

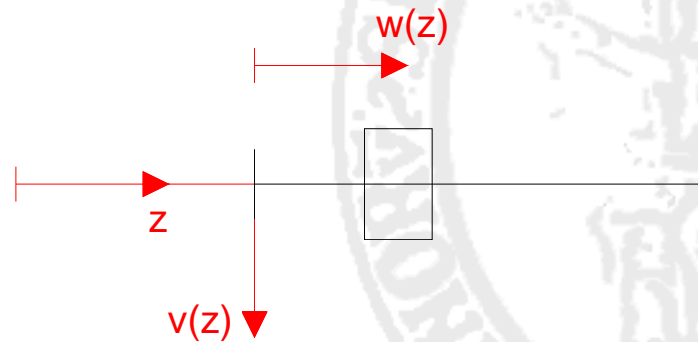


VARIATIONAL FORMULATIONS

PRINCIPLE of VIRTUAL FORCES vs. PRINCIPLE of VIRTUAL DISPLACEMENTS



TIMOSHENKO BEAM



Strains

$$\varepsilon_a = w' = \frac{dw}{dz}$$

Extensional curvature

$$\chi = \varphi' = \frac{d\varphi}{dz}$$

Bending curvature

$$\gamma = v' + \varphi = \frac{dv}{dz} + \varphi$$

Shear strain



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Internal Virtual Work

Let us consider a triplet $\hat{N}, \hat{T}, \hat{M}$ of functions NOT associated with the previously defined strains via constitutive relations.

The expression of the internal virtual work is the following:

$$L_i = \int_0^l \hat{N}(z) \varepsilon_a(z) dz + \int_0^l \hat{T}(z) \gamma(z) dz + \int_0^l \hat{M}(z) \chi(z) dz$$

or equivalently:

$$L_i = \int_0^l \hat{N} w' dz + \int_0^l \hat{T} (v' + \varphi) dz + \int_0^l \hat{M} \varphi' dz$$



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Internal Virtual Work

$$\boxed{[f g]_0^l = \int_0^l f' g dz + \int_0^l f g' dz}$$

Integration Rule by Parts

Integrating by parts the internal virtual work we obtain:

$$L_i = [\hat{N} w]_0^l - \int_0^l \hat{N}' w dz + [\hat{T} v]_0^l - \int_0^l \hat{T}' v dz + \int_0^l \hat{T} \varphi dz + [\hat{M} \varphi]_0^l - \int_0^l \hat{M}' \varphi dz$$

where:

$$[\hat{N} w]_0^l = \hat{N}(l) w(l) - \hat{N}(0) w(0)$$

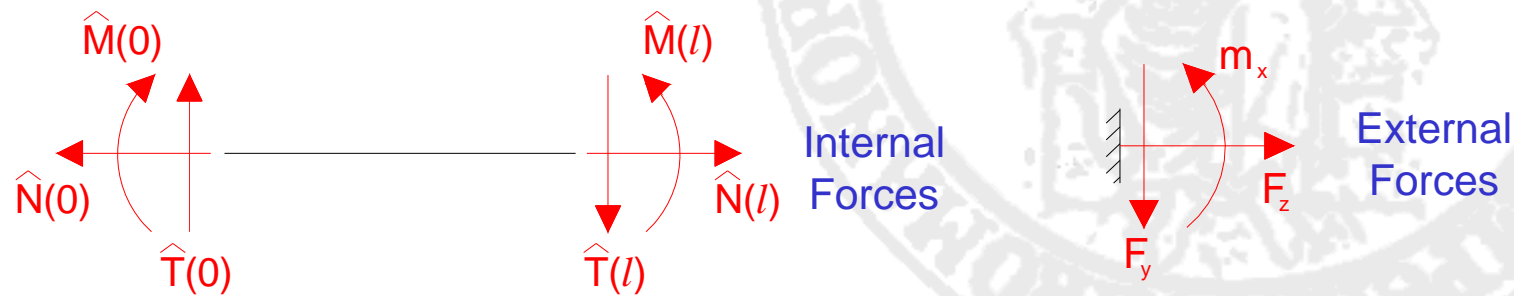
$$[\hat{T} v]_0^l = \hat{T}(l) v(l) - \hat{T}(0) v(0)$$

$$[\hat{M} \varphi]_0^l = \hat{M}(l) \varphi(l) - \hat{M}(0) \varphi(0)$$

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Internal Virtual Work

Positive Internal and External Forces



Internal and external forces at $z = l$ have the same signs, while they are discordant at $z = 0$; thus:

$$F_z(l) w(l) + F_z(0) w(0) = \hat{N}(l) w(l) - \hat{N}(0) w(0)$$

$$F_y(l) v(l) + F_y(0) v(0) = \hat{T}(l) v(l) - \hat{T}(0) v(0)$$

$$m_x(l) \varphi(l) + m_x(0) \varphi(0) = \hat{M}(l) \varphi(l) - \hat{M}(0) \varphi(0)$$



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Internal Virtual Work

Therefore the end reactions are the following:

$$\begin{aligned} R_{A1} &= F_z(0) & R_{B1} &= F_z(l) \\ R_{A2} &= F_y(0) & R_{B2} &= F_y(l) \\ R_{A3} &= m_x(0) & R_{B3} &= m_x(l) \end{aligned}$$

while the dual kinematic parameters are defined by:

$$\begin{aligned} s_{A1} &= w(0) & s_{B1} &= w(l) \\ s_{A2} &= v(0) & s_{B2} &= v(l) \\ s_{A3} &= \varphi(0) & s_{B3} &= \varphi(l) \end{aligned}$$



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We thus obtain the specialization of the *Fundamental Identity of Mechanics* to the case of the Timoshenko beam:

$$\int_0^l \hat{N} \varepsilon_a dz + \int_0^l \hat{T} \gamma dz + \int_0^l \hat{M} \chi dz = \sum_{i=1}^3 \hat{R}_{Ai} s_{Ai} + \sum_{j=1}^3 \hat{R}_{Bj} s_{Bj} - \int_0^l \hat{N}' w dz - \int_0^l \hat{T}' v dz + \int_0^l (\hat{M}' - \hat{T}) \varphi dz$$

$\forall v, w, \varphi \quad \forall \hat{N}, \hat{T}, \hat{M}$

where the functions v, w, φ , as well as $\hat{N}, \hat{T}, \hat{M}$ are completely arbitrary but they must be continuous with their derivatives and possess a degree of continuity sufficient to make the integrals well-defined.

Once more we remind that the hat superimposed on the internal force functions $\hat{N}, \hat{T}, \hat{M}$ emphasizes the fact that they are completed unrelated from the displacement functions, in the sense that the former cannot be obtained from the latter via a constitutive relation.



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From the Fundamental Identity of Mechanics one derives:

- the Principle of Virtual Forces (PVF)
- the Principle of Virtual Displacements (PVD).

The **PVF** or **variational condition of compatibility**

reformulates the compatibility between displacements w , v , φ and associated strains ε_a , γ , χ , usually expressed in differential, or “strong” form,

$$\varepsilon_a = w' \quad \gamma = v' + \varphi \quad \chi = \varphi'$$

into an integral, or “weak”, form represented by the specialization of the Fundamental Identity of Mechanics to special classes of internal forces \hat{N} , \hat{T} , \hat{M} . Their arbitrariness, which is exploited to specialize the Fundamental Identity of Mechanics to a form useful for applications, justifies the term “virtual” which is commonly appended to the name of the principle.



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PRINCIPLE OF VIRTUAL FORCES (PVF)

Recalling that

$$\hat{N}' = -\hat{q}_a \quad \hat{T}' = -\hat{q}_f \quad \hat{M}' - \hat{T} = -\hat{m}_f$$

it is natural to choose an arbitrary set of self-equilibrated internal forces, that is a set of internal forces fulfilling the conditions

$$\hat{N}' = 0 \quad \hat{T}' = 0 \quad \hat{M}' = 0$$

Thus the Fundamental Identity of Mechanics specializes to

$$\int_0^l \hat{N} w' dz + \int_0^l \hat{T} (v' + \varphi) dz + \int_0^l \hat{M} \varphi' dz = \sum_{i=1}^3 \hat{R}_{Ai} s_{Ai} + \sum_{j=1}^3 \hat{R}_{Bj} s_{Bj} \quad \forall \hat{N}, \hat{T}, \hat{M} \text{ self-equilibrated}$$

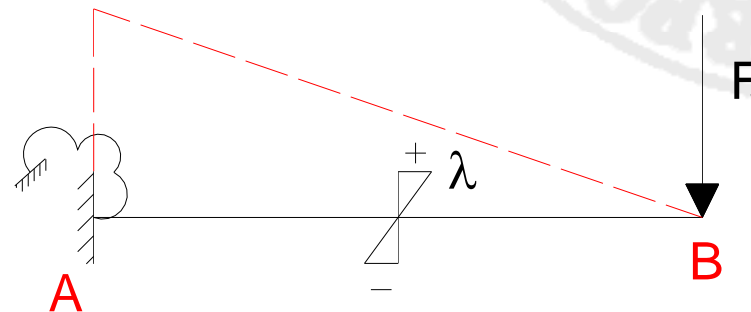


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PRINCIPLE OF VIRTUAL FORCES (PVF)

In order to distinguish between real and virtual schemes we superimpose a hat \wedge on the parameters pertaining to the virtual scheme.

To fix the ideas let us apply the PVF to a cantilever beam with an angular spring placed at the left end, a vertical force acting at the right end and a thermal distortion λ . The unknown which is looked for is the rotation φ_B .



Displacements system
(or real scheme)



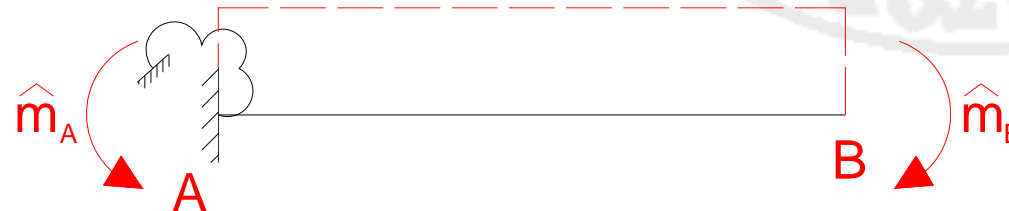
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PRINCIPLE OF VIRTUAL FORCES (PVF)

For simplicity we shall make reference to an Euler-Bernoulli model, so that:

$$\gamma = v' + \varphi = 0 \Rightarrow v' = -\varphi$$

To determine the rotation φ_B we choose a convenient self-equilibrated system, that is a system of forces in equilibrium with the restraints.



Forces system
(or virtual scheme)



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PRINCIPLE OF VIRTUAL FORCES (PVF)

Denoting by k the spring stiffness or, equivalently, by $c = 1/k$ its flexibility it turns out to be:

$$\varphi' = \frac{M}{EJ} + \frac{\alpha\Delta t}{h_s}$$

$$\varphi_A = -c \cdot m_A$$

Being also $N = 0$, the PVF specializes to the following:

$$\int_0^l \hat{M} d\varphi = \hat{m}_B \varphi_B + \hat{m}_A \varphi_A$$

that is:

$$\int_0^l \hat{M} \frac{M dz}{EJ} + \int_0^l \hat{M} \frac{\alpha\Delta t}{h_s} dz = \hat{m}_B \varphi_B + \hat{m}_A (-c \cdot m_A)$$



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PRINCIPLE OF VIRTUAL FORCES (PVF)

Observing that the bending moment function is provided by:

$$\hat{M}(z) = -\hat{m}_B \quad M(z) = -F \cdot (l - z)$$

the last equation becomes, having assumed couples positive if clockwise:

$$\int_0^l \hat{m}_B \frac{F(l-z)dz}{EJ} + \int_0^l (-\hat{m}_B) \frac{-\alpha |\Delta t|}{h_s} dz = \hat{m}_B \varphi_B + (-\hat{m}_B)[-c \cdot (-F \cdot l)]$$

where a linear dependence upon \hat{m}_B of all addends in the previous relation is observed. This means that any value of \hat{m}_B can be chosen; for this reason in the application it is usually set $\hat{m}_B = 1$. **More generally, the set of self-equilibrated forces to be used in the application of the PVF for the evaluation of an arbitrary displacement is associated with a unitary value of the force dual of the displacement which is looked for.**



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PRINCIPLE OF VIRTUAL FORCES (PVF)

Notice that φ_B is certainly positive whenever $\alpha = 0$. However, this does not mean that φ_B is counter-clockwise according to the usual assumption on the positive convention on rotation. Since φ_B has been derived by the PVF, a positive value of φ_B implies simply that φ_B makes the virtual force, in this case \hat{m}_B , perform a (virtual) positive work. Thus, being \hat{m}_B clockwise, a positive φ_B represents a clockwise rotation as well. On the other hand the geometry of the beam and the loading conditions clearly indicate that a clockwise rotation φ_B must be expected.

In conclusion, we get:

$$\int_0^l \frac{F(l-z)dz}{EJ} + \int_0^l \frac{\alpha|\Delta t|dz}{h_s} = \varphi_B - c \cdot (F \cdot l)$$

so that φ_B is obtained by evaluating the integrals placed on the left-hand side.



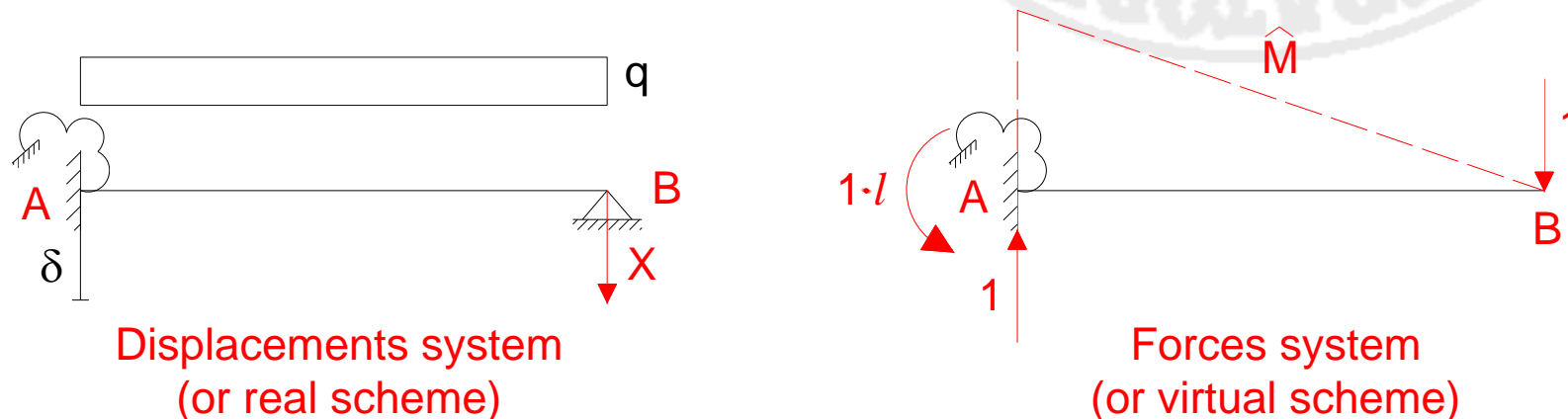
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PRINCIPLE OF VIRTUAL FORCES (PVF)

Let us now apply the PVF to the solution of a statically indeterminate structural scheme, namely a beam elastically clamped at the left end and with a roller at the right end.

The loads applied on the beam are a uniform vertical load and a given vertical displacement at the clamped end.

The real and virtual schemes are follows:



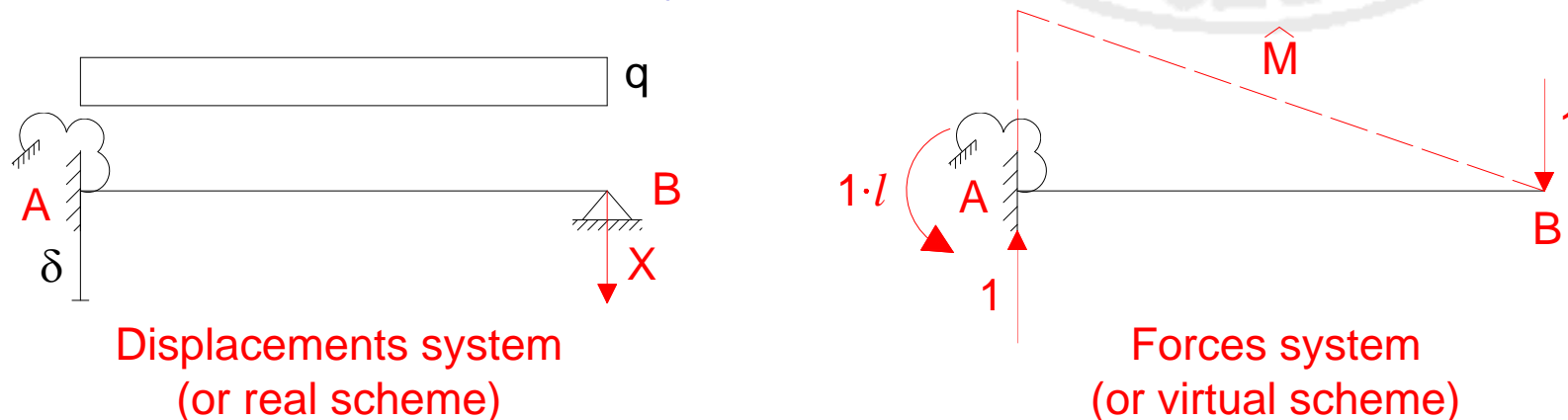
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PRINCIPLE OF VIRTUAL FORCES (PVF)

Thus, the equation representing the PVF yields:

$$\int_0^l \hat{M} \frac{M dz}{EJ} = -1 \cdot \delta + l \cdot (-c \cdot m_A)$$

The real bending moment on the real scheme can be expressed as the sum of the moment M_0 associated with the applied loads, evaluated on the isostatic scheme resulting from the elimination of the redundant supports, and of the virtual moment \hat{M} amplified by X .





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PRINCIPLE OF VIRTUAL FORCES (PVF)

Thus, it turns out to be:

$$M(z) = M_0(z) + X \cdot \hat{M}(z)$$

And the PVF equation becomes:

$$\int_0^l \hat{M} \frac{(M_0 + X \cdot \hat{M})}{EJ} dz = -1 \cdot \delta + l \cdot (-c \cdot m_A)$$

Considering that the expression of the bending moment pertaining to the virtual scheme is given by:

$$\hat{M}(z) = -1 \cdot (l - z)$$

the PVF equation can be rewritten in the form:

$$\int_0^l -1 \cdot (l - z) \frac{M}{EJ} dz = -\delta + l \cdot (-c \cdot m_A) \quad (\text{PVF1})$$



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PRINCIPLE OF VIRTUAL FORCES (PVF)

Furthermore, the value of the bending moment at A is given by:

$$m_A = -M_0(0) - X \cdot \hat{M}(0)$$

so that the equation of PVF can be expressed in the form:

$$\int_0^l -1 \cdot (l-z) \frac{(M_0 + X \cdot \hat{M})}{EJ} dz = -\delta + l \cdot [-c \cdot (-M_0(0) - X \cdot \hat{M}(0))]$$

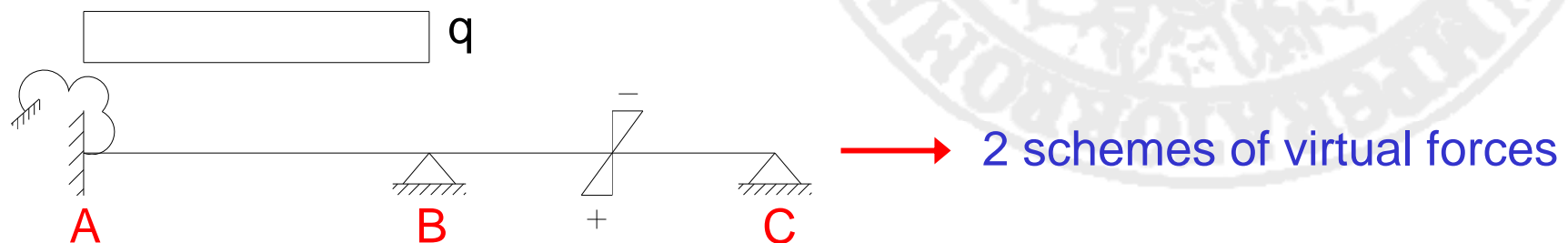
Thus, the following equation in the unknown quantity X is derived:

$$\int_0^l -1 \cdot (l-z) \frac{M_0}{EJ} dz + X \cdot \int_0^l -1 \cdot (l-z) \frac{\hat{M}}{EJ} dz = -\delta + c \cdot l \cdot M_0(0) + X \cdot l \cdot \hat{M}(0)$$

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PRINCIPLE OF VIRTUAL FORCES (PVF)

If the structure to solve is n -times statically indeterminate, it is necessary to write a system of n equations because n independent compatibility conditions have to be imposed on a suitably defined statically determinate scheme.



From a computational point of view, the PVF is not particularly convenient for hyperstatic structures since the choice of redundant supports can be completely arbitrary and cannot be easily automatized.



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PRINCIPLE OF VIRTUAL FORCES (PVF)

Mechanical interpretation of PVF:

It has been already emphasized that the PVF represents an alternative formulation, expressed in integral or “weak” form, of the compatibility relation between displacements and strains; this relation is naturally expressed in differential or “strong” form.

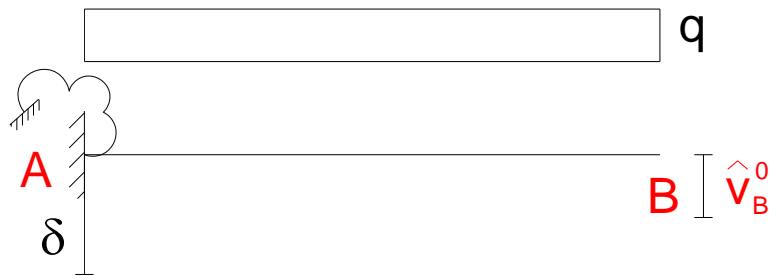
Thus, to give physical insight of the PVF equation, we are going to show that it actually represents a compatibility condition expressed in terms of absolute or relative displacements.



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PRINCIPLE OF VIRTUAL FORCES (PVF)

To this end let us consider the following schemes:



and write the PVF equation:

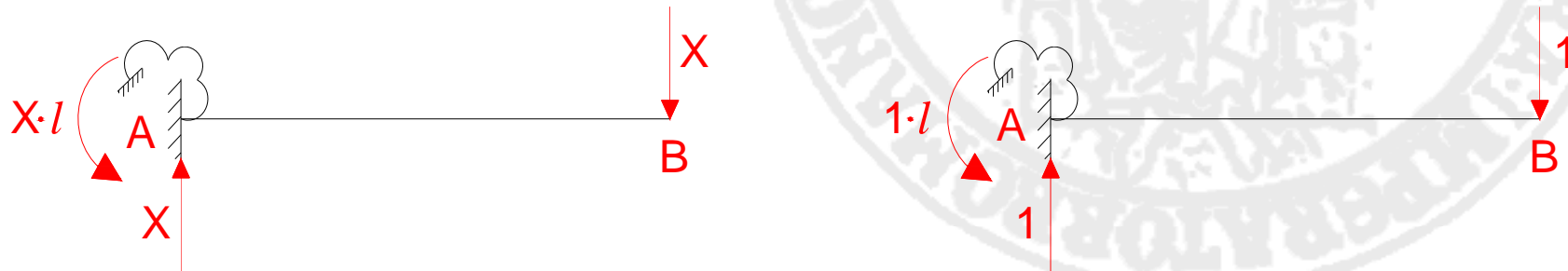
$$1 \cdot v_B^0 - 1 \cdot \delta + l \cdot (-c \cdot m_A) = \int_0^l \hat{M} \varphi' dz = \int_0^l \hat{M} \frac{M_0}{EJ} dz \quad (\text{PVF2})$$



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PRINCIPLE OF VIRTUAL FORCES (PVF)

We now apply the PVF to the following schemes:



in order to evaluate v_B^X :

$$1 \cdot v_B^X - l \cdot (-c \cdot m_A^X) = \int_0^l \hat{M} \frac{M^X}{EJ} dz = \int_0^l X \frac{\hat{M} \cdot \hat{M}}{EJ} dz \quad (\text{PVF3})$$



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PRINCIPLE OF VIRTUAL FORCES (PVF)

Summing up the equations PVF2 and PVF3 and comparing the resulting expression with PVF1 we ultimately infer that:

$$v_B^0 + v_B^X = 0$$

Thus, it has been shown that the application of the PVF actually amounts to imposing a compatibility equation expressed in terms of absolute (or relative) displacements.



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PRINCIPLE OF VIRTUAL DISPLACEMENTS (PVD)

The Principle of Virtual Displacements represents the theoretical basis of the traditional Finite Element Method (FEM), i.e. the one based on the so-called displacement approach.

We start by using the Fundamental Identity of Mechanics to enforce in weak form the equilibrium conditions between the applied loads on the beam q_a , q_t , m and the relevant internal forces N , T , M :

$$N' = -q_a$$

$$T' = -q_t$$

$$M' - T = -m$$

In other words we are expressing a variational condition of the equilibrium existing between external and internal loads on the real scheme, i.e. on the structural scheme we are going to solve.



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PRINCIPLE OF VIRTUAL DISPLACEMENTS (PVD)

The Fundamental Identity of Mechanics becomes:

$$\int_0^l N \hat{w}' dz + \int_0^l T (\hat{v}' + \hat{\phi}) dz + \int_0^l M \hat{\phi}' dz = \sum_{i=1}^3 R_{Ai} \hat{S}_{Ai} + \sum_{j=1}^3 R_{Bj} \hat{S}_{Bj} + \int_0^l q_a \hat{w} dz + \int_0^l q_t \hat{v} dz + \int_0^l m \hat{\phi} dz$$

$\forall \hat{v}, \hat{w}, \hat{\phi}$

where, similarly to PVF, a hat has been superimposed to the “virtual” fields, i.e. to fields which are suitably chosen in order to enforce the condition, in this case equilibrium, on the actual scheme.

The “virtual” fields v , w and ϕ are usually termed “test functions” since they are used “to test” the equilibrium conditions. From the mathematical standpoint they have to possess a suitable degree of continuity in order to make the integrals well defined. In particular the derivatives \hat{v}' , \hat{w}' and $\hat{\phi}'$ need to be at least piecewise continuous.



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PRINCIPLE OF VIRTUAL DISPLACEMENTS (PVD)

Clearly, the arbitrariness of \hat{v} , \hat{w} and $\hat{\phi}$ can be suitably specialized to simplify the previous equation. For instance, in the case of fixed restraints or restraints with imposed displacements the virtual displacements can be assumed to be zero at the restrained sections of the beam so that the sums in the previous equation vanish.

Stated equivalently, the virtual displacements can be conceived as “variations” of the actual displacements, what motivates the symbols $\hat{v} = \delta v$, $\hat{w} = \delta w$, $\hat{\phi} = \delta \phi$ commonly encountered in the literature. This interpretation further motivates the previous assertion according to which “virtual” displacements can be assumed to be zero at the restrained sections where zero or given displacements are assigned. Actually, if the real displacements assume such values at the restrained sections, their variations, i.e. the “virtual” displacements, must be zero at these sections.



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PRINCIPLE OF VIRTUAL DISPLACEMENTS (PVD)

Remark:

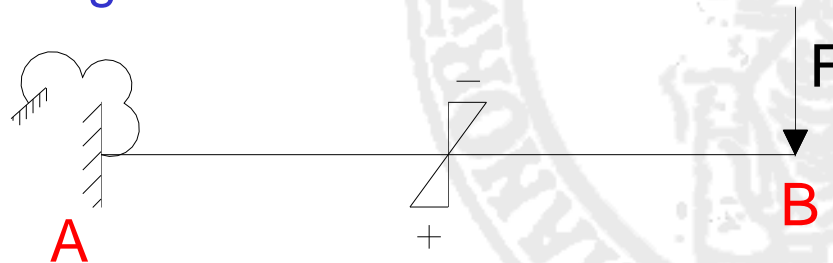
We emphasize once more, although this aspect has been already pointed out with reference to the PVF, that the addends of the previous equation do not represent expended work but only a mathematical interaction between dual quantities, the real and the virtual ones, which comes out with terms having dimension of work.

Let us now apply the PVD to an Euler-Bernoulli beam model. Thus, the shear strain γ vanishes, $T = M'$ and the distributed couples m can be eliminated since they have no dual kinematic counterpart, i.e. there is no kinematic quantity which can be associated with m so as to produce “virtual” work.

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PRINCIPLE OF VIRTUAL DISPLACEMENTS (PVD)

We consider the following scheme



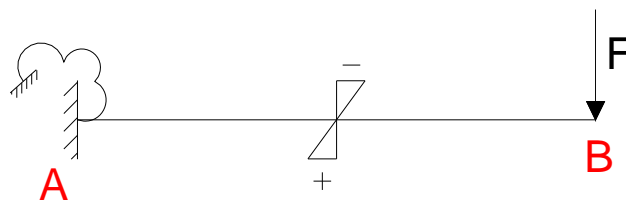
in which the rotation φ_B is looked for. Recalling that the virtual displacements can be considered as a variation of the actual ones, the “virtual” scheme has to be defined so that \hat{v}, \hat{w} and $\hat{\varphi}$ can be expressed as function of displacement parameters assigned at B. In other words we characterize the virtual displacement \hat{v}, \hat{w} and $\hat{\varphi}$ as variations of the actual ones by assigning virtual displacement fields which are expressed as functions of parameters representing arbitrary displacements at B. In turn, such parameters can be considered as variations of the actual, and unknown, values of the displacements at B.



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PRINCIPLE OF VIRTUAL DISPLACEMENTS (PVD)

Clearly, in order to characterize the virtual displacements as specified above we need to imagine the beam clamped at B and subject to the virtual parameters, namely the vertical, and horizontal displacement as well as the rotation at B. This means that the application of PVD requires to take into account a new structural model in which additional constraints have been added. Thus, for hand calculations, the PVD is not particularly suited for statically determinate structures; actually, in this case, the solution can be obtained by solving a system of equations whose order is equal to the number of additional restraints imposed to the original, real, structure.



Forces system
(real scheme)



Displacements system
(virtual scheme)

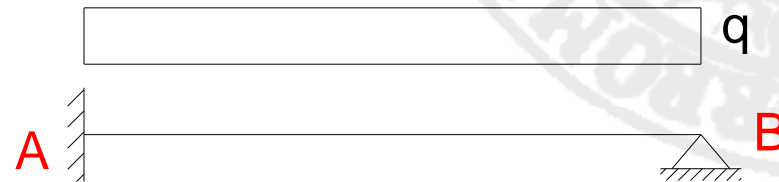


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PRINCIPLE OF VIRTUAL DISPLACEMENTS (PVD)

For these reasons we show the application of the PVD by making reference to a simple statically indeterminate structural model.

Let us consider a beam clamped at one end and hinged at the other one, subject to a uniform vertical load:



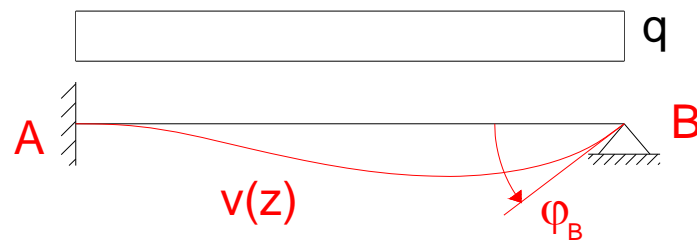
The virtual scheme to consider is, in a certain sense, in perfect duality with the one considered in the PVF. In this last one we assumed a statically determinate system by eliminating the redundant supports. On the contrary, in the PVD, we add a number of supports so as to transform the original structural scheme in a clamped-clamped beam and considering a number of

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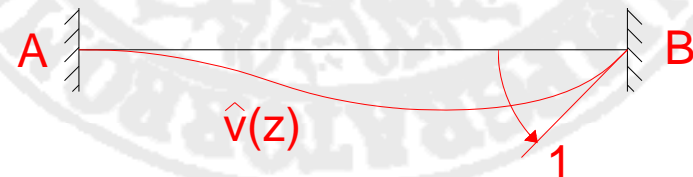
PRINCIPLE OF VIRTUAL DISPLACEMENTS (PVD)

unit-value imposed displacements corresponding to the restrained degrees of freedom.

Thus, to solve our problem, we consider the following schemes



Forces system
(real scheme)



Displacements system
(virtual scheme)

where a unit rotation is imposed at the support at B and $\hat{v}(z)$ is the relevant displacement functions.

Let us further denote by $v(z)$ the displacement function of the real scheme on which the equilibrium between the actual external and internal loads is enforced.



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PRINCIPLE OF VIRTUAL DISPLACEMENTS (PVD)

Therefore, the general equation of PVD assumes the simplified form:

$$\int_0^l M \hat{\phi}' dz = \int_0^l q \hat{v} dz$$

The bending moment M can be expressed in terms of the real displacement $v(z)$ as:

$$M(z) = -EJ \cdot v''(z)$$

so as to obtain the equation:

$$\int_0^l (-EJv'')(-\hat{v}'') dz = \int_0^l q \hat{v} dz$$

where it has been employed the well-known relation:

$$\hat{\phi}' = -\hat{v}''$$

holding for the Euler-Bernoulli beam model.



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PRINCIPLE OF VIRTUAL DISPLACEMENTS (PVD)

On the other hand, it can be set:

$$v(z) = v_0(z) + \varphi_B \cdot \hat{v}(z)$$

where the function $v_0(z)$ represents the displacement function on the clamped-clamped beam.

Substituting the previous expression in the equation supplied by the PVD one obtains:

$$\int_0^l EJ v_0'' \hat{v}'' dz + \varphi_B \cdot \int_0^l EJ \hat{v}'' \hat{v}'' dz = \int_0^l q \hat{v} dz$$

i.e. a linear equation in φ_B .

Notice that the first integral vanishes as a result of the Colonnetti's theorem.



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PRINCIPLE OF VIRTUAL DISPLACEMENTS (PVD)

Remark:

The result based on Colonnetti's theorem clearly shows that the value of φ_B does not depend on v_0 . Hence, the evaluation of the function $v_0(z)$, that is the displacement function on the clamped-clamped structural scheme, is completely unnecessary for evaluating the internal force in the beam.

Actually, disposing of the displacements and rotations at the end of the beam, it is possible to evaluate the end reactions via the stiffness matrix. Consequently, the internal forces at any section of the beam can be obtained by equilibrium.



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PRINCIPLE OF VIRTUAL DISPLACEMENTS (PVD)

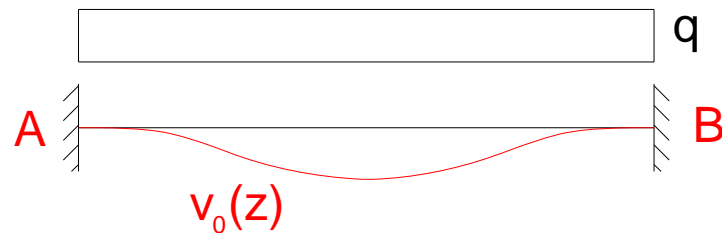
On the other hand the knowledge of $v_0(z)$ is necessary if one wants to evaluate exactly the displacements and rotations along the beam.

The previous considerations concerning $v_0(z)$ carry over directly to the 2D or 3D case in which the knowledge of $v_0(z)$ is not only unnecessary for evaluating the displacement parameters of the FEM model but it is also impossible to achieve.

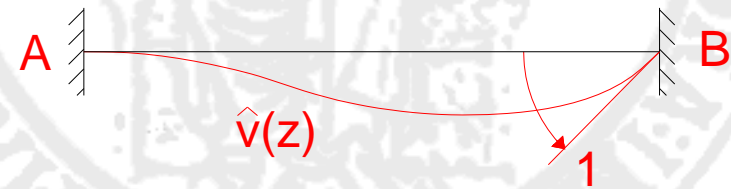
This represents a minor problem since the displacement field in each element is anyway approximated; hence the distinction between the amount due to the external loads on the fixed-end scheme and the one due to the fixed-end reactions, yet each of them impossible to determine, becomes inessential.

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PRINCIPLE OF VIRTUAL DISPLACEMENTS (PVD)



Displacements system



Forces system

Determination of the test function:

$$v^{IV}(z) = \frac{q}{EI} = 0 \Rightarrow \begin{cases} \hat{v}(z) = A + Bz + Cz^2 + Dz^3 \\ \hat{v}'(z) = B + 2Cz + 3Dz^2 \end{cases}$$



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PRINCIPLE OF VIRTUAL DISPLACEMENTS (PVD)

From the boundary conditions:

$$\hat{v}(0) = 0$$

$$\hat{v}(l) = 0;$$

$$-\hat{v}'(0) = \varphi_A = 0$$

$$-\hat{v}'(l) = \varphi_B = 1$$

the values of the integration constants are:

$$A = 0$$

$$B = 0$$

$$C = -D \cdot l = \frac{1}{l}$$

$$D = -\frac{1}{l^2}$$

Thus, it turns out to be:

$$\hat{v}(z) = \frac{1}{l} \cdot z^2 - \frac{1}{l^2} \cdot z^3$$

$$\hat{v}'(z) = \frac{2}{l} \cdot z - \frac{3}{l^2} \cdot z^2$$



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Check of the expression thus obtained:

$$\hat{v}(z) = l - \frac{l^3}{l^2} = 0 \quad \longrightarrow \quad \text{Right!!!}$$
$$\hat{v}'(z) = 2 - 3 = -1$$

Let us further evaluate the second derivate of the test function:

$$\hat{v}''(z) = \frac{2}{l} - \frac{6}{l^2} \cdot z$$

Now it is possible to calculate the previous integral:

$$\int_0^l EJ \hat{v}'' \hat{v}'' dz = EJ \cdot \int_0^l \left(\frac{2}{l} - \frac{6}{l^2} \cdot z \right)^2 dz = EJ \cdot \int_0^l \left(\frac{4}{l^2} + \frac{36}{l^4} \cdot z^2 - \frac{24}{l^3} \cdot z \right) dz =$$



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$$= EJ \cdot \left[\frac{4}{l^2} \cdot z + \frac{36}{l^4} \cdot \frac{z^3}{3} - \frac{24}{l^3} \cdot \frac{z^2}{2} \right]_0^l = EJ \cdot \left[\frac{4}{l^2} \cdot l + \frac{12}{l^4} \cdot l^3 - \frac{12}{l^3} \cdot l^2 \right] = \frac{4EJ}{l}$$

Thus, it has been obtained a stiffness coefficient dual of the bending moment;
Analogously, the integral:

$$\int_0^l q \hat{v} dz$$

represents the well-known *perfectly-fixed joint reaction* of the beam.



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PRINCIPLE OF VIRTUAL DISPLACEMENTS (PVD)

Mechanical interpretation of PVD:

To give physical insight of the PVD equation, it is possible to show that it actually represents an equilibrium condition expressed in terms of the static quantity dual of the unknown displacement, in this case a rotation.

Thus, by applying twice the PVD to the structures reported in the next slide, the PVD equation amounts to enforcing the rotational equilibrium of the reactive couples exerted by the fictitiously added constraint at B.

$$M_B^0 + \varphi_B \cdot \hat{M}_B = 0$$

Stated equivalently, the reaction of the rotational support which has been fictitiously added at B has to be zero!!!



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PRINCIPLE OF VIRTUAL DISPLACEMENTS (PVD)

To prove the previous result it is sufficient to apply twice the PVD to the schemes on the left by considering the scheme on the right as displacement system.

