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Strategic behavior under partial cooperation

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Abstract We investigate how a group of players might cooperate with each other within the setting of a non-cooperative game. We pursue two notions of partial cooperative equilibria that follow a modification of Nash's best response rationality rather than a core-like approach. Partial cooperative Nash equilibrium treats non-cooperative players and the coalition of cooperators symmetrically, while the notion of partial cooperative leadership equilibrium assumes that the group of cooperators has a first-mover advantage. We prove existence theorems for both types of equilibria. We look at three well-known applications under partial cooperation. In a game of voluntary provision of a public good we show that our two new equilibrium notions of partial cooperation coincide. In a modified Cournot oligopoly, we identify multiple equilibria of each type and show that a non-cooperator may have a higher payoff than a cooperator. In contrast, under partial cooperation in a symmetric Salop City game, a cooperator enjoys a higher return.

Keywords Cooperation · Non-cooperative games

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1 Introduction

In this article, we investigate the direct implementation of partial cooperation into standard, normal form games. A large literature has emerged on coalition formation and cooperation in strategic environments, but there has been done relatively little work on the direct implementation of a coalition of cooperators into the framework of a standard normal form game.[1] Our study, which draws on existing ideas in game theory and its applications, tries to address this deficiency.

We introduce a coalition of cooperators into the description of a game directly. We consider two extensions of the standard Nash equilibrium concept to capture the strategic behavior of such a coalition of cooperators. The first concept is a direct extension of the Nash equilibrium concept in which the coalition of cooperators acts as a single player with a utilitarian collective payoff function. The second concept incorporates leadership descriptors into a static setting.

In our *partial cooperative equilibrium* concept, we endow the coalition of cooperators with a utilitarian collective payoff function and have it select a collective best response to the actions of all other players in the game. This notion is reminiscent of the concepts considered in Salant et al. (1983), Chander and Tulkens (1995, 1997). While the former set of authors do not derive a general existence result, the latter study a core-like concept in which, if a coalition breaks away from the grand coalition, the remaining players respond with individually best responses. This fundamental principle is implemented in our equilibrium concept into the setting of a normal form strategic non-cooperative game rather than a cooperative framework.

We show that a large class of normal form games admit a partial cooperative equilibrium. The required conditions are only slightly stronger than the well-accepted conditions in the seminal equilibrium existence theorem of Nash (1951).

Our second equilibrium concept is founded on the idea that the coalition of cooperators has a strategic leadership position with respect to the other players. Thus, these non-cooperators play individual best responses to the actions selected by all players in the game, and the coalition of cooperators anticipates the best response selections of these non-cooperators. Our *partial cooperative leadership equilibrium* concept captures this sentiment into a purely static framework based on a maximin formulation.[2]

In previous study, Mallozzi and Tijs (2008a) introduced the partial cooperative leadership equilibrium concept on the restricted class of symmetric potential games (Monderer and Shapley 1996). With certain strong restrictions on the strategy set and payoff functions, they show that the group of non-cooperators chooses an equilibrium which is both symmetric and unique for every strategy that is selected by the

[1] Other authors have studied possibilities for coordination in repeated normal form games, e.g., Lau and Mui (2008).

[2] We emphasize that the maximin approach is at the foundation of decision theory, in particular, decision making under uncertainty (Gilboa 2009).

coalition of cooperators. This implies that the coalition of cooperators can therefore perfectly anticipate the resulting Nash equilibrium for any of the available strategic choices; and, hence, the payoffs of its members for every strategy choice. Now, the cooperators maximize a composite payoff function that includes as arguments, both their strategy and the unique Nash equilibrium strategy that results from their strategic choice. A maximizer of this function—provided it exists—now determines the corresponding equilibrium. Mallozzi and Tijs (2008a) provide conditions for which such an equilibrium exists.

Following their initial contribution in Mallozzi and Tijs (2008a), subsequent study in Mallozzi and Tijs (2007) and Mallozzi and Tijs (2008b) extended the partial cooperation framework to certain games in which the non-cooperating players select from multiple best responses. Mallozzi and Tijs (2007) consider symmetric aggregative games (Dubey et al. 1980) and assume that the non-cooperative players coordinate on the *symmetric* Nash equilibrium that yields the highest payoff to them, and, thus, do not consider any non-symmetric Nash equilibrium that, possibly, might result in higher payoff for all. Mallozzi and Tijs (2008b), on the other hand, assume that the non-cooperating players coordinate on the Nash equilibrium with the greatest or lowest strategy vector, irrespective of the payoff attained by the non-cooperators in this equilibrium.

Mallozzi's and Tijs's assumptions are unduly restrictive so as to be inapplicable to most strategic situations. However, at the same time they are hard to do away with. As long as the game is symmetric, identical strategies will lead to identical payoffs. Hence, the choice of a strategy that maximizes individual payoffs in the coalition of cooperators also maximizes joint payoffs if we assume that all members of the coalition of cooperators select the same strategy. Hence, cooperation is sustainable without payoff sharing as long as cooperation yields higher payoffs compared to the purely non-cooperative situation in which a Nash equilibrium outcome is attained, provided it exists.[3] If the game is not symmetric, however, the selection of the same strategy by the coalition of cooperators need not confer identical payoffs to all members of the group.

Our solution is to simply let the coalition of cooperators choose a strategy that maximizes its joint, utilitarian payoffs. We assume that the coalition of cooperators is risk-averse and chooses a maximin strategy. Hence, if there are multiple best responses given a strategic agreement of the coalition of cooperators, then the coalition of cooperators takes into account only the worst possible outcome.

The equilibrium concepts that we propose are close in nature to the Nash equilibrium concept. Note that the coalition of cooperators acts as a "single" player within the two formulated equilibrium concepts. In particular, the cooperators' collective payoff function assigns the sum of their payoffs under the prevailing strategy tuple.

[3] Here, we implicitly require that cooperative players are perfectly foresighted. A cooperative player who is better-off by deviating from the cooperatively agreed strategy provided that the rest of the coalition of cooperators sustain cooperation, realizes that should she deviate, the other members of the coalition of cooperators will also deviate, leading to a purely non-cooperative outcome.

We show that our leadership equilibrium concept exists under weaker conditions than previously established; in particular, we establish its existence under conditions equal to the seminal equilibrium existence theorem of Nash (1951).

Finally, we compute our equilibrium concepts for partial cooperation in some well-known applications, in particular, the voluntary provision of a public good, an extension of the standard model of Cournot oligopoly incorporating technological innovation, and Salop's City model of competition. The three applications are chosen to illustrate the versatility of our partial cooperative equilibrium concepts. To this end we choose to analyze symmetric games (Salop City and Cournot) as well as a non-symmetric game (public good provision).

In the game of voluntary provision of a public good, we show that both types of partial cooperative outcomes coincide, while these equilibria differ in the other two applications. In the Salop city model, we obtain uniqueness of both types of equilibria; in the modified Cournot game there is a multiplicity of equilibria of each type. In particular, in the analysis of the modified Cournot game, we utilize the maximin approach embedded in our notion of partial cooperative leadership equilibrium. These applications also differ in terms of the relative equilibrium payoffs between an arbitrary cooperator and an arbitrary non-cooperator. While a cooperator has a greater equilibrium payoff than a non-cooperator in both types of partial cooperative equilibria in the Salop city model, the opposite holds in the partial cooperative Nash equilibria of the modified Cournot game and in the partial cooperative equilibria in the non-trivial cases of the public good provision game.[4]

2 Equilibrium concepts under partial cooperation

We consider normal form games in which one given coalition of players cooperates and writes binding agreements, while the other players act purely non-cooperatively. Throughout we let $C = \{1, \ldots, k\}$ be the *coalition of cooperators* and $N = \{1, \ldots, n\}$ be the set of non-cooperative players. Thus, there are $k \geqslant 1$ cooperating players and $n \geqslant 1$ purely non-cooperative players. In the sequel we indicate a generic cooperator by $i \in C$, while a generic non-cooperator is denoted by $j \in N$.

Each cooperator $i \in C$ is endowed with an action set X_i and each non-cooperator $j \in N$ with an action set Y_j. We do not make any assumptions on these action sets unless explicitly indicated in the statement of the various theorems. We now denote $X = \prod_{i \in C} X_i$, respectively $Y = \prod_{j \in N} Y_j$, as the set of action tuples for the coalition of cooperators C, respectively the set of non-cooperators N. A generic action tuple for C is denoted by $x = (x_1, \ldots, x_k) \in X$ and a generic action tuple for N by $y = (y_1, \ldots, y_n) \in Y$. The total set of action tuples is denoted by $A = X \times Y = \{(x, y) \mid x \in X,\ y \in Y\}$.

In addition, following accepted conventions in the literature, for any cooperator $i \in C$ and action tuple $x \in X$, we write $x_{-i} = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_k)$ for the

[4] A similar analysis in the context of a standard Cournot model with general demand and cost functions is done by Okuguchi (1999) who compares profits between the Stackelberg leader, the Stackelberg follower and Cournot duopolists. More recently, Lafay (2010) studies the robustness of the Stackelberg's results in a model of more than two firms. These authors, however, do not discuss issues of partial cooperation.

actions assigned all cooperators in $C \setminus \{i\}$. Similarly, for any non-cooperator $j \in N$ and action tuple $y \in Y$ we introduce $y_{-j} = (y_1, \ldots, y_{j-1}, y_{j+1}, \ldots, y_n)$.

Furthermore, each cooperator $i \in C$ is endowed with a payoff function $\varphi_i : A \to \mathbb{R}$ and every non-cooperator $j \in N$ with a payoff function $\pi_j : A \to \mathbb{R}$. We also use $\varphi = (\varphi_1, \ldots, \varphi_k)$ and $\pi = (\pi_1, \ldots, \pi_n)$.

A *partial cooperative game* is now given as a list $\Gamma = \langle C, N, X, Y, \varphi, \pi \rangle$. The case where $k = 1$, i.e., C is a singleton, reverts to a standard normal form game. Therefore, we usually assume that $k \geqslant 2$.

We take as a benchmark the notion of Nash equilibrium (Nash 1950). In the setting of a partial cooperation game we reformulate this as follows:

Definition 2.1 An action tuple $(x^*, y^*) \in A$ is a Nash equilibrium in the partial cooperative game $\Gamma = \langle C, N, X, Y, \varphi, \pi \rangle$ if for every cooperator $i \in C$: $\varphi_i(x^*, y^*) \geqslant \varphi_i(x_i, x^*_{-i}, y^*)$ for all $x_i \in X_i$, and for every non-cooperator $j \in N$: $\pi_j(x^*, y^*) \geqslant \pi_i(y_j, x^*, y^*_{-j})$ for every $y_j \in Y_j$.

The Nash equilibrium simply assumes that all cooperators act purely non-cooperatively, so it does not capture the specific abilities of C. The familiar, main existence result for Nash equilibria can be stated for partial cooperative games as follows:

Lemma 2.2 (Nash 1951) *Assume that*

- *all action sets X_i $(i \in C)$ and Y_j $(j \in N)$ are non-empty, compact and convex subsets of a Euclidean space;*
- *all payoff functions φ_i $(i \in C)$ and π_j $(j \in N)$ are continuous on the space of action tuples A;*
- *for every cooperator $i \in C$, the payoff function φ_i is quasi-concave on X_i;*
- *and for every non-cooperator $j \in N$, the payoff function π_j is quasi-concave on Y_j.*

Then, the partial cooperative $\Gamma = \langle C, N, X, Y, \varphi, \pi \rangle$ admits at least one Nash equilibrium.

2.1 Partial cooperative Nash equilibrium

One approach to the modeling of partial cooperation in non-cooperative games is to modify the well-accepted concept of Nash equilibrium using a collective formulation of the objective of the coalition of cooperators. This approach was seminally developed in Chander and Tulkens (1995, 1997), as mentioned in the introduction. Given a non-cooperative game, Chander and Tulkens devise the notion of partial agreement equilibrium with respect to a given coalition (Chander and Tulkens 1997, Definitions 6 and 7).

Similarly, we assume that the coalition of cooperators C collectively maximizes their joint payoffs given the strategy choices of the remainder of the players. The non-cooperators $j \in N$ act as singletons, instead maximizing their individual payoff given the strategy choices of the cooperators and the other non-cooperative players.

Definition 2.3 An action tuple $(x^*, y^*) \in A$ is a *partial cooperative Nash equilibrium* in the partial cooperative game $\Gamma = \langle C, N, X, Y, \varphi, \pi \rangle$ if

$$\sum_{i \in C} \varphi_i(x^*, y^*) \geqslant \sum_{i \in C} \varphi_i(x, y^*)$$

for every action tuple $x \in X$, and for every non-cooperator $j \in N$:

$$\pi_j(x^*, y^*) \geqslant \pi_j(x^*, y_j, y^*_{-j})$$

for every $y_j \in Y_j$.

The existence of partial cooperative Nash equilibria can be guaranteed under only slightly stronger conditions than required for existence of Nash equilibrium:

Theorem 2.4 *Assume that*

- *all action sets X_i $(i \in C)$ and Y_j $(j \in N)$ are non-empty, compact, and convex subsets of a Euclidean space;*
- *all payoff functions φ_i $(i \in C)$ and π_j $(j \in N)$ are continuous on the space of action tuples A;*
- *for every cooperator $i \in C$, the payoff function φ_i is concave on X;*
- *and for every non-cooperator $j \in N$, the payoff function π_j is quasi-concave on Y_j.*

Then, the partial cooperative game $\Gamma = \langle C, N, X, Y, \varphi, \pi \rangle$ admits at least one partial cooperative equilibrium.

Proof We convert the partial cooperative game Γ into a standard normal form game Γ', which Nash equilibria correspond to partial cooperative equilibria in Γ. Existence of Nash equilibria in Γ' thus implies the desired existence of partial cooperative equilibria in Γ.
First, consider the function $\phi \colon A \to \mathbb{R}$ given by

$$\phi(x, y) = \sum_{i \in C} \varphi_i(x, y).$$

By continuity of φ_i, $i \in C$, it follows that ϕ is continuous on A. Also, since all φ_i, $i \in C$, are concave on X, it is easy to determine that ϕ is concave on X (Simon and Blume 1994, p. 519 and 523).

Now consider the normal form game $\Gamma' = \langle N', A, \phi, \pi \rangle$, where $N' = N \cup \{\gamma\}$, X is the action set of artificial player γ, Y_j is the action set for every $j \in N$, ϕ is the payoff function of artificial player γ, and π_j is the payoff function of player $j \in N$. It follows immediately that Γ' satisfies the conditions of Lemma 2.2 and, thus, Γ' admits a Nash equilibrium.[5]

[5] Remark here that X is a non-empty, compact, and convex subset of some Euclidean space, since it is the Cartesian product of such sets.

Finally, it can be verified that every Nash equilibrium (x^*, y^*) in Γ' indeed corresponds directly to a partial cooperative equilibrium in Γ through the reinterpretation of the artificial player γ as the coalition of cooperators C in Γ. □

It should be noted that the notion of partial cooperative Nash equilibrium is not related to some other existing notions of partial cooperation. First, consider the hybrid solution introduced by Zhao (1992). In its nature the hybrid solution combines the purely non-cooperative concept of the Nash equilibrium with the cooperative solution concept of the α-core. While the hybrid solution is a rather complex notion, it is easy to see that it does not coincide with the partial cooperative Nash equilibrium.[6] For instance, if $N = \emptyset$, the hybrid solution becomes the α-core of the original game, while the partial cooperative equilibrium becomes the Benthamite, or utilitarian, solution.

Second, Ray and Vohra (1997) introduce an equilibrium concept, denoted as the "coalitional equilibrium", that is explicitly developed for non-cooperative games in which coalitions can write binding agreements. Their equilibrium concept considers that coalitions can block the execution of certain action tuples by writing a binding agreement that strictly improves the payoffs of all members of the blocking coalition. Like Zhao (1992), Ray and Vohra (1997) introduce core-like considerations within the setting of a non-cooperative game, which is not the case in our equilibrium notion. Moreover, the coalitional equilibrium is explicitly formulated in terms of non-transferable utilities. This implies that all members of the coalition of cooperators have to agree to a certain binding agreement that deviates from the status quo. In many applications this is unnecessary and one can assume that payoffs are essentially transferable, as is done here.

2.2 Partial cooperative leadership equilibrium

An alternative approach to modeling partial cooperation was first suggested by Mallozzi and Tijs (2008a). In their concept, for every potential agreement $x \in X$, the players in C are assumed to be able to perfectly anticipate the responses of the players in N. The cooperators can thus optimize the selection of their agreement x taking the responses of the players in N into account.

To generalize the equilibrium notion of Mallozzi and Tijs (2008a), we introduce some auxiliary notation. For any action tuple $x \in X$, we denote by $\Gamma^x = \langle N, Y, \omega^x \rangle$ the standard normal form game given by player set N, and for every non-cooperating player $j \in N$ there are given action set Y_j as well as a conditional payoff function $\omega_j^x : Y \to \mathbb{R}$ defined as

$$\omega_j^x(y) = \pi_j(x, y).$$

Γ^x is denoted as the *conditional partial cooperative game* for all $x \in X$. The set of Nash equilibria of the conditional game Γ^x is denoted by $E_x \subset Y$.[7]

[6] A detailed technical note on the relation between the hybrid solution as defined by Zhao (1992) and the partial cooperative Nash equilibrium is available upon request from the authors.

[7] Note that in principle it can be the case that $E_x = \varnothing$ for certain action tuples $x \in X$.

It should be noted that the notion of partial cooperation as defined by Mallozzi and Tijs (2008a) assumes that for every tuple $x \in X$ the conditional game Γ^x obtains a unique Nash equilibrium, i.e., $\#E_x = 1$. Thus, this equilibrium notion can only be used in a very limited class of games. Therefore, a desirable generalization should broaden the applicability of this notion to games with multiple Nash equilibria in the conditional game Γ_x defined above.

We assume that for those $x \in X$ with $E_x \neq \varnothing$, the cooperators are risk-averse and select a maximin strategy.[8] Furthermore, the cooperators do not select strategies for which there is no Nash equilibrium in the conditional game. To formalize this notion, we introduce

$$\psi(x) = \min_{y \in E_x} \sum_{i \in C} \varphi_i(x, y)$$

$$\widetilde{X} = \left\{ x^\star \in X \;\middle|\; \psi(x^\star) = \max_{x \in X} \psi(x) \right\} = \arg\max \psi$$

The function ψ assigns the minimum payoff level to the coalition of cooperators C that can be achieved when the non-cooperators do a best response to all players' actions. Now $\widetilde{X}$ is the collection of action tuples that maximize this minimum payoff ψ over X, i.e., these are C's best responses for the collective payoff function ψ. We can now define the partial cooperative leadership equilibrium concept in Γ as follows:

Definition 2.5 An action tuple $(x^*, y^*) \in A$ is a *partial cooperative leadership equilibrium* in the partial cooperative game $\Gamma = \langle C, N, X, Y, \varphi, \pi \rangle$ if $x^* \in \widetilde{X}$ with $E_{x^*} \neq \varnothing$ and

$$y^* \in \arg \min_{y \in E_{x^*}} \sum_{i \in C} \varphi_i(x^*, y). \tag{1}$$

Our main existence theorem can now be stated as follows:

Theorem 2.6 *Assume that*

- *all action sets X_i $(i \in C)$ and Y_j $(j \in N)$ are non-empty, compact, and convex subsets of a Euclidean space;*
- *all payoff functions φ_i $(i \in C)$ and π_j $(j \in N)$ are continuous on the space of action tuples A;*
- *and for every non-cooperator $j \in N$, the payoff function π_j is quasi-concave on Y_j.*

Then, the partial cooperative game $\Gamma = \langle C, N, X, Y, \varphi, \pi \rangle$ admits at least one partial cooperative leadership equilibrium.

Proof Let $\Gamma = \langle C, N, X, Y, \varphi, \pi \rangle$ be a partial cooperative game that satisfies the stated conditions. Then, we can state the following assertion:

[8] A similar formulation based on the optimistic maximax approach is also feasible. It is a straightforward exercise to show that the conditions for existence of a partial cooperative leadership equilibrium when the maximax principle is followed are the same as those identified in Theorem 2.6.

Claim *The correspondence* $\mathbf{E}\colon X \to 2^Y$ *given by* $\mathbf{E}(x) = E_x \subset Y$ *is non-empty as well as compact valued and upper hemi-continuous.*

To prove this claim, we first remark that Lemma 2.2 implies that Γ^x admits Nash equilibria for every action tuple $x \in X$. This implies that the map $\mathbf{E}$ is non-empty valued.

Then, we show that $\mathbf{E}$ is a closed correspondence. Let $x_p \to x$ be a convergent sequence in X. Take $y_p \in E_{x_p} \subset Y$ with $y_p \to y$. We need to show that $y \in E_x$. Note that by the definition of y_p as a Nash equilibrium in Γ^{x_p} it holds that

$$\pi_j(x_p, y_p) - \pi_j(y'_j, x_p, y_{p,-j}) \geqslant 0$$

for any $y'_j \in Y_j$. By continuity of π_j and taking $p \to \infty$ it easily follows that

$$\pi_j(x, y) - \pi_j(y'_j, x, y_{-j}) \geqslant 0$$

for an arbitrary $y'_j \in Y_j$. This, in turn, implies that y indeed is a Nash equilibrium in Γ^x. Thus, $y \in E_x$ means that $\mathbf{E}$ is a closed correspondence.
In addition, the closedness of $\mathbf{E}$ implies that $\mathbf{E}(x) = E_x$ is a closed set for every $x \in X$. Since Y is a compact set, this in turn implies that $E_x \subset Y$ is compact for every $x \in X$. Finally, since every closed correspondence whose co-domain is compact is upper hemi-continuous (Border 1985, p. 56), the claim is proven.
Now for any $x \in X$, consider the function $\Psi_x\colon E_x \to \mathbb{R}$ given by

$$\Psi_x(y) = \sum_{i \in C} \varphi_i(x, y). \tag{2}$$

This function Ψ_x is well defined due to the claim and, moreover, Ψ_x is continuous. Since by the claim E_x is compact, Ψ_x attains a minimum on E_x. The minimum is exactly given by $\psi(x)$. This indeed shows that the function $\psi(x)$ is well defined on X.

Then, we note that from Berge's maximum theorem (Berge 1997, p. 115) it can be concluded that for any upper semi-continuous function $f\colon A \to \mathbb{R}$ the function $\Phi\colon X \to \mathbb{R}$ defined by

$$\Phi(x) = \sup\{f(x, y) \mid y \in E_x\}$$

is well defined and upper semi-continuous on X.[9]
To show the existence of a partial cooperative leadership equilibrium, consider the construction above and let $f(x, y) = -\Psi_x(y) = -\sum_{i \in C} \varphi_i(x, y)$. Then $\Phi(x) = \psi(x)$, implying that ψ is upper semi-continuous on a compact set. thus, ψ admits a maximum. This maximum corresponds to a partial cooperative leadership equilibrium. □

[9] This is based on the fact that the introduced multi-valued correspondence $\mathbf{E}$ which maps $x \to E_x$ is non-empty and compact valued as well as upper hemi-continuous.

Clearly, a partial cooperative equilibrium is introduced by Mallozzi and Tijs (2008a), if it exists, is a partial cooperative leadership equilibrium, though the converse is not true.

The assumption that cooperators are risk-averse is also employed in many other equilibrium notions in game theory. For example, by construction, the α-core (Aumann and Peleg 1960) assumes that players outside a given coalition will choose the coalition structure that result in the worst possible outcome for the players in that coalition. Also, we would like to point out some similarities between the minimax approach adopted here and the notion of the pessimistic recursive core of Kóczy (2007). Both notions assume that in the presence of multiple best responses to a given strategy profile by a coalition of players, the remainder of the players will coordinate on that focal point which yields the minimum payoff for the former group. Certainly, this assumption does not go without loss of generality in terms of the resulting equilibrium outcome as discussed by Kóczy. Despite its limitations we show in the next section that the partial cooperative leadership equilibrium outcomes may vary the relative payoff of a cooperator versus that of a non-cooperator depending on the game specification. This shows that our equilibrium notion is non-trivial in nature.

3 Some applications

In this section, we discuss three applications of our notions of partial cooperative Nash and partial cooperative leadership equilibrium. We study the voluntary provision of a public good, an extension of the Cournot model incorporating technological innovation and investment, and Salop's City model of spatial competition.

3.1 Public good provision with voluntary contributions

We consider a standard public good provision game on the player sets $C = \{1, \ldots, k\}$ and $N = \{1, \ldots, n\}$. Each player has an endowment which can be invested in a private account or contributed to a public good. Players differ in terms of their endowments such that each cooperator has an endowment of w^c and each non-cooperator an endowment of w^n.

Each unit invested in the private account returns one unit, while the public good gives a return of $\rho \in (1, k+n)$ per invested unit towards the whole group. These public returns are distributed equally among all the players. For all $i \in C$ we denote by $x_i \in [0, w^c]$ player i's contribution to the public good and, similarly, for all $j \in N$ we denote by $y_j \in [0, w^n]$ player j's contribution. We let $V(x, y) = \sum_{i=1}^{k} x_i + \sum_{j=1}^{n} y_j$ be the total contributions made, then players' payoff functions are given by

$$\varphi_i(x, y) = \frac{\rho}{k+n} V(x, y) + (w^c - x_i) \quad \text{for all } i \in C \tag{3}$$

$$\pi_j(x, y) = \frac{\rho}{k+n} V(x, y) + (w^n - y_j) \quad \text{for all } j \in N. \tag{4}$$

Clearly, the public good provision game with voluntary contributions satisfies all assumptions of Theorems 2.4 and 2.6.

As a benchmark we consider the outcome obtained in the unique Nash equilibrium. Irrespective of the strategy profiles of all other players, for any given player the private benefit from contributing to the public good, $\frac{\rho}{k+n}$, is lower than the private cost of contribution, equal to 1. This implies that there is a unique Nash equilibrium with $x_i^{\mathrm{NE}} = y_j^{\mathrm{NE}} = 0$ for all $i \in C$ and $j \in N$.

Under partial cooperation, the analysis of the non-cooperating players is analogous to that of the Nash equilibrium above. Indeed, the derivative of $\pi_j(x, y)$ given in (4) with respect to y_j yields $\frac{\rho}{k+n} - 1 < 0$. Moreover, $\frac{\partial^2 \pi_j}{\partial y_j \partial x_i} = 0$ for all $j \in N$ and $i \in C$. Therefore, the non-cooperators make a contribution of 0 in both the partial cooperative equilibrium and partial cooperative leadership equilibrium.

For the cooperators, the relevant objective function is

$$\phi(x, y) = \sum_{i=1}^{k} \varphi_i(x, y) \tag{5}$$

where $\varphi_i(x, y)$ is given in (3). The derivative of (5) with respect to x_i for $i \in C$ is

$$\frac{\partial \phi(x, y)}{\partial x_i} = \frac{k \cdot \rho}{k + n} - 1.$$

Therefore, the equilibrium strategies for the cooperators coincide in any partial cooperative equilibrium as $\frac{\partial^2 \phi(x,y)}{\partial x_i \partial y_j} = 0$ for all $i \in C$ and $j \in N$. The vector of equilibrium contributions is straightforward to derive and it depends on the parameter values as summarized below.

Proposition 3.1 *Consider the public good provision with voluntary contributions game described above.*

(a) *If $(\rho - 1) \cdot k < n$ there is a unique partial cooperative equilibrium and a unique partial cooperative leadership equilibrium. These two equilibria coincide with the standard Nash equilibrium with $x_i^* = y_j^* = 0$ for all $i \in C$ and $j \in N$. In any partial cooperative equilibrium the payoff of a cooperating player is higher than that of a player who is not cooperating if and only if $w^c > w^n$.*

(b) *If $(\rho - 1) \cdot k = n$ there is a multiplicity of partial cooperative equilibria and partial cooperative leadership equilibria. The two sets of equilibria coincide and they are given by $x_i^* \in [0, w^c]$ $(i \in C)$ and $y_j^* = 0$ $(j \in N)$.*

(c) *If $(\rho - 1) \cdot k > n$ there is a unique partial cooperative equilibrium and a unique partial cooperative leadership equilibrium. These two equilibria coincide and are given by $x_i^* = w^c$ $(i \in C)$ and $y_j^* = 0$ $(j \in N)$. In any partial cooperative equilibrium the payoff of a cooperator is lower than that of a non-cooperator, with $\varphi_i(x^*, y^*) = \frac{k \cdot \rho}{k+n} w^c$ and $\pi_j(x^*, y^*) = \frac{k \cdot \rho}{k+n} w^c + w^n$, respectively.*

Notably, only when the coalition of cooperators is sufficiently large compared to that of the non-cooperators, i.e., if $(\rho - 1) \cdot k > n$, does partial cooperation lead to higher payoffs for all players than the purely non-cooperative ones. These payoffs are higher for the non-cooperating players who free-ride on the public good contributions of the cooperators.

As described above, in this voluntary contribution game the partial cooperative equilibrium and partial cooperative leadership equilibrium coincide. In contrast, these equilibria differ for the Cournot and Salop City games considered next.

3.2 Cost-reducing technological innovation in a Cournot oligopoly

Mallozzi and Tijs (2008a) have already pointed out that a partial cooperative leadership equilibrium exists in the homogeneous Cournot model. Here, we consider a modification in which firms can choose to make a cost-reducing investment simultaneously with their choice of output levels. The introduction of such cost-reducing investments provides a rationale for cooperation among the competitors in a Cournot oligopoly.

Such a cost-reducing investment can be either done in terms of reducing labor costs or in terms of enhancing technological efficiency. From a firm's point of view the two types of cost-reducing investments are perfect substitutes. Investment in technology, however, entails positive spillovers to the other firms in the market and thus reduces the competitors' marginal costs as well.

We denote by $C = \{1, \ldots, k\}$ the set of firms which sign a binding agreement on the output level and investment type and by $N = \{1, \ldots, n\}$ the set of firms who act non-cooperatively with respect to all other players. We consider a linear demand formulation:

$$p = \alpha - \beta \cdot \sum_{i \in C \cup N} q_i, \tag{6}$$

where p represents the market price and q_i represents firm $i \in C \cup N$'s produced quantity.

Before making a cost-reducing investment, all firms $h \in C \cup N$ are assumed to have a common standard marginal cost level, $\gamma > 0$. For each firm $h \in C \cup N$ we introduce two strategic variables, $\ell_h \geqslant 0$ and $t_h \geqslant 0$, representing investments in cost reduction. Here, ℓ_h is the reduction of marginal cost due to labor-enhancing efficiency, and t_h is the reduction of marginal cost due to technology-enhancing efficiency. The two types of investment ℓ_h and t_h are perfect substitutes in terms of marginal cost reduction and must satisfy the resource constraint that $\ell_h + t_h \leqslant I$, where $I > 0$ is a given upper bound on the total investment in cost reduction.

For each formulated investment plan $(\ell_h, t_h)_{h \in C \cup N}$ there results a marginal cost parameter for each firm $h \in C \cup N$ given by

$$\gamma_h = \gamma - \ell_h - t_h - \sum_{j \in \{C \cup N\} \setminus \{h\}} t_j,$$

Here, the labor-reduction ℓ_h is only effective for firm h, while the technology-enhancing t_h spills over to the other firms in the market.

Hence, the profit of firm $h \in C \cup N$ can now be formulated as

$$\pi_h = q_h \left(\alpha - \beta \sum_{j \in C \cup N} q_j - \left(\gamma - \ell_h - \sum_{j \in C \cup N} t_j \right) \right) \tag{7}$$

In this modified Cournot oligopoly, each firm $h \in C \cup N$ selects an output quantity q_h, a labor-cost-reducing investment ℓ_h, and a technology-enhancing investment t_h simultaneously to maximize her profits. The game satisfies all assumptions of Theorems 2.4 and 2.6 with the sole exception of the fact that quantities are not bounded above. Without loss of generality, however, we may further assume that firms choose quantities in the range $[0, \alpha]$. This implies that both equilibrium types exist for this case.

As a benchmark again we take the purely non-cooperative behavior prescribed by the Nash equilibrium in which a player $h \in C \cup N$ maximize profits π_h given the choices of the remainder of the players. The first-order conditions with respect to q_h and t_h yield, noting that $\ell_h + t_h = 1$ in the optimum,

$$\frac{\partial \pi_h}{\partial q_h} = \alpha - \beta \textstyle\sum_{j \in C \cup N} \overline{q}_j - \left(\gamma - (I - t_h) - \sum_{j \in C \cup N} \overline{t}_j \right) - \beta q_h \leqslant 0 \tag{8}$$

$$\frac{\partial \pi_h}{\partial t_h} = q_h(-1 + 1) = 0 \leqslant 0 \tag{9}$$

Condition (9) is satisfied for any $t_h \in [0, I]$, which implies the existence of a multiplicity of Nash equilibria. The equilibrium output level can be derived in a similar fashion to that of the standard Cournot oligopoly where (8) defines a best response function. The set of Nash equilibria are thus given by $(q_h^{\mathrm{NE}}, t_h^{\mathrm{NE}})_{h \in C \cup N}$ with $t_i^{\mathrm{NE}} \in [0, I]$ and

$$q_h^{\mathrm{NE}} = \frac{\alpha - \gamma + I + 2 \sum_{j \in C \cup N} t_j^{\mathrm{NE}} - (1 + k + n) t_h^{\mathrm{NE}}}{\beta(k + n_1)}.$$

The profits in the Nash Equilibrium $(q_h^{\mathrm{NE}}, t_h^{\mathrm{NE}})_{h \in C \cup N}$ for a player $h \in C \cup N$ are given by

$$\pi_h^{\mathrm{NE}} = \frac{(\alpha - \gamma + I + 2 \sum_{j \in C \cup N} t_j^{\mathrm{NE}} - (1 + k + n) t_h^{\mathrm{NE}})^2}{\beta(k + n_1)^2}$$

A partial cooperative equilibrium in this modified Cournot game can be derived in a similar fashion: each non-cooperator plays a standard best response and the cooperative output level is determined by an optimization of the collective payoff. The following two points should be kept in mind, however. First, as the two types of cost-reducing investments are perfect substitutes in the payoff function of a non-cooperator π_j for all $j \in N$, any combination of ℓ_j and t_j such that $l_j + t_j \leqslant I$ will be supported in any type of partial cooperative equilibrium. Second, as the cooperators maximize the sum of their payoff functions, they can internalize the positive spillover effect of investment in technology and set $t_i^* = I$ for all $i \in C$ in any partial cooperative equilibrium.[10] These equilibrium outcomes are summarized in Proposition 3.2 below.

The derivation of partial cooperative leadership equilibria follows the steps of deriving a Stakelberg equilibrium in the standard Cournot game. However, for any given strategy profile of the cooperators, there is a multiplicity of Nash equilibria in the conditional game among the non-cooperators. Thus, the group of cooperators is assumed to use a maximin approach when selecting their strategy in any partial cooperative leadership equilibrium. This analysis is outlined below.

Consider a given strategy profile for the group of cooperators $\overline{q}^c = (\overline{q}_1, \ldots, \overline{q}_k)$ and $\overline{t}^c = (\overline{t}_1, \ldots, \overline{t}_k)$. Then a firm $j \in N$, maximizes the conditional profits

$$\omega_j^{(\overline{q}^c,\overline{t}^c)} = q_j \left(\alpha - \beta \sum_{i \in C} \overline{q}_i - \beta \sum_{j \in N} q_j - \left(\gamma - \ell_k - \sum_{i \in C} \overline{t}_i - \sum_{j \in N} t_j \right) \right)$$
$$\text{subject to } \ell_j + t_j \leqslant I.$$

The first-order conditions with respect to q_j and t_j yield

$$\frac{\partial \omega_j^{(\overline{q}^c,\overline{t}^c)}}{\partial q_j} = \alpha - \beta \sum_{i \in C} \overline{q}_i - \beta \sum_{j \in N} q_j - \left(\gamma - \ell_j - \sum_{i \in C} \overline{t}_i - \sum_{j \in N} t_j \right) - \beta q_j \leqslant 0 \quad (10)$$

$$\frac{\partial \omega_j^{(\overline{q}^c,\overline{t}^c)}}{\partial t_j} = q_j(-1+1) \leqslant 0 \quad (11)$$

As discussed above condition (11) is satisfied for any $t_j \in [0, I]$; therefore, there is a multiplicity of Nash equilibria in the conditional game. Clearly, the Nash equilibrium that yields the minimum payoff for the cooperators is the one in which $t_j = 0$ for all $j \in N$. Condition (10) defines a best-response function for a non-cooperator. The group of cooperators takes these reaction functions into account when determining their output and investment levels. Thus, the group of cooperators maximizes

$$\sum_{i \in C} \psi_i = \sum_{i \in C} \overline{q}_i \left(\alpha - \beta \sum_{i \in C} \overline{q}_i - \beta \sum_{j \in N} q_j(\overline{q}^c, \overline{t}^c) - \left(\gamma - \overline{\ell}_i - \sum_{i \in C} \overline{t}_i \right) \right)$$
$$\text{subject to } \overline{\ell}_i + \overline{t}_i \leqslant I \text{ for all } i \in C.$$

For all $i \in C$, the following first-order conditions obtain:

$$\frac{\partial \psi_i(\overline{q}^c, \overline{t}^c)}{\partial \overline{q}_i} = \frac{\alpha - 2\beta \sum_{i \in C} \overline{q}_i - \gamma + I - (1+n)\overline{t}_i + \sum_{i \in C} \overline{t}_i}{n+1} \leqslant 0 \quad (12)$$

[10] This analysis assumes that $k \geq 2$.

$$\frac{\partial \psi_i(\overline{q}^c, \overline{t}^c)}{\partial \overline{t}_i} = \frac{\sum_{i \in C} \overline{q}_i}{n+1} - q_i \leqslant 0 \tag{13}$$

Condition (12) implies that for all $i \in C$, $\sum_{i \in C} \overline{q}_i^* = \frac{\alpha - \gamma + I - (n+1)\overline{t}_i^* + \sum_{i \in C} \overline{t}_i^*}{2\beta}$, therefore, $\overline{t}_i^* = \overline{t}_j^*$ for all $i, j \in C$. Condition (13) implies that $\overline{t}_i^* = I$, if $\frac{\sum_{i \in C} \overline{q}_i^*}{n+1} > \overline{q}_i^*$; and $\overline{t}_i^* = 0$, if $\frac{\sum_{i \in C} \overline{q}_i^*}{n+1} < \overline{q}_i^*$. The equilibrium levels of output and technology-enhancing investment for the non-cooperators can be obtained by substituting $(\overline{q}^{c*}, \overline{t}^{c*})$ into (12). The outcome of this analysis is summarized below.

Proposition 3.2 *Consider the Cournot oligopoly game with cost-reducing technological innovation described above.*

(a) *There is a multiplicity of partial cooperative equilibria given by*

$$q_i^* = \frac{\alpha - \gamma + (k-n) \cdot I + 2 \sum\limits_{j \in N} t_j^*}{\beta \cdot k(2+n)}, \quad t_i^* = I, \quad \ell_i^* = 0,$$

for all $i \in C$;

$$q_j^* = \frac{\alpha - \gamma + (2+k) \cdot I + 2 \sum\limits_{j \in N} t_j^* - (2+n) \cdot t_j^*}{\beta(2+n)}, \quad t_i^* \in [0, I];$$

$$\ell_i^* = I - t_j^*, \quad \textit{for all } j \in N.$$

In a partial cooperative equilibrium, the profit of a cooperating player $i \in C$ *is given by* $\pi_i^* = \frac{(\alpha - \gamma + (k-n) \cdot I + 2\sum_{j \in N} t_j^*)^2}{\beta \cdot k(2+n)^2}$ *and the profit of a non-cooperating player* $j \in N$ *is given by* $\pi_j^* = \frac{(\alpha - \gamma + (k+2) \cdot I + 2\sum_{j \in N} t_j^* - (2+n) \cdot t_j^*)^2}{\beta \cdot (2+n)^2}$. *If* $t_j^* = 0$ *for some* $j \in N$, $\pi_i^* < \pi_j^*$ *for all* $i \in C$.

(b) *The set of partial cooperative leadership equilibria is given by a tuple* $(\tilde{q}^*, \tilde{t}^*)$ *with*

$$\sum_{i \in C} \tilde{q}_i^* = \frac{\alpha - \gamma + I - (n+1-k)\tilde{t}^{c*}}{2\beta}, \quad \tilde{t}_i^* = \tilde{t}^{c*}, \quad \tilde{\ell}_i^* = I - \tilde{t}^{c*}$$

for all $i \in C$,

$$\textit{where } \tilde{t}^{c*} = \begin{cases} I & \textit{if } \frac{\sum_{i \in C} \tilde{q}_i^*}{n+1} > \tilde{q}_i^*, \quad \textit{for all } i \in C \\ [0, I] & \textit{if } \frac{\sum_{i \in C} \tilde{q}_i^*}{n+1} = \tilde{q}_i^*, \quad \textit{for all } i \in C \\ 0 & \textit{if } \frac{\sum_{i \in C} \tilde{q}_i^*}{n+1} < \tilde{q}_i^*, \quad \textit{for all } i \in C \end{cases}$$

$$\tilde{q}_j^* = \frac{\alpha - \gamma + I + 4\sum_{j \in N} \tilde{t}_j^* + (1+k+n)\tilde{t}^{c*} - 2(1+n) \cdot \tilde{t}_j^*}{2\beta(1+n)},$$

$$\tilde{t}_j^* \in [0, I], \quad \tilde{\ell}_j^* = I - t_j^*, \quad \textit{for all } j \in N.$$

Consider a symmetric partial cooperative leadership equilibrium with $\tilde{q}_i^* = \frac{\sum_{i \in C} \tilde{q}_i^*}{k}$ *for all* $i \in C$ *and* $\tilde{t}_j^* = 0$ *for all* $j \in N$. *At this symmetric partial* *cooperative leadership equilibrium the profits for a cooperative player* $i \in C$ *are given by* $\tilde{\pi}_i = \frac{(\alpha-\gamma+I+(k-(1+n))\tilde{t}^{c*})^2}{4k\beta(1+n)}$ *and the profits for a non-cooperative player* $j \in N$ *are given by* $\tilde{\pi}_j = \frac{(\alpha-\gamma+I+(k+(1+n))\tilde{t}^{c*})^2}{4\beta(1+n)^2}$. *If the group of the cooperators is larger in size, i.e.,* $k > 1+n$, *then* $\tilde{t}^{c*} = I$ *and* $\tilde{\pi}_i < \tilde{\pi}_j$ *for all* $i \in C$ *and all* $j \in N$. *Instead, if* $k < 1+n$, *then* $\tilde{t}^{c*} = 0$ *and* $\tilde{\pi}_i > \tilde{\pi}_j$ *for all* $i \in C$ *and all* $j \in N$.

The results above suggest that a non-cooperator attains higher payoffs than a cooperator in a partial cooperative equilibrium; however, the opposite may occur in a symmetric partial cooperative leadership equilibrium if the size of the group of cooperators is relatively small.

Because of the multiplicity of equilibria, a comparison of payoffs across the equilibrium concepts is less clear. To make this comparison more straightforward, we consider the specific equilibrium profiles in which all cooperators make only a technology-enhancing investment and all non-cooperators make only a labor-enhancing investment. Notice that for this to be part of partial cooperative leadership equilibrium $k > 1+n$. With this type of equilibrium strategies, the profits for a cooperator may be the highest under partial cooperative leadership equilibrium if $(k+n)^2 > 4k(1+n)$, otherwise they will be the highest under the Nash equilibrium. For a non-cooperator, clearly the worst outcome is that of Nash equilibrium and the comparison between the partial cooperative equilibrium and the partial cooperative leadership equilibria depends on whether $\frac{(\alpha-\gamma+(k+2)I)^2}{\beta(2+n)^2} > \frac{(\alpha-\gamma+(k+n+2)I)^2}{4\beta(1+n)^2}$ in which case the partial cooperative equilibrium yields higher profits, or whether the reverse inequality holds in which case partial cooperative leadership equilibrium is preferred by such players.

3.3 Salop's City model of spatial competition

Our final application considers a modification of the model of the circular city due to Salop (1979) describing spatial competition between neighboring firms. Consumers are located uniformly on a circle whose circumference is equal to 1. Firms are located equidistant along the unit circle and compete in prices.[11] Each consumer consumes one unit of the good and derives an utility equal to V minus the transportation cost. Transportation cost is given by t per unit of distance. Hence, for a consumer who purchases her unit from a firm located at distance d, the attained utility is

$$\pi = V - p - d \cdot t \tag{14}$$

where p is the price charged by the firm. Each firm has a marginal cost of c.

[11] Salop (1979) originally discusses a two stage game of competing firms on a circular space: in the first stage firms decide whether or not to enter in the first stage; and in the second stage, the entered firms compete in prices. Here, we abstract from the analysis of the first stage to preserve the required symmetry in our modification of Salop's original game.

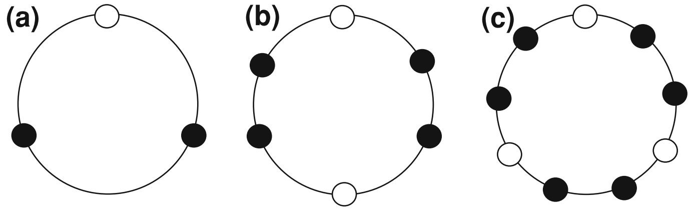

Fig. 1 Symmetric Salop City games with (**a**) 3, (**b**) 6, and (**c**) 9 firms

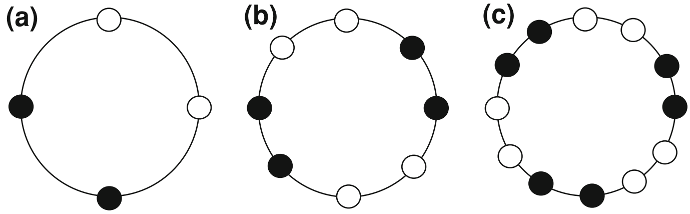

Fig. 2 Symmetric Salop City games with (**a**) 4, (**b**) 8, and (**c**) 12 firms

It can be shown that a Nash equilibrium results in a price charged equal to $c + \frac{t}{k+n}$ where $k+n$ is the total number of firms on the market and equilibrium profits $\frac{t}{(k+n)^2}$.[12]

Now, consider the setting in which there are cooperative, $C = \{1, \ldots, k\}$, as well as non-cooperative, $N = \{1, \ldots, n\}$, firms. Assume that only adjacent firms may sign a cooperation agreement. Firms are located in a way that the game is symmetric, i.e., each cooperator and non-cooperator has the same number of adjacent cooperative and non-cooperative firms and that there are at least two cooperative firms which are adjacent.[13] Note that Salop City games that satisfy these requirements are games in which the number of firms is either a multiple of three or a multiple of four. Figure 1 presents three games in which the number of firms is a multiple of three, and Fig.2 presents three games in which the number of firms is a multiple of four.

It should be clear that the Salop City satisfies all assumptions of Theorems 2.4 and 2.6 with the exception of the fact that the prices are not bounded from above.[14] The profit function is continuous and strictly concave, and hence quasi-concave, in its own price. The Nash equilibrium of the conditional game is a continuous function of the price chosen by the coalition of cooperators. Furthermore, note that for every strategic choice of the coalition of cooperators, a unique Nash equilibrium exists so a partial cooperative leadership equilibrium may be derived in a straightforward manner.

First, consider a symmetric game in which the number of firms is a multiple of three. Here, every non-cooperative firm is adjacent to two cooperators, and every cooperator is adjacent to a cooperator and a non-cooperative firm. On the assumption that V is sufficiently large, there is no consumer participation constraint. So, there are $n = \frac{k+n}{3}$

[12] For details we refer to Tirole (1988, p. 282).

[13] If no cooperative firms are adjacent, then there will be no benefit from cooperation and the game resumes its standard non-cooperative nature.

[14] However, there is no loss of generality if we assume that firms choose prices in the interval $[0, c + \frac{2t}{n}]$.

non-cooperative firms and $k = \frac{2(k+n)}{3}$ cooperators with $k + n$ equal the total number of firms. Both types of partial cooperative equilibria are straightforward to derive, and the results are summarized below.

Proposition 3.3 *Consider the symmetric Salop City discussed above with $k + n = 3s$ where s is a positive integer.*

(a) *The unique partial cooperative equilibrium results in an equilibrium price of $p_i = c + \frac{5t}{3(k+n)}$ with $\pi_i = \frac{25t}{9(k+n)^2}$ for all $i \in C$; an equilibrium price of $p_j = c + \frac{4t}{3(k+n)}$ with $\pi_j = \frac{16t}{9(k+n)^2}$ for all $j \in N$.*
(b) *The unique partial cooperative leadership equilibrium results in an equilibrium price of $\tilde{p}_i = c + \frac{5t}{2(k+n)}$ with $\tilde{\pi}_i = \frac{25t}{8(k+n)^2}$ for all $i \in C$; and an equilibrium price of $\tilde{p}_j = c + \frac{7t}{4(k+n)}$ $\tilde{\pi}_j = \frac{49t}{16(k+n)^2}$ for all $j \in N$.*

Then, we consider the case that the number of firms is a multiple of four. Here, every firm is adjacent to a cooperator and a non-cooperative player. So, the number of non-cooperative and cooperative firms is the same ($k = n$). A straightforward derivation results into the full determination of the unique partial cooperative equilibrium and the unique partial cooperative leadership equilibrium:

Proposition 3.4 *Consider the symmetric Salop City discussed above with $k + n = 4s$ where s is a positive integer.*

(a) *The unique partial cooperative equilibrium results in $p_i = c + \frac{8t}{5(k+n)}$ with $\pi_i = \frac{64t}{25(k+n)^2}$ for all $i \in C$; and $p_j = c + \frac{6t}{5(k+n)}$ with $\pi_j = \frac{36t}{25(k+n)^2}$ for all $j \in N$.*
(b) *Let $k + n = 4s$ where s is a positive integer. The unique partial cooperative leadership equilibrium results in $\tilde{p}_i = c + \frac{2t}{(k+n)}$ with $\tilde{\pi}_i = \frac{8t}{3(k+n)^2}$ for all $i \in C$; and $\tilde{p}_j = c + \frac{4t}{3(k+n)}$ with $\tilde{\pi}_j = \frac{16t}{9(k+n)^2}$ for all $j \in N$.*

As expected both cooperating an non-cooperating players earn higher profits in a situation of partial cooperation under both the partial cooperative equilibrium and the partial cooperative leadership equilibrium. Interestingly in the Salop City game, unlike in the public good and the Cournot games, in both types of symmetric games and both types of partial cooperative equilibria, the profits of a cooperator are higher than those of a non-cooperator.

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