European Journal of Operational Research 216 (2012) 469–476

Contents lists available at ScienceDirect

European Journal of Operational Research

journal homepage: www.elsevier.com/locate/ejor

Interfaces with Other Disciplines

The evolutionary dynamics of audit

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ARTICLE INFO

Article history:
Received 11 June 2010
Accepted 4 June 2011
Available online 22 June 2011

Keywords:
Auditing
Decision analysis
Dynamical systems
Economics
Evolutionary computations
Game theory

ABSTRACT

A critical issue in auditing is provisioning of reasonable assurance that the financial reports are free from material misstatements. The auditing detection problem can be viewed as a two-player game between the auditor and the auditee where the auditor aims at eliminating misstatements, reducing at the same time his audit efforts, while the auditee aims at benefiting from fraudulent financial reporting and defalcation.

In this paper, the auditing/fraud detection problem is modeled employing evolutionary game theory. It is proved that, given that the players have accurate information for the parameters involved in the problem, the auditing/fraud detection game is stable but not asymptotically stable. The case of the auditor being partially informed about the auditee firm is also studied and it is concluded that if the auditor is partially informed about the auditee firm, a more comprehensive audit is necessary to guarantee quality of audit. Finally, analytical results are derived concerning the impact of audit tenure on audit quality.

1. Introduction

An audit in accordance with International Standards of Auditing (ISA) should be designed to provide reasonable assurance that the financial statements as a whole are free from material misstatements (ISA 200, 2009). Misstatements in the financial statements can arise either by fraudulent reporting or inadvertent error (ISA 240, 2009). The former appear when, the government or the auditee firm manipulate on purpose financial reports in order to gain an illegal advantage, while the latter appear due to unintentional mistakes or misinterpretations in gathering or processing of data. Even though, actions should be taken for both cases to be avoided the most critical one, however, that should be detected and confronted is fraudulent financial reporting.

The issues of identification and deterioration of fraudulent financial reporting are complex and many approaches have been proposed in the literature of Business Economics. Early approaches were focused on evidence gathering for detecting fraud using statistical sampling theory (Matsamura and Tucker, 1992). Then, in the light of corporate scandals and due to the increasing demands for improvements in audit quality, questions are raised about how and to what extent this target could be achieved. Researchers recognized that in order to answer these questions, it is important to understand what an audit means to stakeholders, boards of directors, regulators and other third parties. This need led to the adoption of agency theory. For a detailed review on the subject the reader may refer to ICAEW (2005). Finally, motivated by the pioneer paper of Fellingham and Newmam (1985) researchers began to analyze the strategic interactions between auditors and clients in such a way that the auditor's strategy has an impact on the behavior of the auditee. Specifically, in this scheme, the client has the option to choose between high and low effort to eliminate misstatements in financial reports, while the auditor may select between high and low audit effort to detect misstatements. Inspired by this approach significant research has been carried out on the fraud detection problem employing game theory (Cook et al., 1997; Coates et al., 2002; Patterson and Noel, 2003; Fischbacher and Stefani, 2007; Carpenter, 2007; Bowlin et al., 2009; Wilks and Zimbelman, 2004).

A non-cooperative game theory-based auditing model studying internal control and substantive testing is presented in Cook et al. (1997). It is shown that in order to obtain the presumed socially desirable outcome of high and honest effort by all, it is necessary to adjust the cost structure in such a way that optimizes the costs of not-qualifying erroneous accounts under the condition that the auditor can prove having worked hard. A comparison with a cooperative game formulation of this problem leads to the conclusion that there is a region of values for the parameters involved in the game where both cooperative and non-cooperative approaches lead to this socially desirable outcome.

In Coates et al. (2002), the client/auditor financial reporting auditing problem is modeled based on the game of chicken. This approach is mainly focused on the client's misreporting and its

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doi:10.1016/j.ejor.2011.06.003

detection by the auditor. In addition to the game of chicken, a welfare game-based approach is also proposed to model the client/auditor problem. The same problem is also studied in Patterson and Noel (2003), where the auditee has the opportunity to commit various types of fraud. Unlike previous studies, the auditee can misappropriate assets (defalcation), misreport financial information (fraudulent financial reporting), or misreport financial information in combination with defalcation.

In Fischbacher and Stefani (2007), experimental results for a simple bi-matrix game between a manager and an auditor are reported. The paper investigates how an increase to the relative proportion of perfectly honest auditors affects the audit quality. It is proved that the quality of audited reports is higher if computerized auditors, who always perform a high audit effort, are available. Other applications of game theory in auditing may be found in Carpenter (2007) and Bowlin et al. (2009). Specifically, in Carpenter (2007) the authors study how audit team brainstorming sessions help auditors detect fraud, whereas in Bowlin et al. (2009) the authors perform an experimental examination of the influence of audit experience on subsequent reporting decisions when the auditors become managers of audited firms. Finally, for a detailed review on theoretical and empirical research concerning game theory and its application in practical auditing issues can be found in Wilks and Zimbelman (2004).

In the above approaches the auditing/fraud detection problem is formulated employing classic game theory. However, these approaches fail to specify:

(a) how the players arrive at equilibrium,
(b) whether this equilibrium is stable or not and,
(c) what is the long term behavior of the auditor's tenure on the quality of auditing.

In the current paper, a novel formulation of the auditing/fraud detection game using evolutionary game theory (EGT) is introduced to combat the liabilities of the classic game theory approach. To the authors' knowledge this is the first time in the literature that EGT is adopted to study the auditing/fraud detection problem. The present study extents the literature in several ways. First, an answer to the question on how the auditor and the client select their strategies and arrive at equilibrium is given. This is achieved by analyzing the dynamics of auditing strategies using the model of replicator dynamics. Second, the stability properties of the equilibrium are investigated. Based on *Lyapunov's First* (*or Indirect*) *Method* it is proved that, if the players have accurate information for the parameters involved in the problem, then the audit fraud detection game is stable but not asymptotically stable. It is also proved that if the auditor is partially informed about the auditee firm, a more comprehensive audit is essential to guarantee quality of audit. Third, analytical results are derived concerning the impact of audit tenure on the probability of failed audit. Unlike previous studies that are based on logistic regression models to test the relation between audit firm tenure and audit quality (Carcello and Nagy, 2004; Ghosh and Moon, 2005; Fargher et al., 2008; Jenkins and Velury, 2008; Jackson et al., 2008), in this paper, a new approach on the subject based on EGT is proposed. The results obtained comply with the IFAC Code of Ethics for Professional Accountants that requires the key audit partner to be rotated after a predefined period.

The rest of the paper is organized as follows. In Section 2, the basic principles of EGT are presented, giving emphasis on the concepts of replicator dynamics and evolutionary stability. A generalized version of the audit/fraud detection game using classic game theory in presented in Section 3, whereas in Section 4, the same problem is formulated employing evolutionary game theory. The stability of the proposed model using a dynamical system of equations is proved in Section 4, while Section 5 concludes the paper.

2. Preliminaries in evolutionary game theory

In a conventional game, the objective of a rational player is to choose the strategy which maximizes its payoff. Instead, in the frame of EGT, the game is played repeatedly by agents[1] randomly drawn from a large population (Weibull, 1996; Nowak, 2006). In general, an evolutionary process combines two significant mechanisms: a *mutation mechanism* which provides varieties and a *selection mechanism* which favors some varieties over others. The role of mutation is highlighted by the notion of evolutionary stable strategies (ESS) – which is a refinement of the Nash Equilibrium (NE) – while selection is associated to the replicator dynamics model, which assumes that a subpopulation grows (declines) when it plays strategies that are better (worse) than average.

2.1. Evolutionary stable strategies

ESS is a key concept in EGT. A population following such a strategy is invincible. Specifically, suppose that the initial population is programmed to play a certain pure or mixed strategy[2] x (the incumbent strategy). Then, let a small population share of agents $\epsilon \in (0,1)$ play a different pure or mixed strategy y (the mutant strategy). Hence, if an individual is drawn to play the game, the probabilities that its opponent plays the incumbent strategy x and the mutant strategy y are $1-\epsilon$ and ϵ, respectively. The payoff of such a game is the same as that of a game where the individual plays the mixed strategy $w = \epsilon y + (1-\epsilon)x$. The payoffs of strategies x and y given that the opponent adopts strategy w are denoted by $u(x,w)$ and $u(y,w)$, respectively.

Definition 1. A strategy x is called *evolutionary stable* if, for every strategy $y \neq x$, a certain $\bar{\epsilon} \in (0,1)$ exists, such that the inequality

$$u[x, \epsilon y + (1-\epsilon)x] > u[y, \epsilon y + (1-\epsilon)x], \tag{1}$$

holds for all $\epsilon \in (0, \bar{\epsilon})$.

A weaker notion of evolutionary stability also exists, called *neutral stability*. Instead of requiring that all mutant strategies gain less than the incumbent strategy x, in neutral stability it is required that no mutant strategy prospers, in the sense that it gains a higher payoff than the incumbent strategy.

Definition 2. . A strategy x is called *neutral stable* if, for every strategy $y \neq x$, a certain $\bar{\epsilon} \in (0,1)$ exists, such that inequality

$$u[x, \epsilon y + (1-\epsilon)x] \geqslant u[y, \epsilon y + (1-\epsilon)x], \tag{2}$$

holds for all $\epsilon \in (0, \bar{\epsilon})$.

Using the linearity property of the payoff function, (1) yields:

$$(1-\epsilon)u(x,x) + \epsilon u(x,y) > (1-\epsilon)u(y,x) + \epsilon u(y,y). \tag{3}$$

If ϵ is close to zero, (3) yields either:

$$u(x,x) > u(y,x) \tag{4}$$

or

$$u(x,x) = u(y,x) \quad and \quad u(x,y) > u(y,y) \tag{5}$$

Hence, it becomes obvious that an ESS must be a NE; otherwise, (4) does not hold (Fudenberg and Levin, 1998).

2.2. Single population replicator dynamics

The replicator dynamics, first proposed by Taylor and Jonker (1978), specify how population shares associated with different

[1] Players in EGT.

[2] In terms of auditing, a strategy is characterized as pure (or mixed) when a single (or multiple) path(s) is (are) selected to route the information to the destination.

pure strategies evolve over time. In contrast to evolutionary stability, in replicator dynamics agents are programmed to play only pure strategies. To define the replicator dynamics, let us consider a large but finite population of agents all programmed to play pure strategy $k \in \mathbf{K}$, where $\mathbf{K}$ is the set of strategies. At any instant t, let $\lambda_k(t) \geqslant 0$ be the number of agents programmed to play pure strategy k. The total population of agents is given by $\lambda(t) = \sum_{k\in\mathbf{K}} \lambda_k(t)$. Let $x_k(t) = \lambda_k(t)/\lambda(t)$ be the fraction of agents using pure strategy k at time t. The associated *population state* is defined by the vector $\mathbf{x}(t) = [x_1(t), \ldots, x_k(t), \ldots, x_K(t)]$. Then, the expected payoff of using pure strategy k given that the population is in state x is $u(k,x)$. The *average payoff* of the population x, that is the payoff of an agent drawn randomly from the population, is $u(x,x) = \sum_{k=1}^{K} x_k \cdot u(k,x)$.

Suppose that payoffs are proportional to the reproduction rate of each individual and, furthermore, that a strategy profile is inherited. This leads to the following dynamics for the population shares x_k:

$$\dot{x}_k = x_k \cdot [u(k,x) - u(x,x)], \tag{6}$$

where $\dot{x}_k$ is the time derivative of x_k. Eq. (6) states that the proportion of individuals using strategy k increases (decreases) if its payoff is bigger (smaller) than the average payoff in the population.

However, there are cases when even a *strictly dominated strategy* may gain more than average. Hence, it is not *a priori* clear whether such strategies get wiped out in the replicator dynamics. The following theorem answers this question (Weibull, 1996):

Theorem 1. *If a pure strategy k is strictly dominated then* $\xi_k(t,x^0)_{t\to\infty} \to 0$*, where* $\xi_k(t,x^0)$ *is the population at time t and* x^0 *is the initial state.*

On the other hand, it should be noted that the ratio x_k/x_ℓ of two population shares $x_k > 0$ and $x_\ell > 0$ increases with time if the strictly dominated strategy k gains a higher payoff than the strictly dominated strategy ℓ. This is a direct result of (6) and may be expressed analytically via

$$\frac{d}{dt}\left[\frac{x_k}{x_\ell}\right] = [u(k,x) - u(\ell,x)]\frac{x_k}{x_\ell}. \tag{7}$$

From (7) it is evident that even suboptimal strategies could temporarily increase their share before being wiped out in the long run. However, there is a close connection between the steady states of the replicator dynamics, that is states where the population shares do not change their strategy over time, and NE. Thus, since in NE all strategies have the same average payoff, every NE is a steady state. The reverse is not always valid: Steady states are not necessarily NE, e.g., any state where all agents use the same pure strategy is a steady state, but, it is not necessarily stable (Weibull, 1996).

3. Problem formulation

3.1. The two-player audit game

The auditing/fraud detection problem can be modeled using non-cooperative game theory. The auditing game belongs to the general class of coordination games, which are a formalization of the notion of a coordination problem well known in social sciences, including economics. It includes situations in which all parties can realize mutual gains, but only by making mutually consistent decisions (Russell, 1998).

The game under consideration consists of two players, the auditor and the client. According to ISA 240 (2009), the auditor conducting an audit should obtain reasonable assurance that the financial statements are free from material misstatement, caused either by fraud or inadvertent error. In case material misstatements are caused unintentionally, audit procedures should be effectively used for error detection. However, these procedures may be proved inappropriate for the identification of material misstatements caused by fraud. For this reason, according to the guidelines of ISA 315 (2009), a set of risk assessment procedures should be applied in that case by the auditor in order to obtain an understanding of the entity and its environment that will assist him to assess the risks of misstatements resulting from fraudulent financial reporting. As a result, these procedures require additional efforts from the auditor, and consequently, are more costly.

Each player in the game has two available strategies. The client has the choice either to commit fraud (F) or not (NF). Hence, the client's pure-strategy set is defined as $S_C = \{NF, F\}$. On the other hand, the auditor has the choice to adopt an audit plan that is based either on a basic set of audit procedures (Basic Audit – BA) or on a set of risk assessment procedures according to ISA 315 (Extended Audit – EA). In this case, the auditor's pure-strategy set is defined as $S_A = \{BA, EA\}$. In the current problem formulation it is assumed that material misstatements due to inadvertent error can be detected by both auditing procedures. However, intentional errors (frauds) can be detected only by EA.

Table 1 presents the auditing/fraud detection game between engagement parts. If the client does not commit fraud, the profit (payoff in terms of game theory) for the auditor is a_{11}, a_{21} and for the client is b_{11}, b_{21} in case the auditor selects BA and EA, respectively. It is clear that $a_{21} < a_{11}$ since EA is more costly compared to BA due to additional auditing efforts. On the contrary, a scrutinized audit that will not reveal misstatements will improve client's reputation, therefore, b_{21} is assumed to be greater to b_{11}.

On the other hand, if the client commits a fraud and the auditor selects BA then, the material misstatements will be not detected. Thus, the client will receive a payoff $b_{12} > b_{11}$. The auditor, due to the failure of the auditing procedures to unveil frauds will pay a_{12} ($a_{12} < 0$) as penalty. However, if the auditor selects EA then, the frauds committed by client will be detected. In this case, a penalty b_{22} ($b_{22} < 0$) will be imposed to the client, while the profit for the auditor is a_{22}. The payoff matrices of the auditor and the client are denoted by $\mathbf{A}$ and $\mathbf{B}$, respectively, and are given by:

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \tag{8a}$$

$$\mathbf{B} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}. \tag{8b}$$

The auditing/fraud detection game has no pure strategy equilibria. Specifically, the best response of the auditor to the strategies NF and F is BA and EA, respectively, while the best response of the client to the strategies BA and EA is F and NF, respectively. Therefore, a pair of strategies in which each strategy is the best response compared to the other one in that pair does not exist (Dutta, 1999).

Even though the game has no pure strategy equilibria, it has a single mixed strategy equilibrium. A mixed strategy for player i, $i = A, C$ is a probability distribution over his set S_i of pure strategies (Weibull, 1996). Since the set S_i, $i = A, C$ is finite, any mixed strategy $\mathbf{x}$ and $\mathbf{y}$ for the auditor and the client, respectively, may be represented as vector with their hth coordinate, x_h, y_h denoting the probability assigned by the auditor and the client to pure strategy $h \in S_A$

Table 1
Two-player audit/fraud detection game.

		Client	
		No Fraud (NF)	Fraud (F)
Auditor	Basic Audit (BA)	(a_{11}, b_{11})	(a_{12}, b_{12})
	Extended Audit (EA)	(a_{21}, b_{21})	(a_{22}, b_{22})

and $h \in S_C$, respectively. It should be also noted that $x_h, y_h \geqslant 0$ and $\sum_{h\in S_A} x_h = 1,\ \sum_{h\in S_C} y_h = 1$. The expected payoffs of the two players are denoted by $u_A(\mathbf{x},\mathbf{y}) = \mathbf{x}\cdot\mathbf{Ay}$, and $u_C(\mathbf{x},\mathbf{y}) = \mathbf{y}\cdot\mathbf{B}^T\mathbf{x}$ and are given by:

$$u_A(\mathbf{x},\mathbf{y}) = x_1[a_{11}y_1 + a_{12}(1-y_1)] + (1-x_1)[a_{21}y_1 + a_{22}(1-y_1)], \tag{9a}$$

$$u_C(\mathbf{x},\mathbf{y}) = y_1[b_{11}x_1 + b_{12}(1-x_1)] + (1-y_1)[b_{21}x_1 + b_{22}(1-x_1)]. \tag{9b}$$

The auditor and the client choose x_1 and y_1 that maximize (9a) and (9b), respectively. Their first order conditions are:

$$\frac{\partial u_A}{\partial x_1} = a_{11}y_1 + a_{12}(1-y_1) - a_{21}y_1 - a_{22}(1-y_1) = 0, \tag{10a}$$

$$\frac{\partial u_C}{\partial y_1} = b_{11}y_1 + b_{12}(1-x_1) - b_{21}x_1 - b_{22}(1-x_1) = 0. \tag{10b}$$

Solving the set of Eq. (10) for y_i and x_i, respectively, the following mixed strategy equilibrium is derived:

$$(x_1^*, x_2^*) = \left(\frac{\gamma}{\gamma+\delta}, \frac{\delta}{\gamma+\delta}\right), \tag{11a}$$

$$(y_1^*, y_2^*) = \left(\frac{\alpha}{\alpha+\beta}, \frac{\beta}{\beta+\alpha}\right), \tag{11b}$$

where

$$\alpha = a_{22} - a_{12} > 0, \tag{12a}$$
$$\beta = a_{11} - a_{21} > 0, \tag{12b}$$
$$\gamma = b_{22} - b_{12} < 0, \tag{12c}$$
$$\delta = b_{11} - b_{21} < 0. \tag{12d}$$

Given that, $a_{12} < 0$, $a_{22} > 0$, $a_{11} > a_{21} > 0$, $b_{12} > 0$ and $b_{22} < 0$, therefore, $\alpha > 0$, $\beta > 0$, $\gamma < 0$ and $\delta < 0$. From this solution the following conclusions are drawn

i. It is verified again that the game has no pure strategy equilibria since the probabilities for the auditor and the client to select BA or EA and NF and F, respectively, always take values in the range (0, 1).
ii. The probability for the auditor to select BA increases with γ and decreases with δ. Recall that γ is an increasing function of the penalty that is imposed to the client when a fraudulent report is detected, while δ depends on the reputation that a company earns in case that at the end of an extended audit frauds are not detected by the auditor.
iii. The probability for the client to select not to fraud increases with α, where α depends on the bonus an auditor earns when a fraudulent report is detected.

Finally, since an audit fails when the auditor selects BA and the client F, the probability for this event to take place is:

$$P_{FA} = x_1 \cdot (1 - y_1). \tag{13}$$

At the equilibrium point, substituting (11a) and (11b) into (15), yields:

$$P_{FA}^* = \frac{\beta\gamma}{(\alpha+\beta)(\gamma+\delta)}. \tag{14}$$

An audit in accordance with ISAs is designed to provide reasonable assurance[3] that the financial statements taken as a whole are free from material misstatement. In practice, the probability of failed audit should be kept below a specific threshold, P_{FA}^{th}, that usually takes values 5–7%. Thus, the payoffs of both players should be selected in a way that satisfies the following inequality expressing the quality of audit:

$$P_{FA}^* = \frac{1}{(\alpha/\beta+1)(\delta/\gamma+1)} \leqslant P_{FA}^{th}. \tag{15}$$

From (15) it is deduced that the equilibrium probability of failed audit is a decreasing function of α, δ and an increasing function of β, γ. The contour plot of parameters α/β and δ/γ for different values of P_{FA}^{th} is depicted in Fig. 1. It is observed that a stricter threshold for the probability of failed audit may be achieved by increasing the reward gained by the auditor when a fraudulent report is detected or the bonus of the client when the financial reports are free of material misstatements. Moreover, it is clear that, given P_{FA}^{th}, a trade-off exists between α/β and δ/γ. It is also observed from Fig. 1 that the probabilities for the client to select *NF* and for the auditor *EA* increase with α/β and δ/γ, respectively. Therefore, it is deduced that the more determined the auditor is to reveal fraud, the less probable is for the client to commit a fraud.

3.2. Problem formulation using evolutionary game theory

In this section, the audit/fraud detection game in modeled using Evolutionary Game Theory. At this point it should be noted that the present study is limited only to the case where the expected payoff for the engagement parts is linear function of the employed strategies. Discussions concerning nonlinear models are out of the scope of the current analysis. For possible extensions of the proposed model to handle exponential payoff functions the reader is referred to Li and Liu (1989) and Li (1988).

Following the notation presented in Section 3.1, let again **A** be the payoff matrix of the auditor and **B** that of the client as described via (8a) and (8b). Thus a_{hk} is the payoff to the auditor and b_{hk} the payoff to the client when the auditor uses pure strategy $h \in S_A$ and the client pure strategy $k \in S_C$.

In this notation the standard two-population replicator equations based on (6) may be written as follows:

$$\dot{x}_h = [\mathbf{e^h Ay} - \mathbf{xAy}]x_h = \left[\sum_{k\in S_C} a_{hk}y_k - \sum_{j\in S_A}\sum_{k\in S_C} x_j a_{jk} y_k\right] x_h, \tag{16a}$$

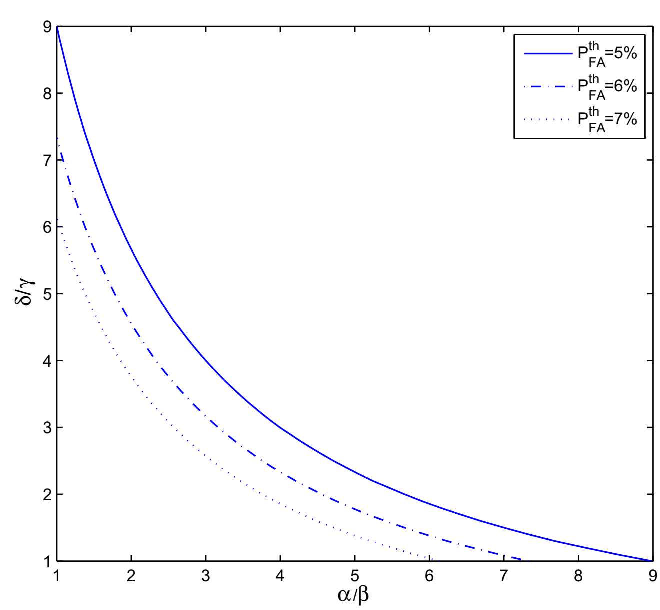

Fig. 1. Payoff values for α/β and δ/γ for different values of P_{FA}^{th}.

[3] A high but not absolute level of assurance.

$$\dot{y}_k = [\mathbf{e^k B^T x} - \mathbf{y B^T x}]y_k = \left[\sum_{h\in S_A} b_{hk}y_h - \sum_{j\in S_C}\sum_{h\in S_A} y_j b_{hj} x_h\right]y_k. \quad (16b)$$

After straightforward algebra, the system of differential equation in (16a) and (16b) is transformed into:

$$\dot{x}_1 = [-\alpha + (\alpha+\beta)y_1](1-x_1)x_1, \quad (17a)$$
$$\dot{y}_1 = [-\gamma + (\gamma+\delta)x_1](1-y_1)y_1. \quad (17b)$$

Then, the stability of the proposed auditing scheme is investigated using *Lyapunov's First* (*or Indirect*) *Method*. It is proved that after a number of auditing rounds the auditor selects EA with probability $\frac{\delta}{\gamma+\delta}$ while the client proceeds to fraudulent financial reporting with probability $\frac{\beta}{\beta+\alpha}$. The following theorem holds:

Theorem 2. *The audit/fraud detection game has a unique stable point, (x^*, y^*), with*

$$x^* = \left(\frac{\gamma}{\gamma+\delta}, \frac{\delta}{\gamma+\delta}\right),$$
$$y^* = \left(\frac{\alpha}{\alpha+\beta}, \frac{\beta}{\beta+\alpha}\right).$$

Proof. To proof the claim, the critical points of the system are determined by setting (17a) and (17b) equal to zero and solving the resulting system of algebraic equations under the following constraints:

$$x_k \geqslant 0, \quad k \in \mathbf{S_A}, \quad \sum_{k\in\mathbf{S_A}} x_k = 1,$$
$$y_k \geqslant 0, \quad k \in \mathbf{S_C}, \quad \sum_{k\in\mathbf{S_C}} y_k = 1.$$

The set of critical points of the system are $(x_0,y_0) = \{(1,0),(0,1),(1,1),(0,0),(\gamma/(\gamma+\delta),\alpha/(\alpha+\beta))\}$. The stability of the nonlinear system at (x_0,y_0) is studied after linearizing it at $(0,0)$ employing the transformation $x_1' = x_1 - x_0$ and $y_1' = y_1 - y_0$ and then analyzing the corresponding eigenvalues, λ_i, of the matrix:

$$\mathbf{\Phi} = \begin{bmatrix} \frac{\partial x_1'}{\partial x_1} & \frac{\partial x_1'}{\partial y_1} \\ \frac{\partial y_1'}{\partial x_1} & \frac{\partial y_1'}{\partial y_1} \end{bmatrix}, \quad (18)$$

at $(x_1', y_1') = (0,0)$.

It is proved in the Appendix A that at least one eigenvalue of the matrix $\mathbf{\Phi}$ corresponding to critical points (1,0), (0,1), (1,1), (0,0) is positive and, therefore, these solutions are instable. As far as the critical point $(\gamma/(\gamma+\delta),\alpha/(\alpha+\beta))$ is concerned, both eigenvalues are complex with zero real part. Thus, the nonlinear system is stable but not asymptotical stable. Also, the trajectories of the strategies followed by the auditor and the client are closed curves, while near the equilibrium point the trajectories are ellipses of the form:

$$\left(\frac{x_1 - x_1*}{A_{x_1}}\right)^2 + \left(\frac{y_1 - y_1*}{A_{y_1}}\right)^2 = 1, \quad (19)$$

where A_{x_1}, A_{y_1} are an increasing function of the values of x_1', y_1' at $t = 0$ defined in the Appendix A. Note that $x_1'(0)$, $y_1'(0)$ actually depend on the distance of the players' strategies from the equilibrium point. □

A typical numerical implementation illustrating the behavior of the players' strategies near the equilibrium point is depicted in Fig. 2 where the phase diagram as well as x_1 and y_1 are drawn as a function of time for $\alpha = 1$, $\beta = 4$, $\gamma = -1$, $\delta = -3$. It is observed that the trajectory is an ellipse with center the equilibrium point $(x_1^*, y_1^*) = (0.25, 0.2)$. Furthermore, x_1 y_1 oscillate around 0.25 and 0.2, respectively, with period $2\pi/\omega$, where ω is the angular frequency defined in the Appendix A and in the current numerical example has been calculated equal to $\omega = 0.77$ radian/seconds. Moreover, as depicted in Fig. 2c the probability of failed audit, P_{FA}, also oscillates around $P_{FA}^* = 0.2$ with period $2\pi/\omega$.

Furthermore, to provide reasonable assurance that financial reports are free of material misstatements, the probability of failed audit should be kept below P_{FA}^{th} during the whole auditing procedure. Since x_1, y_1 take values in the range $[x_1^* - A_{x1}, x_1^* + A_{x1}]$ and $[y_1^* - A_{y1}, y_1^* + A_{y1}]$, respectively, the following inequality constraint should be satisfied:

$$(1 - y_1^* + A_{y1})(x_1^* + A_{x_1}) \leqslant P_{FA}^{th}, \quad (20)$$

or

$$P_{FA}^* \leqslant P_{FA}^{th'}, \quad (21)$$

where $A_{FA} = (1 - y_1^*)A_{x1} + x_1^* A_{y1} + A_{y1}A_{x1}$ and $P_{FA}^{th'} = P_{FA}^{th} - A_{FA}$.

If A_{x1}, A_{y1} take values close to zero, A_{FA} tends also to zero and, therefore, reasonable assurance might be achieved by slightly decreasing P_{FA}^{th} and solving again (15) in order to determine the updated payoff values. Unfortunately, this procedure cannot be applied when large values are assigned to A_{x1}, A_{y1}. In this case, to satisfy quality of audit constraints, zero values should be assigned to $P_{FA}^{th'}$ above specific values of parameters A_{x1} and A_{y1}. However, this is infeasible since parameters α, β, γ and δ take finite and nonzero values. The above is illustrated in Fig. 3, where the impact of A_{x1} and A_{y1} on $P_{FA}^{th'}$ is drawn. To overcome this issue, in the following section a novel algorithm is proposed that assures quality of audit under inaccurate information.

4. Design of Auditing strategies under inaccurate information

So far the audit/fraud detection game has been modeled using classic and evolutionary game theory. The main vulnerability of the current problem formulation is the assumption that the players are aware of the information related to payoffs matrices. From the client's perspective this assumption might be rational since the cost of the auditing procedure either for the basic or the extended case is *a priori* agreed between the two parties. Unfortunately, from the auditor's point of view, information related to the payoff gained by the client in case of undetected fraud might not be avail-

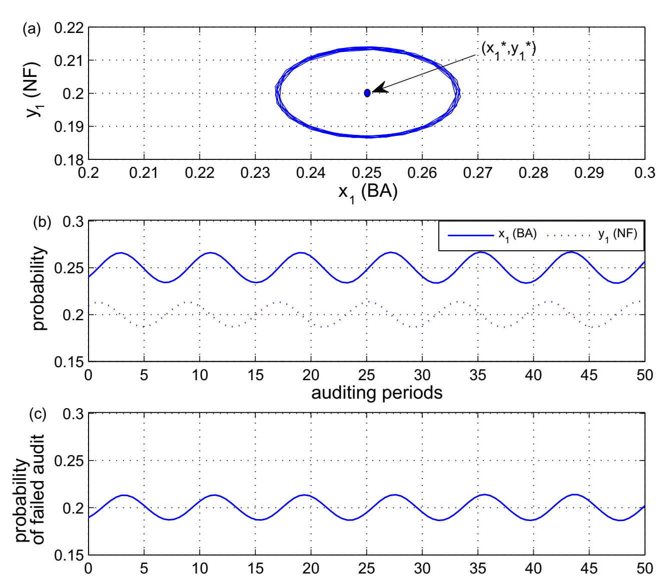

Fig. 2. Trajectories for $\lambda_{1,2} = \pm i0.77$ (a) Phase diagram. (b) x_1, y_1 as a function of time. (c) Probability of failed audit as a function of time ($\alpha = 1$, $\beta = 4$, $\gamma = -1$, $\delta = -3$, $x_1(0) = 0.24$, $y_1(0) = 0.21$).

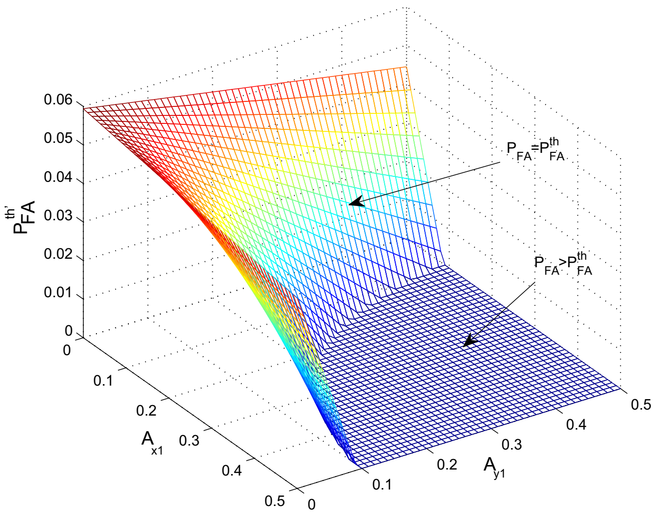

Fig. 3. $P_{FA}^{th'}$ for different values of A_{x1} and A_{y1} $\left(P_{FA}^{th} = 6\%\right)$.

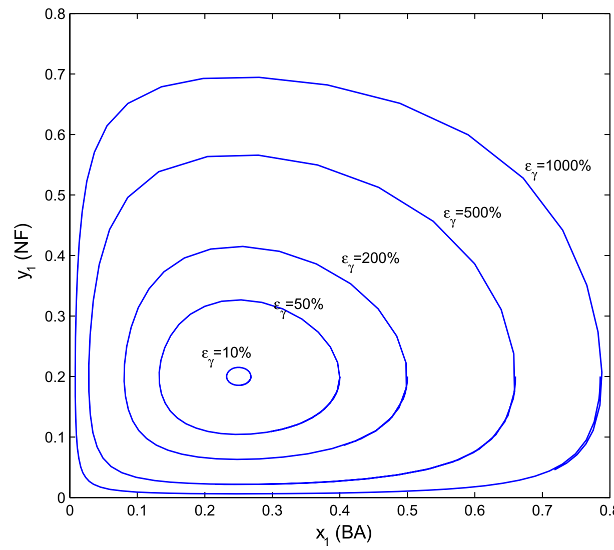

Fig. 4. Trajectories for $\lambda_{1,2} = \pm i0.77$ – Phase diagram for different percentage values of erroneous estimation of γ ($\alpha = 1$, $\beta = 4$, $\gamma = -1$, $\delta = -3$).

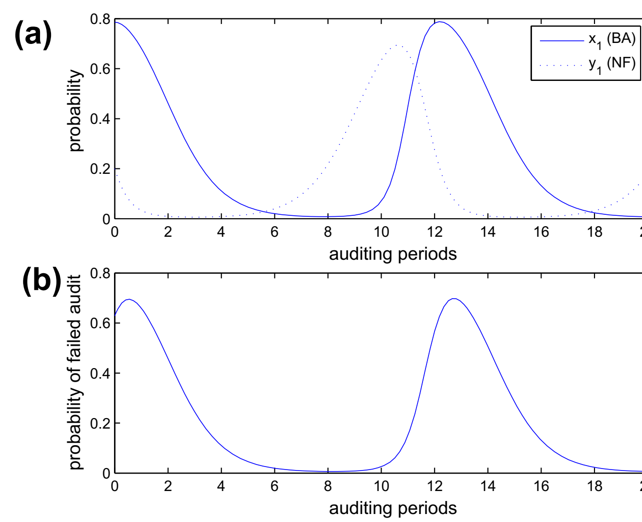

Fig. 5. (a) Evolution of auditing strategies under inaccurate information. (b) Evolution of the probability of failed audit.

able. This issue becomes of utmost importance especially in cases of misreporting and defalcation of financial reporting. For example, when a firm exhibits good performance, the auditee has the opportunity to embezzle a significant amount of money and report low asset values. Then, the client in order to cover-up defalcation might proceed to misreporting of firm's performance (Patterson and Noel, 2003). In such cases, the auditor might have a totally wrong estimation of parameter γ and, consequently, all calculations related to the determination of equilibrium point are inaccurate.

The impact of erroneous estimation of parameter γ, namely ϵ_γ on the strategies employed by the engagement parts is depicted in Fig. 4, where it is observed that an increase of ϵ_γ leads to an increase of the distance of the trajectories from the center. For low values of ϵ_γ, the strategies employed by the players are close to the equilibrium and, consequently, their curves are described by (19). Assuming that the client has an accurate knowledge of the auditor's payoffs, he selects NF and F with probabilities y^* and $1 - y^*$, respectively. Therefore, employing (20), reasonable assurance can be provided if:

$$(1 - y_1^*)x_1^* + (1 - y_1^*)x_1'(0)\sqrt{\left|\frac{c_1}{c_2}\right|} \leqslant P_{FA}^{th}. \tag{22}$$

However, for higher values of ϵ_γ, (22) cannot be satisfied and reasonable assurance cannot be provided during the whole auditing procedure. The evolution of the auditing strategies as well as their impact on the probability of failed audit under inaccurate information is depicted in Fig. 5. Due to an erroneous estimation of parameter γ, the auditor believes at fault that the client will not commit a fraud. For this reason, at the initiation of the auditing procedure the auditor selects *BA* with high probability. On the other hand, the client after observing the auditor's choice, selects fraud with higher probability due to its belief that his illegal actions will not be detected. Thus, the probability of failed audit significantly increases. After few auditing rounds the auditor understands client's illegal actions and increases significantly the probability of extended auditing, thus reducing the probability of failed audit. Unfortunately, even though the auditor will finally detect fraudulent reporting, for a large period the auditing procedure failed to provide reasonable assurance that financial reports were free of material misstatements.

To avoid the undesirable effects of the above scenario, during the initialization phase of the auditing procedure, the auditor should examine whether he is able or not to accurately determine the payoffs assigned to players. If the auditor can accurately determine the parameters involved in the calculations then, the players' strategies will oscillate very close around to the equilibrium point. If this is not feasible the auditor should perform the following actions.

1. Initially, the auditor should always begin the auditing procedure selecting *EA*. The reason is twofold; first, this strategy will reveal possible cases of fraudulent financial reporting and, second, inhibit illegal behavior.
2. Since long auditor tenure is problematic, after a few auditing periods the auditor should stop auditing the same client

The impact of the auditor's tenure on P_{FA} is depicted in Fig. 6. Initially, the players select *EA* and *NF* with probabilities 10^{-3} and 0.2, respectively. Since all fraudulent financial reporting efforts will be detected by the auditor, the *NF* probability is increased. On the other side, the auditor knows that the client will play *NF* for few

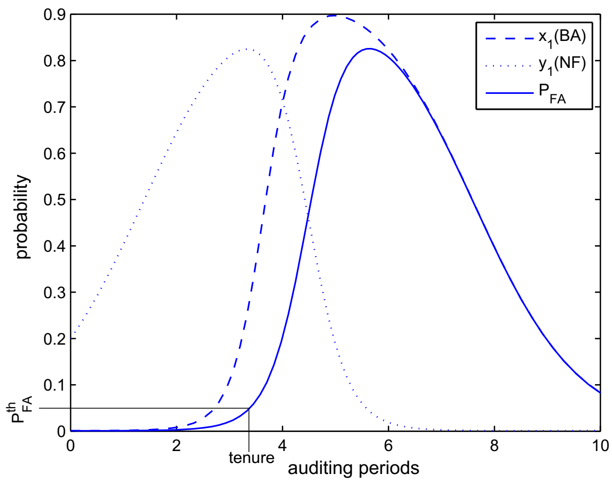

Fig. 6. Impact of the auditor's tenure on $P_{FA}\left(P_{FA}^{th}=5\%\right)$.

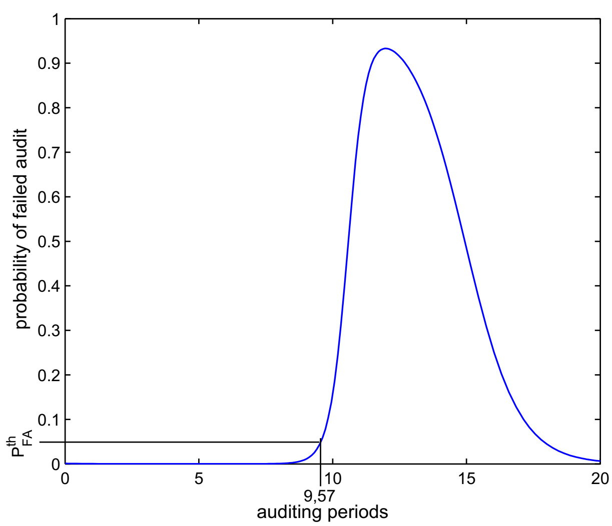

Fig. 7. Impact of the auditor's tenure on $P_{FA}\left(P_{FA}^{th}=5\%\right)$.

more round, therefore, he decreases its auditing effort. During this time interval quality of audit is assured. However, after few auditing rounds the auditor should be replaced and a new auditing procedure starts. Note that, given P_{FA}^{th}, the optimal replacement time may be directly determined from Fig. 6. In the current numerical result, where the Risk of material misstatement[4] is 80%, the auditor should be replaced approximately after three years' tenure.

Finally, in Fig. 7 the accuracy of the proposed scheme is tested on the same data set as used in Carcello and Nagy (2004). In Carcello and Nagy (2004), the authors identified instances of fraudulent financial reporting by examining SEC Accounting and Auditing Enforcement Releases (AAERs) issued between 1990 and 2001. Their final sample consists of 68195 non-fraud observations and 267 companies that are subject to AAERs alleging a violation. Thus, using these values as input to the EGT based model, Fig. 7 may be directly determined. It is observed that the replacement time that is obtained from the analytical model is 9.57 auditing periods. This value is very close to the measured average auditor tenure of 9.5 years that has been also recognized in Davis et al. (2009).

[4] According to ISA 2009 the Risk of material misstatement is defined as the risk that the financial statements are materially misstated prior to audit.

5. Concluding remarks

An audit in accordance with ISAs should be designed to provide reasonable assurance that the financial statements are clear from material misstatements caused either by fraudulent reporting or inadvertent error. The problem of auditing/fraud detection was traditionally modeled employing classic game theory, failing to determine the way the game reaches equilibrium, whether this equilibrium is stable or not and what is the long term impact of the auditor's tenure on the quality of auditing.

The present study sought to answer questions about how and why the auditors/auditees strategies evolve over time. The objective of the current approach was to provide analytical solutions to a series of problems in auditing including optimal auditor replacement (or auditor tenure), assessment and prediction of fraud. To this end the evolutionary dynamics of audit were analyzed using a set of differential equations, namely replicator dynamics.

Initially, the stability of the equilibrium point was examined based on *Lyapunov's First* (*or Indirect*) *Method*. It was proved that, under the condition that the players have accurate information for the parameters involved in the problem, the auditing/fraud detection game was stable but not asymptotically stable. Useful conclusions were drawn in case that the auditor is partially informed about the auditee firm and it was proved that in this case a more comprehensive audit is necessary to guarantee quality of audit. Furthermore, analytical results were derived concerning the impact of audit tenure on the probability of failed audit. Unlike previous studies that were based on logistic regression models to test the relation between audit firm tenure and audit quality, in this paper a new approach on the subject based on EGT was proposed. The results obtained comply with the IFAC Code of Ethics for Professional Accountants that requires the key audit partner to be rotated after a predefined period. Finally, the accuracy of the proposed analytical model was tested on a real data set.

Appendix A. Stability analysis of the auditing game

The set of critical points of the system are $(x_0,y_0)=\{(1,0),(0,1),(1,1),(0,0),(\gamma/(\gamma+\delta),\alpha/(\alpha+\beta))\}$. The following cases are examined

1. For $(x_0,y_0)=(0,1)$, Φ is evaluated as

$$\Phi=\begin{bmatrix}\beta & 0\\ 0 & -\gamma\end{bmatrix}. \tag{A.1}$$

The eigenvalues of matrix Φ are $\lambda_1=\beta$ and $\lambda_2=-\gamma$. Since both eigenvalues are positive, $(0,1)$ is unstable for the nonlinear system.

2. For $(x_0,y_0)=(1,0)$, Φ is evaluated as

$$\Phi=\begin{bmatrix}\alpha & 0\\ 0 & \delta\end{bmatrix}. \tag{A.2}$$

The eigenvalues of matrix Φ are $\lambda_1=\alpha$ and $\lambda_2=\delta$. Similarly, $(0,1)$ is unstable since the eigenvalue λ_1 is positive.

3. For $(x_0,y_0)=(1,1)$, Φ is evaluated as

$$\Phi=\begin{bmatrix}-\beta & 0\\ 0 & -\delta\end{bmatrix} \tag{A.3}$$

The eigenvalues of matrix Φ are $\lambda_1=-\beta$ and $\lambda_2=-\delta$ and, therefore, $(1,1)$ is unstable since the eigenvalue λ_2 is positive.

4. For $(x_0, y_0) = (0, 0)$, Φ is evaluated as

$$\Phi = \begin{bmatrix} -\alpha & 0 \\ 0 & -\gamma \end{bmatrix}. \tag{A.4}$$

The eigenvalues of matrix A are $\lambda_1 = -\alpha$ and $\lambda_2 = -\gamma$. The critical point (0,0) is unstable since the eigenvalue λ_2 is positive.

5. For $(x_0, y_0) = (\gamma/(\gamma+\delta), \alpha/(\alpha+\beta))$, Φ is evaluated as

$$\Phi = \begin{bmatrix} 0 & c_1 \\ c_2 & 0 \end{bmatrix}, \tag{A.5}$$

where

$$c_1 = \frac{\gamma\delta(\alpha+\beta)}{(\gamma+\delta)^2},$$
$$c_2 = \frac{\alpha\beta(\gamma+\delta)}{(\alpha+\beta)^2}.$$

The eigenvalues of Φ are given by the solution of the characteristic equation:

$$\det \begin{bmatrix} -\lambda & c_1 \\ c_2 & -\lambda \end{bmatrix} = 0, \tag{A.6}$$

or

$$\lambda^2 = c_1 c_2. \tag{A.7}$$

Since $\alpha, \beta > 0$ and $\gamma, \delta < 0$, $c_1 c_2$ is negative and, therefore, the eigenvalues are

$$\lambda_{1,2} = \pm i\sqrt{|c_1 c_2|} = \pm i\sqrt{\frac{\alpha\beta\gamma\delta}{(\alpha+\beta)|\gamma+\delta|}}. \tag{A.8}$$

Since both eigenvalues are complex the nonlinear system at $(\gamma/(\gamma+\delta), \alpha/(\alpha+\beta))$ is stable but not asymptotical stable.

Extending the analysis, the eigenvector $\Xi = (\xi_1, \xi_2)^T$ corresponding to the eigenvalue $\lambda_1 = +i\sqrt{|c_1 c_2|}$ will be determined from the solution of

$$\Phi\Xi = \lambda_1 \Phi. \tag{A.9}$$

Then, based on (A.5) and (A.9) translates into

$$c_1\xi_2 = i\sqrt{|c_1 c_2|}\xi_1,$$
$$c_2\xi_1 = i\sqrt{|c_1 c_2|}\xi_2.$$

An eigenvector is

$$\Xi = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} \sqrt{|c_1/c_2|} \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + i\begin{pmatrix} \sqrt{|c_1/c_2|} \\ 0 \end{pmatrix}, \tag{A.10}$$

Hence,

$$\begin{pmatrix} x_1'(t) \\ y_1'(t) \end{pmatrix} = \begin{pmatrix} -y_1'(0)\sqrt{\left|\frac{c_1}{c_2}\right|}sin(\omega t) + x_1'(0)cos(\omega t) \\ y_1'(0)cos(\omega t) + x_1'(0)\sqrt{\left|\frac{c_2}{c_1}\right|}sin(\omega t) \end{pmatrix}, \tag{A.11}$$

where $\omega = \sqrt{|c_1 c_2|}$ and $x_1'(0) y_1'(0)$ are the values of $x_1'(t)$ and $y_1'(t)$, respectively, at $t = 0$. Finally, (A.11) is written as

$$\begin{pmatrix} x_1'(t) \\ y_1'(t) \end{pmatrix} = \begin{pmatrix} A_{x_1} sin(\omega t + \phi_1) \\ A_{y_1} sin(\omega t + \phi_2) \end{pmatrix}, \tag{A.12}$$

where

$$A_{x_1} = \sqrt{y_1'(0)^2\left|\frac{c_1}{c_2}\right| + x_1'(0)^2}$$
$$A_{y_1} = \sqrt{y_1'(0)^2 + x_1'(0)^2\left|\frac{c_2}{c_1}\right|},$$

and

$$\phi_1 = arctan\left(-\frac{x_1'(0)}{y_1'(0)}\sqrt{\left|\frac{c_2}{c_1}\right|}\right) + \begin{cases} 0 \; if \; -y_1'(0) \geqslant 0, \\ \pi \; if \; -y_1'(0) < 0, \end{cases}$$

$$\phi_2 = arctan\left(\frac{y_1'(0)}{x_1'(0)}\sqrt{\left|\frac{c_1}{c_2}\right|}\right) + \begin{cases} 0 \; if \; x_1'(0) \geqslant 0, \\ \pi \; if \; x_1'(0) < 0. \end{cases} \tag{A.13}$$

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