi_i(x',u(x'))\frac{\partial u}{\partial x_i}dx' + \int_U \varphi_n(x',u(x'))dx'
$$

$$
= -\sum_{i=1}^{n-1}\int_U \varphi_i(x',u(x'))\frac{\partial u}{\partial x_i}dx' + \int_U \varphi_n(x',u(x'))dx' = -\int_{G_u}\langle\varphi,\nu_u\rangle\, d\mathcal{H}^{n-1}\,.
$$

Notice that from (3.7) and the area formula (2.9) it follows that

$$
P(\mathcal{S}_u; U\times\mathrm{I\!R}) = \int_U \sqrt{1+|\nabla u(x')|^2}\,dx', \qquad D\chi_{\mathcal{S}_u} = \nu_u\mathcal{H}^{n-1}\llcorner G_u\,.
$$

(ii) By the previous example one can easily get that if E is a bounded open subset of $\mathrm{I\!R}^n$ with Lipschitz boundary, then E is a set of finite perimeter and $P(E) = \mathcal{H}^{n-1}(\partial E)$. Moreover, denoting by $\nu(x)$ the interior normal to ∂E at x (which exists for $\mathcal{H}^{n-1}$-a.e. $x\in\partial E$) we have also that $D\chi_E = \nu\mathcal{H}^{n-1}\llcorner\partial E$.
(iii) Let us go back to the first example and, keeping the same notation, let us now assume that $u\in BV(U)$. Let $u_h : U\to\mathrm{I\!R}$, $h\in\mathrm{I\!N}$, be a sequence of smooth functions, such that $u_h\to u$ in $L^1(U)$ and $\mathcal{L}^{n-1}$-a.e. in U and $\int_U |\nabla u_h|dx' \to |Du|(U)$. From (3.8) we get that for any function $\varphi\in C^1_0(U\times\mathrm{I\!R};\mathrm{I\!R}^n)$ with $\|\varphi\|_\infty\le 1$

$$
\begin{aligned}
\int_{\mathcal{S}_{u_h}} \mathrm{div}\varphi(x)\,dx &= -\sum_{i=1}^{n-1}\int_U \varphi_i(x',u_h(x'))\frac{\partial u_h}{\partial x_i}\,dx' + \int_U \varphi_n(x',u_h(x'))dx' \\
&\le \int_U |\nabla u_h|\,dx' + \mathcal{L}^{n-1}(U)
\end{aligned}
$$

and thus, letting h go to ∞ we conclude that

$$
\int_{\mathcal{S}_u} \mathrm{div}\varphi(x)\,dx \le |Du|(U) + \mathcal{L}^{n-1}(U)\,,
$$

hence $\mathcal{S}_u$ is a set of finite perimeter in $U\times\mathrm{I\!R}$. Notice that the inequality above implies in particular that $P(\mathcal{S}_u; U\times\mathrm{I\!R})$ is less than or equal to the right hand side. However, it can be proved that for any Borel set $B\subset U$ the perimeter of $\mathcal{S}_u$ inside $B\times\mathrm{I\!R}$ is given by

$$
P(\mathcal{S}_u; B\times\mathrm{I\!R}) = \int_B \sqrt{1+|\nabla u|^2}\,dx' + |D^s u|(B)\,,
$$

where we have denoted by ∇u the derivative of Du with respect to $\mathcal{L}^{n-1}$ and by $D^s u$ the singular part of Du.

(iv) Let $E = \bigcup_{i=1}^{\infty} B_{1/2^i}(q_i)$, where q_i is a countable, dense set in $\mathbb{R}^n$, with $n > 1$. Notice that $\mathcal{L}^n(E) < \infty$, but $\mathcal{L}^n(\partial E) = \infty$. Let us check that E has finite perimeter. In fact, if $\varphi \in C_0^1(\Omega, \mathbb{R}^n)$, $\|\varphi\|_\infty \leq 1$,

$$\begin{aligned}
\int_{\mathbb{R}^n} \chi_E \operatorname{div}\varphi \, dx &= \int_{\bigcup_{i=1}^{\infty} B_{1/2^i}(q_i)} \operatorname{div}\varphi \, dx = \lim_{N\to\infty} \int_{\bigcup_{i=1}^{N} B_{1/2^i}(q_i)} \operatorname{div}\varphi \, dx \\
&= -\lim_{N\to\infty} \int_{\partial\left(\bigcup_{i=1}^{N} B_{1/2^i}(q_i)\right)} \langle \nu, \varphi \rangle \, d\mathcal{H}^{n-1} \leq \mathcal{H}^{n-1}\Big(\partial\Big(\bigcup_{i=1}^{N} B_{1/2^i}(q_i)\Big)\Big) \\
&\leq \sum_{i=1}^{N} \mathcal{H}^{n-1}(\partial B_{1/2^i}(q_i)) < \sum_{i=1}^{\infty} \frac{n\omega_n}{2^{i(n-1)}} < \infty,
\end{aligned}$$

hence $P(E) < \infty$.

Proposition 3.13 (Elementary properties of perimeters) *Let E, F be measurable sets and Ω an open set in $\mathbb{R}^n$. Then,*
(i) $P(E;\Omega) = P(\mathbb{R}^n \setminus E; \Omega)$;
(ii) $P(E \cup F; \Omega) + P(E \cap F; \Omega) \leq P(E;\Omega) + P(F;\Omega)$;
(iii) if $(E_h)_{h\in\mathbb{N}}$ and E are measurable sets and $\chi_{E_h} \to \chi_E$ in $L^1_{\mathrm{loc}}(\Omega)$, then

$$P(E;\Omega) \leq \liminf_{h\to\infty} P(E_h;\Omega) \,.$$

Proof. (i) and (iii) are straightforward consequences of the definition of perimeter. To prove (ii), let us recall that from Theorem 3.7 it follows that there exist two sequences of $C^\infty(\Omega)$ functions u_h and v_h, converging respectively to χ_E and χ_F in $L^1(\Omega)$ and such that

$$P(E;\Omega) = \lim_{h\to\infty} \int_\Omega |\nabla u_h| \, dx, \qquad P(F;\Omega) = \lim_{h\to\infty} \int_\Omega |\nabla v_h| \, dx \,. \tag{3.9}$$

By truncating the functions u_h, v_h we may also suppose, without loss of generality, that $0 \leq u_h \leq 1$, $0 \leq v_h \leq 1$, and that u_h, v_h are locally Lipschitz continuous in Ω. Then, one can easily check that $u_h v_h$ converges to $\chi_E \chi_F = \chi_{E\cap F}$ in $L^1(\Omega)$ and that $u_h + v_h - u_h v_h$ converges in $L^1(\Omega)$ to $\chi_E + \chi_F - \chi_E\chi_F = \chi_{E\cup F}$. Therefore, since

$$P(E\cup F;\Omega) + P(E\cap F;\Omega) \leq \liminf_{h\to\infty} \int_\Omega |\nabla(u_h + v_h - u_h v_h)| \, dx + \liminf_{h\to\infty} \int_\Omega |\nabla(u_h v_h)| \, dx \,,$$

(ii) follows immediately from (3.9) and from the inequality

$$\int_\Omega |\nabla(u_h + v_h - u_h v_h)| \, dx + \int_\Omega |\nabla(u_h v_h)| \, dx \leq \int_\Omega |\nabla u_h| \, dx + \int_\Omega |\nabla v_h| \, dx \,.$$

□

We have already observed that if E is a set of finite perimeter then the measure $|D\chi_E|$ is concentrated on the boundary of E. However, by using the Besicovitch derivation Theorem 3.2 we may characterize more precisely the support of $|D\chi_E|$.

Definition 3.14 (Reduced boundary) Let E be a measurable set in $\mathbb{R}^n$ and let Ω_0 be the largest open set such that E has (locally) finite perimeter in Ω_0. The *reduced boundary* $\partial^* E$ of E is the collection of all points $x \in \Omega_0$ such that the limit

$$(i) \qquad \lim_{r\to 0} \frac{D\chi_E(B_r(x))}{|D\chi_E(B_r(x))|} = \frac{dD\chi_E}{|dD\chi_E|}(x) = \nu^E(x)$$

exists and moreover

$$(ii) \qquad |\nu^E(x)| = 1\,.$$

From Theorem 3.2 we know that conditions (i) and (ii) in Definition 3.14 are satisfied $|D\chi_E|$ almost everywhere in Ω_0, i.e. $|D\chi_E|(\Omega_0 \setminus \partial^* E) = 0$. The following result (see [3, Theorem 3.59]) contains the main properties of the reduced boundary $\partial^* E$. To this aim, if $x \in \partial^* E$, let us denote by $\pi_{\nu^E(x)}$ the $(n-1)$-hyperplane orthogonal to the normal versor $\nu^E(x)$ and by $H^+_{\nu^E(x)}$ the half-space $\{z : \langle z, \nu^E(x)\rangle > 0\}$.

Theorem 3.15 (De Giorgi's structure theorem) *Let E be a measurable subset of $\mathbb{R}^n$. Then, the reduced boundary $\partial^* E$ of E is an $(n-1)$-rectifiable set and $|D\chi_E| = \mathcal{H}^{n-1} \llcorner \partial^* E$. Moreover, for every $x \in \partial^* E$,*

$$(i) \qquad \mathcal{H}^{n-1} \llcorner \frac{\partial^* E - x}{\varepsilon} \to \mathcal{H}^{n-1} \llcorner \pi_{\nu^E(x)} \quad \textit{weakly* in the sense of measures,}$$

as $\varepsilon \to 0^+$, and

$$(ii) \qquad \chi_{\frac{E-x}{\varepsilon}} \to \chi_{H^+_{\nu^E(x)}} \quad \textit{in } L^1_{\mathrm{loc}}(\mathbb{R}^n)\,.$$

An immediate consequence of the above theorem is that if E is a set of finite perimeter in Ω and $B \subset \Omega$ is any Borel set, then $P(E;B) = |D\chi_E|(B) = \mathcal{H}^{n-1}(B \cap \partial^* E)$. Moreover, from (3.6) we have that for any $\varphi \in C^1_0(\Omega;\mathbb{R}^n)$, then

$$\int_{E\cap\Omega} \mathrm{div}\varphi\, dx = -\int_{\partial^* E\cap\Omega} \langle \varphi, \nu^E\rangle\, d\mathcal{H}^{n-1}\,.$$

By comparing Theorem 2.16 with part (i) of Theorem 3.15 we get that for every $x \in \partial^* E$

$$\pi_{\nu^E_x} \text{ is the approximate tangent plane to } \partial^* E \text{ at } x\,.$$

The following result is a very useful version of the coarea formula for BV functions.

Theorem 3.16 (Coarea formula in BV) *Let Ω be an open subset of $\mathbb{R}^n$ and u a function from $BV(\Omega)$. Then for $\mathcal{L}^1$-a.e. $t \in \mathbb{R}$ the level set $\{u > t\}$ has finite perimeter in Ω and for any Borel set $B \subset \Omega$*

$$|Du|(B) = \int_{-\infty}^{+\infty} P(\{u > t\}; B)\, dt = \int_{-\infty}^{+\infty} \mathcal{H}^{n-1}(B \cap \partial^* \{u > t\})\, dt\,. \tag{3.10}$$

Moreover, if $f : \Omega \to [0, +\infty]$ is a Borel function, then

$$\int_\Omega f(x)\, d|Du|(x) = \int_{-\infty}^{+\infty} dt \int_{\partial^*\{u>t\}\cap\Omega} f(x)\, d\mathcal{H}^{n-1}(x)\,. \tag{3.11}$$

Proof. STEP 1 Let us first assume that $u \in C^\infty(\Omega) \cap BV(\Omega)$. Then, by Sard's theorem, for $\mathcal{L}^1$-a.e. $t \in \mathbb{R}$, $\{u > t\}$ is an open set with C^∞ boundary $\{u = t\}$ and the assertion is an immediate consequence of the coarea formula (2.19). However, we give here a self contained proof of (3.10) and (3.11) which does not use the coarea formula for rectifiable sets (2.18).
To this aim, let us fix a function $f \in C^1_0(\Omega)$ and a function $\psi \in C^1_0(\mathbb{R})$, such that $0 \le \psi(t) \le 1$. We set $\mathcal{S}_u = \{(x, t) \in \mathbb{R}^{n+1} : t < u(x)\}$ and recall that for $\mathcal{L}^1$-a.e. $t \in (\inf u, \sup u)$, the interior normal to the boundary of $\{u > t\}$ is given by $\nabla u/|\nabla u|$. Therefore, by Fubini's theorem and by the classical divergence theorem, we get

$$\begin{aligned}
&\int_{\mathcal{S}_u\cap(\Omega\times\mathbb{R})} \psi(t) \sum_{i=1}^n \frac{\partial}{\partial x_i}\Big(f(x)\frac{\nabla_i u}{|\nabla u|}\Big)\, dxdt \\
&= \int_{-\infty}^{+\infty} \psi(t)\, dt \int_{\{u>t\}} \sum_{i=1}^n \frac{\partial}{\partial x_i}\Big(f(x)\frac{\nabla_i u}{|\nabla u|}\Big) dx = -\int_{-\infty}^{+\infty} \psi(t)\, dt \int_{\{u=t\}} f(x) d\mathcal{H}^{n-1}.
\end{aligned} \tag{3.12}$$

Let $\Phi : \Omega \times \mathbb{R} \mapsto \mathbb{R}^{n+1}$ be the map $\Phi(x,t) = \Big(f(x)\psi(t)\frac{\nabla_1 u}{|\nabla u|}, \dots, f(x)\psi(t)\frac{\nabla_n u}{|\nabla u|}, 0\Big)$ and denote by ν the interior normal to $\mathcal{S}_u$. Since for any point on the graph G_u of u we have $\nu = \Big(\frac{\nabla_1 u}{\sqrt{1+|\nabla u|^2}}, \dots, \frac{\nabla_n u}{\sqrt{1+|\nabla u|^2}}, -\frac{1}{\sqrt{1+|\nabla u|^2}}\Big)$, using the divergence theorem again, we get

$$\begin{aligned}
\int_{\mathcal{S}_u\cap(\Omega\times\mathbb{R}))} \mathrm{div}_{x,t}\Phi\, dxdt &= -\int_{\partial(\mathcal{S}_u\cap(\Omega\times\mathbb{R}))} \langle \Phi, \nu\rangle\, d\mathcal{H}^n \\
&= -\int_{G_u} f(x)\psi(t)\frac{|\nabla u|}{\sqrt{1+|\nabla u|^2}}\, d\mathcal{H}^n(x,t) = -\int_\Omega f(x)\psi(u(x))|\nabla u|\, dx\,.
\end{aligned}$$

Since the left hand side in this equality and the left hand side in (3.12) are equal, we obtain that

$$\int_\Omega f(x)\psi(u(x))|\nabla u|\,dx = \int_{-\infty}^{+\infty}\psi(t)\,dt\int_{\{u=t\}} f(x)\,d\mathcal{H}^{n-1}(x)\,,$$

hence (3.11) follows by letting $\psi \uparrow 1$. The case $f = \chi_A$, where A is any open subset of Ω, then follows by approximating χ_A by an increasing sequence of nonnegative functions $f_h \in C^1_0(\Omega)$.

STEP 2 Let us now assume that u is a BV function and prove (3.10) for an open set $A \subset \Omega$. Since the perimeter of $\{u > t\}$ does not change if we replace u by a function coinciding $\mathcal{L}^n$-a.e. with u, we may assume without loss of generality that u is a Borel function, hence also $(x,t) \to \chi_{\{u>t\}}(x)$ is a Borel function in $A \times \mathbb{R}$. Notice that for any x in A

$$u(x) = \int_0^{+\infty}\chi_{\{u>t\}}(x)\,dt - \int_{-\infty}^0 (1-\chi_{\{u>t\}}(x))\,dt\,.$$

Therefore, for any $\varphi \in C^1_0(A,\mathbb{R}^n)$ with $\|\varphi\|_\infty \le 1$, we get

$$\begin{aligned}\int_A u\mathrm{div}\varphi\,dx &= \int_A \mathrm{div}\varphi\,dx\int_{-\infty}^{+\infty}\chi_{\{u>t\}}(x)\,dt\\ &= \int_{-\infty}^{+\infty}dt\int_A\chi_{\{u>t\}}(x)\mathrm{div}\varphi\,dx \le \int_{-\infty}^{+\infty}P(\{u>t\};A)\,dt\,,\end{aligned}$$

hence, we have

$$|Du|(A) \le \int_{-\infty}^{+\infty}P(\{u>t\};A)\,dt\,. \tag{3.13}$$

To show the opposite inequality, we use Theorem 3.7, thus getting a sequence u_h of smooth functions converging to u in $L^1(A)$ and $\mathcal{L}^n$-a.e. in A, such that $\int_A |\nabla u_h|dx \to |Du|(A)$. Since $\mathcal{L}^n(\{u=t\}) = 0$ for $\mathcal{L}^1$-a.e. t and for all such t the functions $\chi_{\{u_h>t\}}$ converge to $\chi_{\{u>t\}}$ almost everywhere in A, from Fatou's lemma and from Step 1 we get that

$$\begin{aligned}\int_{-\infty}^{+\infty}P(\{u>t\};A)\,dt &\le \int_{-\infty}^{+\infty}\liminf_{h\to\infty}P(\{u_h>t\};A)\,dt\\ &\le \liminf_{h\to\infty}\int_{-\infty}^{+\infty}P(\{u_h>t\};A)\,dt = \lim_{h\to\infty}\int_A|\nabla u_h|dx = |Du|(A)\,.\end{aligned}$$

This inequality, together with (3.13), proves (3.10) when B is an open set.

STEP 3 If $K \subset \Omega$ is a compact set, (3.10) follows at once from Step 2, by approximating K from above by a decreasing sequence of open sets. Let us now assume that $B \subset \Omega$ is a Borel set such that $|Du|(B) = 0$, then there exists a decreasing sequence of open sets $A_h \subset \Omega$, such that $B \subset A_h$ for any h and $\lim_{h\to\infty} |Du|(A_h) = 0$. From Step 2 we then get

$$\lim_{h\to\infty} \int_{-\infty}^{+\infty} P(\{u > t\}; A_h)\, dt = 0$$

and from this equality we immediately have that $P(\{u > t\}; B) = 0$ for $\mathcal{L}^1$-a.e. $t \in \mathbb{R}$. This proves (3.10) when B is a Borel set such that $|Du|(B) = 0$. The general case is obtained by writing any Borel set B in Ω as the union of an increasing sequence of compact sets K_h and of a Borel set B_0 such that $|Du|(B_0) = 0$. Finally, (3.11) immediately follows from (3.10) and Lemma 2.14. □

As a consequence of the coarea formula, we have the following approximation result for sets of finite perimeter.

Theorem 3.17 *Let E be a set of finite perimeter in $\mathbb{R}^n$ with $\mathcal{L}^n(E) < \infty$, $n \geq 2$. Then, there exists a sequence of bounded open sets E_h with C^∞ boundaries, such that $\chi_{E_h} \to \chi_E$ in $L^1(\mathbb{R}^n)$ and $P(E_h) \to P(E)$.*

Proof. By Theorem 3.7 there exists a sequence of functions $u_h \in C^\infty(\mathbb{R}^n) \cap BV(\mathbb{R}^n)$, $0 \leq u_h \leq 1$, converging in $L^1(\mathbb{R}^n)$ to χ_E and such that

$$\lim_{h\to\infty} \int_{\mathbb{R}^n} |\nabla u_h|\, dx = P(E)\,.$$

By approximating each u_h in the norm of $W^{1,1}(\mathbb{R}^n)$ with a smooth function with compact support, it is clear that we may assume that each u_h has compact support. Therefore, by the coarea formula (3.10) we have that

$$\int_0^1 \liminf_{h\to\infty} P(\{u_h > t\})\, dt \leq \liminf_{h\to\infty} \int_0^1 P(\{u_h > t\})\, dt \leq \lim_{h\to\infty} \int_{\mathbb{R}^n} |\nabla u_h|\, dx = P(E)\,.$$

By Sard's theorem, for any h, $\{u_h > t\}$ has C^∞ boundary for $\mathcal{L}^1$-a.e. $t \in (0,1)$. Thus, we get that there exists $t \in (0,1)$ such that $\{u_h > t\}$ is a bounded open set with C^∞ boundary for any h and $\liminf_{h\to\infty} P(\{u_h > t\}) \leq P(E)$. Passing possibly to a subsequence, we may assume that this lim inf is indeed a limit, hence

$$\lim_{h\to\infty} P(\{u_h > t\}) \leq P(E)\,. \tag{3.14}$$

Notice that

$$\begin{aligned}\int_{\mathbb{R}^n}|\chi_E-\chi_{\{u_h>t\}}|\,dx &= \mathcal{L}^n(E\setminus\{u_h>t\})+\mathcal{L}^n(\{u_h>t\}\setminus E)\\ &\le \frac{1}{1-t}\int_{E\setminus\{u_h>t\}}|\chi_E-u_h|\,dx+\frac{1}{t}\int_{\{u_h>t\}\setminus E}|u_h-\chi_E|\,dx\,,\end{aligned}$$

hence, $\chi_{\{u_h>t\}}\to\chi_E$ in $L^1(\mathbb{R}^n)$ and thus, by the lower semicontinuity of perimeters and (3.14), we obtain the assertion with $E_h=\{u_h>t\}$. □

We can now pass to the proof of the isoperimetric inequality in dimension $n>1$. The one dimensional case is in fact a simple consequence of the following characterization of the sets of finite perimeter ([3, Proposition 3.52]).

Proposition 3.18 *Let $E\subset\mathbb{R}$ be a set of finite perimeter. Then, there exist N open intervals (a_i,b_i), $i=1,\ldots,N$, such that $-\infty\le a_1<b_1<a_2<b_2<\cdots<a_N<b_N\le+\infty$, with the property that E is equivalent to $\bigcup_{i=1}^N(a_i,b_i)$. Moreover $P(E)=\#\{i:a_i>-\infty\}+\#\{i:b_i<+\infty\}$.*

Notice that from the above characterization it follows that if E is a set of finite perimeter in $\mathbb{R}$ and $\mathcal{L}^1(E)<\infty$, then $P(E)\ge 2$.

The isoperimetric inequality in higher dimension is a consequence of the classical Sobolev imbedding theorem.

Theorem 3.19 *Let $n>1$. There exists a positive constant $\gamma(n)>0$ such that for any function $u\in W^{1,1}(\mathbb{R}^n)$ the following inequality holds*

$$\left(\int_{\mathbb{R}^n}|u|^{\frac{n}{n-1}}\,dx\right)^{\frac{n-1}{n}}\le\gamma(n)\int_{\mathbb{R}^n}|\nabla u|\,dx\,. \tag{3.15}$$

Theorem 3.20 (Isoperimetric inequality) *Let $E\subset\mathbb{R}^n$ be a set of finite perimeter with finite measure. Then,*

$$\Big[\mathcal{L}^n(E)\Big]^{\frac{n-1}{n}}\le\gamma(n)P(E)\,, \tag{3.16}$$

where $\gamma(n)$ is the constant appearing in (3.15). Moreover the two inequalities (3.15) and (3.16) are equivalent.

Proof. Step 1 By Theorem 3.7 we can approximate χ_E by a sequence u_h of functions from $C^1_0(\mathbb{R}^n)$ such that

$$u_h(x)\to\chi_E(x)\quad\text{for }\mathcal{L}^n\text{-a.e. }x\in\mathbb{R}^n,\qquad\int_{\mathbb{R}^n}|\nabla u_h|\,dx\to|D\chi_E|(\mathbb{R}^n)=P(E)\,.$$

Then, (3.16) follows immediately from inequality (3.15) applied to the functions u_h.
Step 2 To prove that (3.16) implies (3.15), we notice that it is enough to consider only the the case of a nonnegative function $u \in C^1_0(\mathbb{R}^n)$, from which the general case easily follows. To this aim, let us fix u and use the coarea formula (3.10) and the assumption (3.16), thus getting

$$\int_{\mathbb{R}^n} |\nabla u|\, dx = \int_0^\infty P(\{u > t\})\, dt \geq \frac{1}{\gamma(n)} \int_0^\infty \Big[\mathcal{L}^n(\{u > t\})\Big]^{\frac{n-1}{n}} dt\,. \tag{3.17}$$

Let us set, for any $t \geq 0$, $g(t) = \|\min\{u,t\}\|_{L^{n/(n-1)}(\mathbb{R}^n)}$ and notice that g is a non decreasing Lipschitz continuous function. In fact, if $0 \leq s < t$, we have

$$\begin{aligned} 0 \leq g(t) - g(s) &\leq \|\min\{u,t\} - \min\{u,s\}\|_{\frac{n-1}{n}} = \left(\int_{\{u>s\}} |\min\{u,t\} - s|^{\frac{n}{n-1}}\, dx\right)^{\frac{n-1}{n}} \\ &\leq |t-s|\Big[\mathcal{L}^n(\{u > s\})\Big]^{\frac{n-1}{n}} \leq |t-s|(\mathrm{supp} u)^{\frac{n-1}{n}}\,. \end{aligned}$$

Being Lipschitz continuous, g is differentiable for $\mathcal{L}^1$-a.e. $t > 0$. Moreover, from the inequalities above it follows that $0 \leq g'(t) \leq \Big[\mathcal{L}^n(\{u > t\})\Big]^{\frac{n-1}{n}}$, whenever $g'(t)$ exists. Therefore, from (3.17) we get

$$\|u\|_{\frac{n}{n-1}} = \int_0^\infty g'(t)\, dt \leq \int_0^\infty \Big[\mathcal{L}^n(\{u > t\})\Big]^{\frac{n-1}{n}} dt \leq \gamma(n) \int_{\mathbb{R}^n} |\nabla u|\, dx$$

and (3.15) follows. □

It can be proved that if $E \subset \mathbb{R}^n$ is a set of finite perimeter, then either E or $\mathbb{R}^n \setminus E$ has finite measure (see [3, Theorem 3.46]). Therefore, inequality (3.16) can be restated in the form

$$\Big[\min\{\mathcal{L}^n(E), \mathcal{L}^n(\mathbb{R}^n \setminus E)\}\Big]^{\frac{n-1}{n}} \leq \gamma(n) P(E)\,.$$

We conclude this section with a result concerning the one dimensional sections of sets of finite perimeter. Roughly speaking, the theorem below states that if E is a set of finite perimeter and $\nu \in \mathbb{S}^{n-1}$ is a fixed direction then, for $\mathcal{H}^{n-1}$-a.e. point z in the hyperplane π_ν orthoghonal to ν, the one dimensional section $E_{z,\nu}$ is a set of finite perimeter in $\mathbb{R}$. Moreover, the section $(\partial^* E)_{z,\nu}$ of the reduced boundary of E coincides with the reduced boundary $\partial^* E_{z,\nu}$ of the section of E. However, in the statement of the theorem, in order to simplify the notation, we consider only sections parallel to the n-th coordinate axis. To this aim, we denote the points in

$\mathbb{R}^n$ also by (x', y), where $x' \in \mathbb{R}^{n-1}$, $y \in \mathbb{R}$, and if E is any set we denote by $E_{x'}$ the section

$$E_{x'} = \{y \in \mathbb{R} : (x', y) \in E\}.$$

Moreover, if ν is any vector in $\mathbb{R}^n$, we denote by ν_y the n-th component of ν.

Theorem 3.21 (Vol'pert) *Let E be a set of finite perimeter in $\mathbb{R}^n$, $n > 1$. Then, for $\mathcal{L}^{n-1}$-a.e. $x' \in \mathbb{R}^{n-1}$*
(i) $E_{x'}$ is a set of finite perimeter in $\mathbb{R}$;
(ii) $\partial^* E_{x'} = (\partial^* E)_{x'}$;
(iii) $\nu_y^E(x', y) \neq 0$ for every $(x', y) \in \partial^* E_{x'}$; moreover there exists a piecewise constant function $\chi_{x'}^* : \mathbb{R} \to \{0, 1\}$, with $\chi_{x'}^*(y) = \chi(x', y)$ for $\mathcal{L}^1$-a.e. $y \in \mathbb{R}$, such that

$$\tag{3.18} \begin{cases} \lim\limits_{z \to y^-} \chi_{x'}^*(z) = 0, & \lim\limits_{z \to y^+} \chi_{x'}^*(z) = 1 \qquad \textit{if } \nu_y^E(x', y) > 0, \\ \lim\limits_{z \to y^-} \chi_{x'}^*(z) = 1, & \lim\limits_{z \to y^+} \chi_{x'}^*(z) = 0 \qquad \textit{if } \nu_y^E(x', y) < 0 \,. \end{cases}$$

Proof. (i) From Remark 3.8 it follows that there exists a sequence of smooth functions u_h such that

$$\tag{3.19} u_h \to \chi_E \quad \text{in } L^1_{\mathrm{loc}}(\mathbb{R}^n), \qquad \int_{\mathbb{R}^n} |\nabla u_h| \, dx \to |D\chi_E|(\mathbb{R}^n) = P(E)$$

and that

$$\tag{3.20} \lim_{h \to \infty} \int_{\mathbb{R}^n} |\nabla_i u_h| \, dx = |D_i \chi_E|(\mathbb{R}^n) \qquad \text{for all } i = 1, \ldots, n \,.$$

From (3.19), using Fubini's theorem, we have that for any $R > 0$

$$\lim_{h \to \infty} \int_{B_R^{n-1}} dx' \int_{-R}^{R} |u_h(x', y) - \chi_E(x', y)| \, dy = 0 \,.$$

From this equality, by a simple diagonalization argument, it follows that up to a (not relabelled) subsequence $u_h(x', \cdot) \to \chi_E(x', \cdot) = \chi_{E_{x'}}$ in $L^1_{\mathrm{loc}}(\mathbb{R})$ for $\mathcal{L}^{n-1}$-a.e. $x' \in \mathbb{R}^{n-1}$. Hence, by the lower semicontinuity of the total variation, we have

$$P(E_{x'}) = P(E_{x'}; \mathbb{R}) \leq \liminf_{h \to \infty} \int_{\mathbb{R}} |\nabla_y u_h(x', y)| \, dy \,.$$

Integrating this inequality and using Fatou's lemma we have, from (3.20),

$$\tag{3.21} \begin{aligned} \int_{\mathbb{R}^{n-1}} P(E_{x'}) \, dx' &\leq \int_{\mathbb{R}^{n-1}} \liminf_{h \to \infty} \int_{\mathbb{R}} |\nabla_y u_h(x', y)| \, dy dx' \\ &\leq \liminf_{h \to \infty} \int_{\mathbb{R}^n} |\nabla_y u_h(x)| \, dx = |D_y \chi_E|(\mathbb{R}^n) < \infty \,, \end{aligned}$$

hence we get in particular that, for $\mathcal{L}^{n-1}$-a.e. $x' \in \mathbb{R}^{n-1}$, $P(E_{x'}) < \infty$, i.e. $E_{x'}$ has finite perimeter.
(ii) Let us fix $\varphi \in C_0^1(\mathbb{R}^n)$ with $\|\varphi\| \leq 1$. We have

$$
\begin{aligned}
(3.22)\qquad \int_{\mathbb{R}^n} \varphi dD_y \chi_E &= -\int_{\mathbb{R}^n} \chi_E \frac{\partial \varphi}{\partial y} dx \\
&= -\int_{\mathbb{R}^{n-1}} dx' \int_{\mathbb{R}} \chi_{E_{x'}}(y) \frac{\partial \varphi}{\partial y}(x', y) dy \leq \int_{\mathbb{R}^{n-1}} P(E_{x'}) dx'
\end{aligned}
$$

and, passing to the supremum over all φ, we get that

$$
(3.23)\qquad |D_y \chi_E|(\mathbb{R}^n) \leq \int_{\mathbb{R}^{n-1}} P(E_{x'})\, dx' \,.
$$

Therefore from this inequality and from (3.21) we conclude that in (3.23) the equality holds. With exactly the same argument it can be proved that if Ω is any open set in $\mathbb{R}^n$, then

$$
|D_y \chi_E|(\Omega) = \int_{\mathbb{R}^{n-1}} P(E_{x'}; \Omega_{x'})\, dx'
$$

and, arguing as in the Step 3 of the proof of Theorem 3.16, that this equality still holds if we replace Ω by any Borel set $B \subset \mathbb{R}^n$

$$
|D_y \chi_E|(B) = \int_{\mathbb{R}^{n-1}} P(E_{x'}; B_{x'})\, dx' \,.
$$

Since $|\nu_y^E| = \dfrac{d|D_y \chi_E|}{d|D\chi_E|}$, using Theorem 3.15 this equality can be written in the form

$$
\int_{\partial^* E \cap B} |\nu_y^E(x)|\, d\mathcal{H}^{n-1}(x) = \int_{\mathbb{R}^{n-1}} \mathcal{H}^0(\partial^* E_{x'} \cap B)\, dx' \,.
$$

Therefore, using Lemma 2.14, we may conclude that if $f : \mathbb{R}^n \to [0, \infty]$ is any Borel function then

$$
(3.24)\qquad \int_{\partial^* E} f(x) |\nu_y^E(x)|\, d\mathcal{H}^{n-1}(x) = \int_{\mathbb{R}^{n-1}} dx' \int_{\partial^* E_{x'}} f(x', y)\, d\mathcal{H}^0(y) \,.
$$

On the other hand, from the coarea formula (2.20) we have also that

$$
(3.25)\qquad \int_{\partial^* E} f(x) |\nu_y^E(x)|\, d\mathcal{H}^{n-1}(x) = \int_{\mathbb{R}^{n-1}} dx' \int_{(\partial^* E)_{x'}} f(x', y)\, d\mathcal{H}^0(y) \,.
$$

Let us fix $\psi \in C_0(\mathbb{R})$ and take $f(x', y) = g(x')\psi(y)$, where g is any function in $C_0(\mathbb{R}^{n-1})$. By comparing (3.25) with (3.24) we get that for $\mathcal{L}^{n-1}$-a.e. $x' \in \mathbb{R}^{n-1}$

$$
(3.26)\qquad \int_{\partial^* E_{x'}} \psi(y)\, d\mathcal{H}^0(y) = \int_{(\partial^* E)_{x'}} \psi(y)\, d\mathcal{H}^0(y) \,.
$$

Therefore, by taking a dense sequence in $C_0(\mathbb{R})$, we conclude easily that there exists a null set $W_0 \subset \mathbb{R}^{n-1}$, such that for any $x' \in \mathbb{R}^{n-1} \setminus W_0$ (3.26) holds for all $\psi \in C_0(\mathbb{R})$ and this immediately implies that $\partial^* E_{x'} = (\partial^* E)_{x'}$.
(iii) Let us set $Z = \{x \in \partial^* E : \nu_y^E(x) = 0\}$ and let us apply (3.24) with $f = \chi_Z$, thus getting

$$\int_{\mathbb{R}^{n-1}} dx' \int_{\partial^* E_{x'}} \chi_Z(x', y)\, d\mathcal{H}^0(y) = 0\,.$$

Hence, we get that for $\mathcal{L}^{n-1}$-a.e. $x' \in \mathbb{R}^{n-1}$, $\nu_y^E(x', y) \neq 0$ for all y such that $(x', y) \in \partial^* E_{x'}$. To prove the second part of (iii), we fix $\varphi \in C_0^1(\mathbb{R}^n)$, use again the coarea formula (2.20) and the fact that $Z_{x'} = \emptyset$ for $\mathcal{L}^{n-1}$-a.e. x', thus getting

$$\int_{\mathbb{R}^n} \varphi\, dD_y \chi_E = \int_{\partial^* E \setminus Z} \frac{\varphi \nu_y^E}{|\nu_y^E|} |\nu_y^E|\, d\mathcal{H}^{n-1} = \int_{\mathbb{R}^{n-1}} dx' \int_{\partial^* E_{x'}} \varphi \operatorname{sign}(\nu_y^E)\, d\mathcal{H}^0(y)$$

and comparing this equalities with (3.22) we conclude that

$$\int_{\mathbb{R}^{n-1}} dx' \int_{\partial^* E_{x'}} \varphi \operatorname{sign}(\nu_y^E)\, d\mathcal{H}^0(y) = -\int_{\mathbb{R}^{n-1}} dx' \int_{E_{x'}} \frac{\partial \varphi}{\partial y}(x', y)\, dy\,.$$

Hence, arguing as in the final part of the proof of (ii), we conclude that there exists a null set $V_0 \subset \mathbb{R}^{n-1}$ such that for all $x' \notin V_0$ and for all $\psi \in C_0^1(\mathbb{R})$

$$\int_{\partial^* E_{x'}} \psi \operatorname{sign}(\nu_y^E)\, d\mathcal{H}^0(y) = -\int_{E_{x'}} \psi'\, dy\,.$$

Let us fix $x' \notin V_0$. By Proposition 3.18 $E_{x'}$ is equivalent to the union of finitely many disjoint open intervals (a_i, b_i), $i = 1, \dots, N$, with $b_i < a_j$ if $i < j$, and the equality above immediately implies that

$$\nu_y^E(x', a_i) > 0, \quad \nu_y^E(x', b_j) < 0 \quad \text{for all } i, j = 1, \dots, N \text{ such that } -\infty < a_i,\, b_j < \infty\,.$$

Therefore the assertion follows by taking $\chi_{x'}^* = \sum_{i=1}^{N} \chi_{(a_i, b_i)}$. □

4 The isoperimetric theorem

4.1 Steiner symmetrization of sets of finite perimeter

In this section we study the properties of sets of finite perimeter in relation to the Steiner symmetrization. To this aim, let us introduce two quantities which will be relevant for the sequel.

If E is any subset of $\mathbb{R}^n$, for all $x' \in \mathbb{R}^{n-1}$ we set

$$\ell(x') = \mathcal{L}^1(E_{x'}) = \mathcal{L}^1(\{y \in \mathbb{R} : (x', y) \in E\}),$$
$$\pi(E)^+ = \{x' \in \mathbb{R}^{N-1} : \ell(x') > 0\}.$$

Remark 4.1 Since $E_{x'}$ is a null set for all $x' \notin \pi(E)^+$, Theorem 3.21 can be equivalently restated by saying that if E is a set of finite perimeter in $\mathbb{R}^n$, then there exists a Borel set $G_E \subset \pi(E)^+$, such that $\mathcal{L}^{n-1}(\pi(E)^+ \setminus G_E) = 0$ and (i), (ii) and (iii) hold for all $x' \in G_E$. Moreover, if $\mathcal{L}^n(E) < \infty$, we may also assume without loss of generality that $\mathcal{L}^1(E_{x'}) < \infty$ for all $x' \in G_E$.

We discuss some properties of the distribution function ℓ.

Lemma 4.2 *Let $E \subset \mathbb{R}^n$ be a set of finite perimeter, with $\mathcal{L}^n(E) < \infty$. Then, $\ell \in BV(\mathbb{R}^{n-1})$ and for any bounded Borel function $\varphi : \mathbb{R}^{n-1} \to \mathbb{R}$*

$$\int_{\mathbb{R}^{n-1}} \varphi(x')\, dD_i\ell(x') = \int_{\mathbb{R}^n} \varphi(x')\, dD_i\chi_E(x), \qquad i = 1, \dots, n-1. \tag{4.1}$$

Moreover, if $B \subset \mathbb{R}^{n-1}$ is a Borel set,then

$$|D\ell|(B) \le P(E; B \times \mathbb{R}). \tag{4.2}$$

Proof. The fact that $\ell \in L^1(\mathbb{R}^n)$ follows immediately from the assumption $\mathcal{L}^n(E) < \infty$.
Let us fix $\varphi \in C_0^1(\mathbb{R}^{n-1})$ and an increasing sequence ψ_h of functions from $C_0^1(\mathbb{R})$ pointwise converging to 1, such that $0 \le \psi_h \le 1$. By Fubini's theorem, we have for any $i = 1, \dots, n-1$

$$\begin{aligned} \int_{\mathbb{R}^{n-1}} \frac{\partial\varphi}{\partial x_i}(x')\ell(x')\, dx' &= \int_{\mathbb{R}^{n-1}} \frac{\partial\varphi}{\partial x_i}(x')\, dx' \int_{\mathbb{R}} \chi_E(x', y)\, dy \\ &= \int_{\mathbb{R}^n} \frac{\partial\varphi}{\partial x_i}(x')\chi_E(x)\, dx = \lim_{h\to\infty} \int_{\mathbb{R}^n} \frac{\partial\varphi}{\partial x_i}(x')\psi_h(y)\chi_E\, dx \\ &= -\lim_{h\to\infty} \int_{\mathbb{R}^n} \varphi(x')\psi_h(y)\, dD_i\chi_E = -\int_{\mathbb{R}^n} \varphi(x')\, dD_i\chi_E. \end{aligned} \tag{4.3}$$

From these equalities, recalling that $\nu^E = \dfrac{dD\chi_E}{d|D\chi_E|}$, we get that for every $\phi \in C_0^1(\mathbb{R}^{n-1}; \mathbb{R}^{n-1})$ with $\|\phi\|_\infty \le 1$

$$\int_{\mathbb{R}^{n-1}} \ell(x') \sum_{i=1}^{n-1} \frac{\partial\phi_i}{\partial x_i}(x')\, dx' = -\int_{\mathbb{R}^n} \sum_{i=1}^{n-1} \phi_i(x')\nu_i^E(x)\, d|D\chi_E| \le |D\chi_E|(\mathbb{R}^n),$$

hence $\ell \in BV(\mathbb{R}^{n-1})$. Moreover, from (4.3) it follows that for any $\varphi \in C_0^1(\mathbb{R}^{n-1})$

$$\int_{\mathbb{R}^{n-1}} \varphi(x')\, dD_i\ell(x') = -\int_{\mathbb{R}^{n-1}} \frac{\partial \varphi}{\partial x_i}(x')\ell(x')\, dx' = \int_{\mathbb{R}^n} \varphi(x')\, dD_i\chi_E(x)\,,$$

$i = 1, \dots, n-1$, hence, by approximation, (4.1) follows for any bounded continuous function $\varphi : \mathbb{R}^{n-1} \to \mathbb{R}$. To prove the general case of a bounded Borel function, let us introduce the Borel measure μ defined for any Borel set $B \subset \mathbb{R}^{n-1}$ by setting

$$\mu(B) = |D\ell|(B) + |D\chi_E|(B \times \mathbb{R})\,.$$

Given a bounded Borel function $\varphi : \mathbb{R}^{n-1} \to \mathbb{R}$, by Lusin's theorem there exists a continuous function φ_ε such that $\mu(\{x' \in \mathbb{R}^{n-1} : \varphi_\varepsilon(x) \neq \varphi(x'\}) < \varepsilon$ and $\|\varphi_\varepsilon\|_\infty \leq \|\varphi\|_\infty$. Since (4.1) holds for φ_ε, it is easily seen that the absolute value of the difference of the left hand and right hand side of (4.1) for such a φ does not exceed $4\varepsilon\|\varphi\|_\infty$. Thus (4.1) holds also for φ thanks to the arbitrariness of ε.
Finally, (4.2) follows immediately from (4.1) if $B \subset \mathbb{R}^{n-1}$ is an open set and by a simple approximation argument in the general case. □

Let us now give an explicit formula to calculate the absolutely continuous part of $D\ell$ with respect to $\mathcal{L}^{n-1}$.

Lemma 4.3 *Let $E \subset \mathbb{R}^n$ be a set of finite perimeter, with $\mathcal{L}^n(E) < \infty$. Then, for any $i = 1, \dots, n-1$,*

$$\frac{dD_i\ell}{d\mathcal{L}^{n-1}}(x') = \int_{\partial^* E_{x'}} \frac{\nu_i^E(x', y)}{|\nu_y^E(x', y)|}\, d\mathcal{H}^0(y) \quad \text{for } \mathcal{L}^{n-1}\text{-a.e. } x' \in \pi(E)^+\,. \tag{4.4}$$

Proof. Let G_E be the set defined in Remark 4.1. Then for any $\varphi \in C_0(\mathbb{R}^{n-1})$, using (4.1), the fact that $D_i\chi_E = \nu_i^E|D\chi_E|$ and Theorem 3.15, we get

$$\begin{aligned}\int_{\mathbb{R}^{n-1}} \varphi(x')\chi_{G_E}(x')dD_i\ell(x') &= \int_{\mathbb{R}^n} \varphi(x')\chi_{G_E}(x')dD_i\chi_E(x)\\ &= \int_{\partial^* E} \varphi(x')\chi_{G_E}(x')\nu_i^E(x)d\mathcal{H}^{n-1}\,.\end{aligned}$$

Since, for any $x' \in G_E$, $\nu_y^E(x', y) \neq 0$ for all y such that $(x', y) \in \partial^* E$, from coarea formula (2.20) and from the equalities above we have

$$\int_{\mathbb{R}^{n-1}} \varphi(x')\chi_{G_E}(x')\, dD_i\ell(x') = \int_{\mathbb{R}^{n-1}} \varphi(x')\chi_{G_E}(x')\, dx' \int_{\partial^* E_{x'}} \frac{\nu_i^E(x', y)}{|\nu_y^E(x', y)|}\, d\mathcal{H}^0(y)\,.$$

Therefore from the arbitrariness of φ we get that

$$\chi_{G_E}(x')D_i\ell = \chi_{G_E}(x')\left(\int_{\partial^* E_{x'}} \frac{\nu_i^E(x', y)}{|\nu_y^E(x', y)|}\, d\mathcal{H}^0(y)\right)\mathcal{L}^{n-1}\,,$$

hence (4.4) follows, since $\mathcal{L}^{n-1}(\pi(E)^+ \setminus G_E) = 0$. □

Remark 4.4 Let us denote by E^s the Steiner symmetral of E with respect to the y direction, where E is a set of finite perimeter with finite measure. Since

$$E^s = \{(x', y) : -\ell(x')/2 < y < \ell(x')/2\},$$

$E^s = F_1 \cap F_2$, where $F_1 = \{(x', y) \in \mathbb{R}^{n-1} : y < \ell(x')/2\}$ and $F_2\{(x', y) \in \mathbb{R}^{n-1} : y > -\ell(x')/2\}$. From Examples 3.12 (iii) we know that F_1, F_2 are sets of finite perimeter, hence Proposition 3.13 implies that also E^s is a set of finite perimeter. In particular, we may apply Lemma 4.3 to E^s. Since $\pi(E)^+ = \pi(E^s)^+$ and E and E^s have the same distribution function ℓ, from (4.4) we get that for any $i = 1, \dots, n-1$

$$\frac{dD_i\ell}{d\mathcal{L}^{n-1}}(x') = 2\frac{\nu_i^{E^s}(x', \ell(x')/2)}{|\nu_y^{E^s}(x', \ell(x')/2)|} \qquad \textit{for } \mathcal{L}^{n-1}\textit{-a.e. } x' \in \pi(E)^+ . \tag{4.5}$$

Lemma 4.5 *Let $E \subset \mathbb{R}^n$ be a set of finite perimeter, with $\mathcal{L}^n(E) < \infty$. For any Borel set $B \subset \mathbb{R}^{n-1}$*

$$P(E^s; B \times \mathbb{R}) \le P(E; B \times \mathbb{R}) + \int_{\partial^* E^s \cap (B \times \mathbb{R})} |\nu_y^{E^s}|\, d\mathcal{H}^{n-1} . \tag{4.6}$$

Proof. Let us fix an open set $U \subset \mathbb{R}^{n-1}$ and let ℓ_h be a sequence of nonnegative smooth functions converging to ℓ in $L^1(U)$ and $\mathcal{L}^{n-1}$-a.e. in U, such that $\displaystyle\int_U |\nabla \ell_h| dx' \to |D\ell|(U)$. Setting, for any $h \in \mathbb{N}$,

$$E_h = \Big\{(x', y) \in U \times \mathbb{R} : -\frac{\ell_h(x')}{2} < y < \frac{\ell_h(x')}{2}\Big\},$$

we have that $\chi_{E_h}(x) \to \chi_{E^s}(x)$ for $\mathcal{L}^n$-a.e. $x \in U \times \mathbb{R}$. If $\varphi \in C_0^1(U \times \mathbb{R}; \mathbb{R}^n)$, with $\|\varphi\|_\infty \le 1$, we get

$$\begin{aligned}
&\int_{U\times\mathbb{R}} \chi_{E_h}(x) \mathrm{div}\varphi\, dx = \sum_{i=1}^{n-1} \int_U dx' \int_{\frac{-\ell_h(x')}{2}}^{\frac{\ell_h(x')}{2}} \frac{\partial \varphi_i}{\partial x_i}\, dy + \int_{U\times\mathbb{R}} \chi_{E_h}(x) \frac{\partial \varphi_n}{\partial y}\, dx \\
&= \sum_{i=1}^{n-1} \Big\{ \int_U \frac{\partial}{\partial x_i} \Big(\int_{\frac{-\ell_h(x')}{2}}^{\frac{\ell_h(x')}{2}} \varphi_i\, dy \Big) dx' - \frac{1}{2} \int_U \Big[\varphi_i\Big(x', \frac{\ell_h(x')}{2}\Big) - \varphi_i\Big(x', \frac{-\ell_h(x')}{2}\Big) \Big] \frac{\partial \ell_h}{\partial x_i}\, dx' \Big\} \\
&\qquad + \int_{U\times\mathbb{R}} \chi_{E_h} \frac{\partial \varphi_n}{\partial y}\, dx \\
&\le \frac{1}{2} \int_U |\nabla \ell_h| \sqrt{\sum_{i=1}^{n-1} \Big[\varphi_i\Big(x', \frac{\ell_h(x')}{2}\Big) - \varphi_i\Big(x', \frac{-\ell_h(x')}{2}\Big) \Big]^2}\, dx' + \int_{U\times\mathbb{R}} \chi_{E_h} \frac{\partial \varphi_n}{\partial y}\, dx \\
&\le \int_U |\nabla \ell_h|\, dx' + \int_{U\times\mathbb{R}} \chi_{E_h} \frac{\partial \varphi_n}{\partial y}\, dx .
\end{aligned}$$

From the previous inequality, letting h tend to ∞, we get that

$$\int_{U\times\mathbb{R}} \chi_E(x)\mathrm{div}\varphi\,dx \le |D\ell|(U) + \int_{U\times\mathbb{R}} \chi_{E^s}\frac{\partial\varphi_n}{\partial y}\,dx \le |D\ell|(U) + |D_y\chi_{E^s}|(U\times\mathbb{R}),$$

hence, taking the supremum over φ, from (4.2) we obtain (4.6) for the case when B is an open set. Then, the general case easily follows by an approximation argument. $\square$

The following result describes the action of the Steiner symmetrization over sets of finite perimeter.

Theorem 4.6 *Let $E\subset\mathbb{R}^n$ be a set of finite perimeter with $\mathcal{L}^n(E)<\infty$. For any Borel set $B\subset\mathbb{R}^{n-1}$*

$$P(E^s;B\times\mathbb{R}) \le P(E;B\times\mathbb{R}). \tag{4.7}$$

In particular, $P(E^s)\le P(E)$.

Proof. Fix a Borel set B in $\mathbb{R}^{n-1}$. Denoting by G_E and G_{E^s} the sets associated to E and E^s as in Remark 4.1, we set $B_1 = B\setminus(G_E\cap G_{E^s})$, $B_2 = B\cap G_E\cap G_{E^s}$. Notice that using (3.24) and recalling that, by Theorem 3.21, $\mathcal{L}^{n-1}(\pi(E)^+\setminus(G_E\cap G_{E^s}))=0$, we have that

$$\begin{aligned}|D_y\chi_{E^s}|(B_1\times\mathbb{R}) &= \int_{\partial^*E^s}\chi_{B_1\times\mathbb{R}}(x)|\nu_y^{E^s}(x)|\,d\mathcal{H}^{n-1}(x)\\ &= \int_{\mathbb{R}^{n-1}}\chi_{B_1}(x')\mathcal{H}^0(\partial^*(E^s)_{x'})\,dx' = 0.\end{aligned}$$

Therefore, from this equality and from (4.6) we have that

$$P(E^s;B_1\times\mathbb{R}) \le P(E;B_1\times\mathbb{R}). \tag{4.8}$$

By Theorem 3.21, $\nu_y^{E^s}\neq 0$ on $\partial^*E^s\cap(B_2\times\mathbb{R})$, $(\partial^*E^s)_{x'} = \partial^*E^s_{x'}$ (where $E^s_{x'}$ stands for $(E^s)_{x'}$), $(\partial^*E)_{x'} = \partial^*E_{x'}$ and $\mathcal{L}^1(E_{x'})<\infty$, hence $H^0(\partial^*E_{x'})\ge 2$. Thus, using the coarea formula (2.20), (4.5), inequality $H^0(\partial^*E_{x'})\ge 2$ and Minkowski inequality, (4.4) and the coarea formula (2.20) again, we obtain

$$\begin{aligned}P(E^s;B_2\times\mathbb{R}) &= \int_{\partial^*E^s\cap(B_2\times\mathbb{R})} d\mathcal{H}^{n-1} = \int_{\partial^*E^s\cap(B_2\times\mathbb{R})}\frac{1}{|\nu_y^{E^s}|}|\nu_y^{E^s}|\,d\mathcal{H}^{n-1}\\ &= \int_{B_2}dx'\int_{\partial^*E^s_{x'}}\frac{d\mathcal{H}^0(y)}{|\nu_y^{E^s}(x',y)|} = \int_{B_2}dx'\int_{\partial^*E^s_{x'}}\sqrt{1+\sum_{i=1}^{n-1}\left(\frac{\nu_i^{E^s}(x',y)}{|\nu_y^{E^s}(x',y)|}\right)^2}\,d\mathcal{H}^0(y)\end{aligned} \tag{4.9}$$

$$
\begin{aligned}
&= \int_{B_2} 2\sqrt{1+\sum_{i=1}^{n-1}\left(\frac{\nu_i^{E^s}(x',\ell(x')/2)}{|\nu_y^{E^s}(x',\ell(x')/2)|}\right)^2}\, dx' = \int_{B_2}\sqrt{4+\left|\frac{dD\ell}{d\mathcal{L}^{n-1}}\right|^2}\, dx' \\
&\le \int_{B_2}\sqrt{\left(\int_{\partial^* E_{x'}} d\mathcal{H}^0(y)\right)^2+\sum_{i=1}^{n-1}\left(\int_{\partial^* E_{x'}}\frac{\nu_i^E(x',y)}{|\nu_y^E(x',y)|}\, d\mathcal{H}^0(y)\right)^2}\, dx' \\
&\le \int_{B_2} dx' \int_{\partial^* E_{x'}}\sqrt{1+\sum_{i=1}^{n-1}\left(\frac{\nu_i^E(x',y)}{|\nu_y^E(x',y)|}\right)^2}\, d\mathcal{H}^0(y) = \int_{B_2} dx' \int_{\partial^* E_{x'}}\frac{d\mathcal{H}^0(y)}{|\nu_y^E(x',y)|} \\
&= \int_{\partial^* E\cap(B_2\times\mathbb{R})} d\mathcal{H}^{n-1} = P(E;B_2\times\mathbb{R})\,.
\end{aligned}
$$

The assertion then immediately follows from (4.8) and (4.9). □

As a consequence of Theorem 4.6 we obtain the following result for open convex sets. Notice that, since a bounded open convex set has Lipschitz boundary, by Example 3.12 (ii) it is also a set with finite perimeter.

Proposition 4.7 *Let $E\subset\mathbb{R}^n$ an open convex set with finite measure such that $P(E)=P(E^s)<\infty$. Then E is equal to E^s (up to a translation in the y direction).*

Proof. Since E is open and convex, also the projection $\pi(E)$ of E over $\mathbb{R}^{n-1}$ is a convex open set. Moreover, for any $x'\in\pi(E)$, $E_{x'}$ is equal to an interval, say $(y_1(x'),y_2(x'))$, where $(x',y_i(x'))\in\partial E$, $i=1,2$. Moreover, it is easily checked that y_1 is a convex function in $\pi(E)$ and y_2 is concave, hence y_1 and y_2 are locally Lipschitz functions in $\pi(E)$. Let us fix a connected open set $U\subset\subset\pi(E)$. From Theorem 4.6 we have that

$$
P(E^s;U\times\mathbb{R})\le P(E;U\times\mathbb{R}),\qquad P(E^s;(\mathbb{R}^{n-1}\setminus U)\times\mathbb{R})\le P(E;(\mathbb{R}^{n-1}\setminus U)\times\mathbb{R})
$$

and since by assumption $P(E^s)=P(E)$, we have also $P(E^s;U\times\mathbb{R})=P(E;U\times\mathbb{R})$. Since y_1,y_2 and $\ell=(y_2-y_1)/2$ are Lipschitz functions in U, we can write the equality $P(E^s;U\times\mathbb{R})=P(E;U\times\mathbb{R})$ (see Examples 3.12 (i)) as

$$
2\int_U\sqrt{1+\left|\frac{\nabla y_2-\nabla y_1}{2}\right|^2}\, dx' = \int_U\sqrt{1+|\nabla y_2|^2}\, dx' + \int_U\sqrt{1+|\nabla y_1|^2}\, dx'\,.
$$

By the strict convexity of the function $\xi\to\sqrt{1+|\xi|^2}$, the inequality above implies that $\nabla y_2(x')=-\nabla y_1(x')$ for $\mathcal{L}^{n-1}$-a.e. x' in U. Therefore, by the arbitrariety of U we may conclude that there exists a constant c such that $y_2=-y_1+c$. Hence the assertion follows. □

Looking back at the proof of Theorem 4.6, in the case when equality occurs in (4.2), we obtain some nontrivial information.

Proposition 4.8 *Let $E \subset \mathbb{R}^n$ be a set of finite perimeter, with $\mathcal{L}^n(E) < \infty$, and $B \subset \mathbb{R}^{n-1}$ any Borel set. If*

$$P(E^s; B \times \mathbb{R}) = P(E; B \times \mathbb{R})\,, \tag{4.10}$$

then, for $\mathcal{L}^{n-1}$-a.e. $x' \in B$, $E_{x'}$ is equivalent to an interval, say $(y_1(x'), y_2(x'))$; moreover,

$$\begin{aligned} \nu_i^E(x', y_1(x')) &= \nu_i^E(x', y_2(x')) \qquad\qquad \textit{when } i = 1, \dots, n-1\,, \\ \nu_y^E(x', y_1(x')) &= -\nu_y^E(x', y_2(x'))\,. \end{aligned}$$

Proof. Let us define B_1 and B_2 as in the proof of Theorem 4.6. From the assumption (4.10) and from the fact that inequality (4.7) holds in particular for B_1 and B_2 we get that $P(E^s; B_2 \times \mathbb{R}) = P(E; B_2 \times \mathbb{R})$. Therefore, both inequalities in (4.9) are indeed equalities. The fact that the first inequality is an equality yields that $\mathcal{H}^0(\partial^* E_{x'}) = 2$ for $\mathcal{L}^{n-1}$-a.e. $x' \in B_2$, i.e. that $E_{x'}$ is equivalent to an interval $(y_1(x'), y_2(x'))$. Then, since also the second inequality is an equality, we have that for all $i = 1, \dots, n-1$, $\dfrac{\nu_i^E(x', y)}{|\nu_y^E(x', y)|}$, is constant on $\partial^* E_{x'}$, and since $|\nu^E| \equiv 1$ on $\partial^* E_{x'}$, we conclude that $\nu_i^E(x', y_1(x')) = \nu_i^E(x', y_2(x'))$ and $|\nu_y^E(x', y_1(x'))| = |\nu_y^E(x', y_2(x'))|$. Then, the equality $\nu_y^E(x', y_1(x')) = -\nu_y^E(x', y_2(x'))$ follows at once from (3.18). $\square$

Remark 4.9 If E is a set of finite perimeter with finite measure such that $P(E^s) = P(E)$ and E is convex, then by Proposition 4.7 E is equivalent to E^s (up to a translation in the y direction). Notice that this is not true anymore if the convexity assumption is dropped, as shown by the examples represented in the figure below. However, Theorem 4.10, proved in [14], states that these examples are essentially the only cases in which things may go wrong.

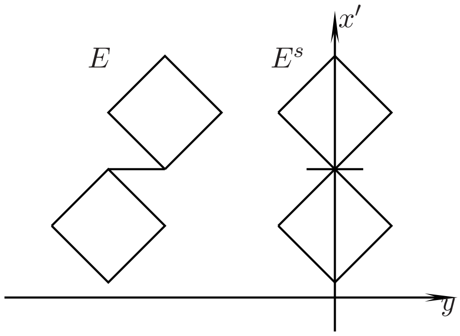
Theorem 4.10 *Let $E \subset \mathbb{R}^n$ be a set of finite perimeter, with $\mathcal{L}^n(E) < \infty$, such that $\pi(E)^+$ is equivalent to a connected open set $U \subset \mathbb{R}^{n-1}$. Assume also that $P(E) = P(E^s)$ and that*

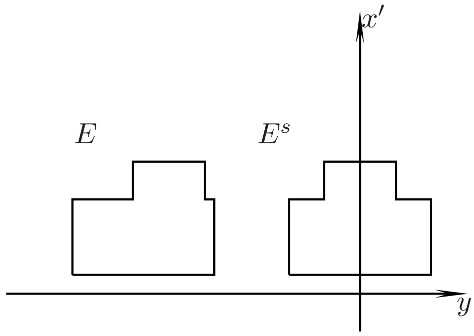
$$\lim_{r \to 0} \frac{1}{r^{n-1}} \int_{B_r^{n-1}(x_0')} \ell(x')\, dx' > 0 \qquad \textit{for } \mathcal{H}^{n-2}\textit{-a.e. } x_0' \in U\,, \tag{i}$$

$$\mathcal{H}^{n-1}\left(\{x \in \partial^* E : \nu_y^E(x) = 0\} \cap (U \times \mathbb{R})\right) = 0. \tag{ii}$$

Then, E is equivalent to E^s (up to a translation in the y direction).

We shall not use this result in the sequel, but we limit ourselves to remark that the set shown in the left picture does not satisfy the assumption (i) of theorem above, while the set on the right does not satisfy assumption (ii).





4.2 Proof of the isoperimetric theorem

We are almost ready for the isoperimetric theorem. But before that, let us recall the notion of density of a point x with respect to a set. To this aim, let E be a measurable set in $\mathbb{R}^n$ and $t \in [0,1]$. We say that a point $x \in \mathbb{R}^n$ has *density t with respect to E* if

$$\lim_{r \to 0} \frac{\mathcal{L}^n(E \cap B_r(x))}{\mathcal{L}^n(B_r(x))} = t\,.$$

We set $E^{(t)} = \{x \in \mathbb{R}^n : x \text{ has density } t \text{ with respect to } E\}$. Notice that $E^{(t)}$ is a Borel set and that $x_0 \in E^{(1)}$ if and only if $\lim_{r \to 0} \frac{1}{r^n} \int_{B_r(x_0)} (1 - \chi_E(x))dx = 0$. Therefore, $E^{(1)}$ is a Borel set such that $\mathcal{L}^n(E \triangle E^{(1)}) = 0$ and

$$x \text{ has density 1 with respect to } E \text{ if and only if } \lim_{r \to 0} \frac{\mathcal{L}^n(E \cap Q_r(x))}{2^n r^n} = 1\,, \tag{4.11}$$

where $Q_r(x)$ is the cube with center x and side length equal to $2r$.

Theorem 4.11 (The isoperimetric theorem) *Let E be a measurable set in $\mathbb{R}^n$ with finite measure. Then,*

$$\left[\mathcal{L}^n(E)\right]^{\frac{n-1}{n}} \le \frac{1}{n\omega_n^{1/n}} P(E)\,, \tag{4.12}$$

where ω_n is the measure of the unit ball. Moreover, if the equality holds in (4.12), then E is equivalent to a ball.

Proof. STEP 1 Let us fix a ball B_R such that $\mathcal{L}^n(B_R) > 1$. We claim that there exists a minimizing set $E \subset B_R$ for the problem

$$\inf\{P(E) : E \subset B_R, \mathcal{L}^n(E) = 1\}. \tag{4.13}$$

In fact, denoting by m the infimum in (4.13), there exists a sequence E_h of measurable sets contained in B_R, such that $\mathcal{L}^n(E_h) = 1$ for any h and $P(E_h) \to m$. Then, by the compactness Theorem 3.10, we may assume that χ_{E_h} converge strongly in $L^1(B_R)$ to some function $u \in BV(B_R)$. Moreover, $u = \chi_E$ for some measurable set $E \subset B_R$, $\mathcal{L}^n(E) = 1$ and $m \leq P(E) \leq \liminf_{h\to\infty} P(E_h) = m$. This proves the existence of a minimizer of (4.13).

STEP 2 Let us prove that if E is a minimizer in (4.13), then E is equivalent to a ball. To this aim let us fix a direction ν and consider the Steiner symmetral of E in the direction ν, E^s_ν, defined by (2.5). Since E^s_ν is contained in B_R and $\mathcal{L}^n(E^s_\nu) = 1$, from Theorem 4.6 and the minimality of E we get that $P(E^s_\nu) = P(E)$. Therefore, Proposition 4.8 yields that for $\mathcal{H}^{n-1}$-a.e. $z \in \pi_\nu$, where π_ν is the plane orthogonal to ν, the section $E_{z,\nu}$ is equivalent to a segment. Denoting by F the set $E^{(1)}$ of the points of density 1 with respect to E, from Lemma 4.12 below we get that, for all $z \in \pi_\nu$, $F_{z,\nu}$ is an interval. Since this property holds for any direction ν we may conclude that F is a convex set. Therefore E is equivalent to a convex set and thus, without loss of generality, we may assume that E is a convex open set. Using again the fact that $P(E^s_\nu) = P(E)$ for any direction ν from Proposition 4.7 we get that E is symmetric with respect to any direction ν, hence E is a ball. Notice that in particular we have proved that

$$\min\{P(E) : E \subset B_R, \mathcal{L}^n(E) = 1\} = n\omega_n^{1/n}.$$

STEP 3 Let us now take any measurable set E with $\mathcal{L}^n(E) = 1$; if E is bounded then E is contained in some convenient ball B_R and from Step 2 we get immediately that $P(E) \geq n\omega_n^{1/n}$ and that if equality holds then E is equivalent to a ball. Since inequality (4.12) is invariant by rescaling, this proves the assertion for the case of a bounded set.

To prove (4.12) for an unbounded set E it is enough to observe that the inequality is trivial if $P(E) = \infty$, while if $P(E) < \infty$ it can be deduced from the case of a bounded set via the approximation Theorem 3.17. Finally if E were an unbounded set of finite perimeter verifying the equality in (4.12), by repeating the argument used in Step 2 we would get that E is equivalent to a convex set and, since $\mathcal{L}^n(E) < \infty$, that E is a ball. This contradiction proves the assertion. □

Let us conclude this section with the following technical lemma used in the proof of Theorem 4.11.

Lemma 4.12 *Let E be a measurable set in $\mathbb{R}^n$ such that, for $\mathcal{L}^{n-1}$-a.e. $x' \in \mathbb{R}^{n-1}$, $E_{x'}$ is equivalent to a segment. Then, denoting by F the set of points of density 1 with respect to E, $F_{x'}$ is a segment for every $x' \in \mathbb{R}^{n-1}$.*

Proof. Let $x_1 = (x', y_1)$, $x_2 = (x', y_2)$ be two points in $F_{x'}$ with $y_1 < y_2$. Let us fix $\overline{y} \in (y_1, y_2)$. We claim that $\overline{x} = (x', \overline{y}) \in F_{x'}$. Since x_1 and x_2 have density 1 with respect to E, they have density 1 also with respect to F. Therefore, given $\varepsilon > 0$, there exists r_ε such that, if $0 < r < r_\varepsilon$, then (see (4.11))

$$\frac{\mathcal{L}^n(F \cap Q_r(x_i))}{2^n r^n} > 1-\varepsilon \qquad \text{for } i = 1, 2\,.$$

Therefore by Fubini's theorem we have that

$$\begin{aligned} 2^n r^n (1-\varepsilon) &< \mathcal{L}^n(F \cap Q_r(x_i)) \\ &= \int_{\pi(F\cap Q_r(x_i))^+} \mathcal{L}^1((F \cap Q_r(x_i))_{z'})\, dz' \le 2r\mathcal{L}^{n-1}(\pi(F \cap Q_r(x_i))^+)\,, \end{aligned}$$

and thus

$$\mathcal{L}^{n-1}(\pi(F \cap Q_r(x_i))^+) > 2^{n-1} r^{n-1}(1-\varepsilon) \qquad \text{for } i = 1, 2\,. \tag{4.14}$$

Since the essential projections of $F \cap Q_r(x_1)$ and $F \cap Q_r(x_2)$ are both contained in the same $(n-1)$-cube of side length r, from (4.14) we easily get that

$$\mathcal{L}^{n-1}\Big(\pi(F \cap Q_r(x_1))^+ \bigcap \pi(F \cap Q_r(x_2))^+\Big) > 2^{n-1} r^{n-1}(1-2\varepsilon)\,. \tag{4.15}$$

Now, recall that for $\mathcal{L}^{n-1}$-a.e. $z' \in \pi(F \cap Q_r(x_1))^+ \cap \pi(F \cap Q_r(x_2))^+$ the set $F_{z'}$ is equivalent to a segment such that $\mathcal{L}^1(F_{z'} \cap Q_r(x_i)) > 0$ for $i = 1, 2$. Therefore, if $r < \dfrac{1}{2}\min\{y_1 - \overline{y}, \overline{y} - y_2\}$, we get that

$$\mathcal{L}^1(F_{z'} \cap Q_r(\overline{x})) = 2r\,.$$

This equality, together with (4.15) implies that

$$\mathcal{L}^n(F \cap Q_r(\overline{x})) > 2^n r^n (1-2\varepsilon), \qquad \text{for all } r < r_\varepsilon\,.$$

Therefore, letting first $r \to 0$ and then $\varepsilon \to 0$, we immediately get that also $\overline{x}$ has density 1 with respect to F and thus $\overline{x} \in F$. Hence the result follows. $\square$

References

[1] F.J.Almgren: *Optimal isoperimetric inequalities,* Bull. Amer. Math. Soc., **13** (1985), 123–126.

[2] F.J.Almgren: *Optimal isoperimetric inequalities,* Indiana Univ. Math. J., **35** (1986),451–547.

[3] L.Ambrosio, N.Fusco & D.Pallara: *Functions of bounded variation and free discontinuity problems,* Oxford University Press, Oxford, 2000.

[4] C.Bandle: *Isoperimetric inequalities and applications,* Pitman, London, 2000.

[5] P.Berard & D.Meyer: *Inégalités isopérimétriques et applications,* Ann. Sci. École Norm. Sup., **15** (1982), 513–541.

[6] W.Blaschke: *Vorlesungen über Differentialgeometrie,* Springer, Berlin, 1930.

[7] W.Blaschke: *Kreis und Kugel,* Chelsea, New York, 1949.

[8] T.Bonnesen: *Les problèmes des isopérimètres et isèpiphanes,* Gauthier–Villars, Paris, 1929.

[9] Y.D.Burago & V.A.Zalgaller: *Geometric inequalities,* Springer, Berlin, 1988.

[10] R.Caccioppoli: *Misura e integrazione sugli insiemi dimensionalmente orientati I,* Rend. Accad. Naz. Lincei Cl. Sci. Fis. Mat. Nat., **12** (1952), 3–11.

[11] R.Caccioppoli: *Misura e integrazione sugli insiemi dimensionalmente orientati II,* Rend. Accad. Naz. Lincei Cl. Sci. Fis. Mat. Nat., **12** (1952), 137–146.

[12] C.Carathéodory & E.Study: *Zwei Beweise des Satzes, dass der Kreis unter allen Figuren gleichen Umfanges den grössten Inhalt hat,* Math. Ann., **68**, 133–140.

[13] I.Chavel: *On Hurwitz' method in isoperimetric inequalities,* Proc. Amer. Math. Soc., **71** (1978), 275–279.

[14] M.Chlebík, A.Cianchi, N.Fusco: *Perimeter inequalities for Steiner symmetrization: cases of equalities,* Annals of Mathematics, to appear.

[15] E.De Giorgi: *Su una teoria generale della misura* $(r-1)$*-dimensionale in uno spazio a* r *dimensioni,* Ann. Mat. Pura Appl., (4) **36** (1954), 191–213.

[16] E.De Giorgi: *Nuovi teoremi relativi alle misure $(r-1)$-dimensionali in uno spazio a r dimensioni,* Ricerche Mat. **4** (1955), 95–113.

[17] E.De Giorgi: *Sulla proprietà isoperimetrica dell'ipersfera, nella classe degli insiemi aventi frontiera orientata di misura finita,* Atti Accad. Naz. Lincei Mem. Cl. Sci. Fis. Mat. Nat. Sez. I,**5** (1958), 33–44.

[18] F.Edler: *Vervollständigung der Steinerschen elementargeometrichen Beweise für den Satz, das der Kreis grösseren Flächeninhalt besitz als jede andere ebene Figur gleich grossen Umfangs,* Nachr. Ges. Wiss. Göttingen, (1882), 73–80 (French translation in Bull. Sci.Math., **7** (1883), 198–204).

[19] L.C.Evans & R.F.Gariepy: *Lecture notes on measure theory and fine properties of functions,* CRC Press, 1992.

[20] H.Federer: *Geometric measure theory,* Springer, Berlin, 1969.

[21] S.Gallot: *Inégalités isopérimétriques et analityques sur les variétés riemannienes,* Astérisque, **163–164** (1986), 31–91.

[22] J.Hersch: *Isoperimetric monotonicity: some properties and conjectures (connections between isoperimetric inequalities),* SIAM Rev., **30** (1988), 551-577.

[23] A.Hurwitz: *Sur le problème des isopérimètres,* C.R. Acad. Sci. Paris, **132** (1901), 401–403.

[24] A.Hurwitz: *Sur quelques applications géomètriques des séries de Fourier,* Ann. Sci. École Norm. Sup., **19** (1902), 392–394.

[25] H.Lebesgue: *Leçon sur les séries trigonometriques,* Gauthier–Villars, Paris, 1906.

[26] H.Minkowski: *Volumen und Oberfläche,* Math. Ann., **57** (1903), 459–467.

[27] J.Mossino, *Inégalités isopérimétriques et applications en Physique,* Hermann, Paris, 1984.

[28] R.Osserman: *The isoperimetric inequality,* Bull. Amer. Math. Soc., **84** (1978), 1182–1238.

[29] R.Osserman: *Bonnesen-style isoperimetric inequalities,* Amer. Math. Mon., **86** (1979), 1–29.

[30] L.E.Payne: *Isoperimetric inequalities and their applications,* SIAM Rev., **9** (1961), 453-488.

[31] L.E.Payne: *Some comments over the past fifty years of isoperimetric inequalities,* in "Inequalities", W.N.Everitt Ed., Marcel Dekker, New York (1991), 143–161.

[32] G.Pòlya & G.Szegö, *Isoperimetric inequalities in Mathematical Physics,* Ann. of Math. Studies, **27**, Princeton University Press, Princeton, 1951.

[33] T.Radò: *The isoperimetric inequality and the Lebesgue definition of surface area,* Trans. Amer. Math. Soc., **61** (1947), 530–555.

[34] E.Schmidt: *Über das isoperimetrische Problem in Raum von n Dimensionen,* Math. Z., **44** (1939), 689–788.

[35] E.Schmidt: *Der Brunn–Minkowski Satz und sein Spiegeltheorem sowie die isoperime-trische Eigenshaft der Kugel in der euklidischen und nichteuklidischen Geometrie I,* Math. Nachr., **1** (1948), 81–157.

[36] E.Schmidt: *Der Brunn–Minkowski Satz und sein Spiegeltheorem sowie die isoperime-trische Eigenshaft der Kugel in der euklidischen und nichteuklidischen Geometrie II,* Math. Nachr., **2** (1948), 171–244.

[37] J.Steiner: *Gesammelte Werke,* **2**, Reimer, Berlin, 1882.

[38] G.Talenti: *The standard isoperimetric theorem,* in "Handbook on convex geometry", P.M.Gruber & J.M.Willis Eds., North Holland, Amsterdam (1993), 73–123.

[39] L.Tonelli: *Sulla proprietà di minimo della sfera,* Rend. Circ. Mat. Palermo, **39** (1915), 109–138.