9.4 Yield and fracture criteria

In Section 1.4 we saw that materials in uniaxial tension which exhibit linear elastic stress–strain behavior at moderate stress levels deviate from this behavior once a point on the stress–strain curve known as the *yield point* is reached. In other words, beyond a certain *yield stress* or *yield strain*, the behavior is no longer linear elastic, and the material is said to be *yielding* or behaving plastically. What happens subsequently depends on whether the material is ductile or brittle (typical stress–strain curves for the two types are shown in Fig. 1.25). Ductile materials, at least under static loading, undergo relatively large amounts of plastic deformation before final failure. Necking of specimens of the type shown in Fig. 1.24 is evidence of this deformation. Also, if the appearance of the exposed surfaces after failure at the neck is examined it is found that the separation of layers of material occurs not along the plane at right angles to the axis of the specimen but along a series of small planes inclined (typically at about 45°) to the axis. This suggests that yielding and failure are due to shear stresses causing slipping of the ductile material along oblique planes. On the other hand, brittle materials exhibit little or no plastic deformation before failure occurs, by fracturing along planes normal to the applied load. Under cyclically varying loading conditions all materials, even those which are ductile under steady load, tend to exhibit brittle failure at relatively low stress levels, due to the phenomenon of *fatigue*.

When designing a member of an engineering system we must avoid failure, and we often wish to avoid yielding of the material. Indeed, for the reasons discussed in Section 1.3.4, we often have to maintain a safety factor substantially greater than unity against either yielding or ultimate failure. In other words, we must ensure that the maximum stress anywhere in the member does not exceed some maximum allowable value. This maximum allowable value is some fraction of the yield stress, σ_Y, for a ductile material or some fraction of the ultimate (fracture) stress, σ_U, for a brittle material, where σ_Y or σ_U is obtained from a simple tension test. So long as the state of stress in the member to be designed is that of uniaxial tension the maximum stress in it is straightforward to define. Similarly, if the state of stress is simple shear, the shear stress can in principle be compared with the yield or ultimate stress measured in a shear test.

For the more complex states of stress which occur in most situations of practical interest, we would like to have a way of comparing the severity of the combined effect of the several stress components with that of a simple stress state, usually simple tension. In other words, from a complex state of stress we wish to extract a single value of normal stress which can be compared with σ_Y or σ_U to determine whether the loading is excessive. A particular way of characterizing a complex state of stress by a single stress to test for failure is known as a *yield or fracture criterion*.

A yield or fracture criterion is a theoretical formula, which can only be verified by comparison with appropriate experimental results obtained under complex states of stress. Although a number of criteria have been developed over the past century or so, we will only consider those which have proved to be reliable and are currently in common use. Yield and fracture criteria are most conveniently expressed in terms of principal stresses, which completely deter-

Per mostrare che in regime monoassiale la τ_{max} si ha sui piani a 45° (senza conos. i cerchi di Mohr) posso procedere come segue.

$$\begin{cases}[\sigma_n \operatorname{sen}\alpha + \tau\cos\alpha]\frac{1\cdot c}{\operatorname{sen}\alpha} = \sigma\cdot 1\cdot c \\ \sigma_n\cos\alpha = \tau\operatorname{sen}\alpha\end{cases}$$

$\sigma_n = \tau\frac{\operatorname{sen}\alpha}{\cos\alpha}$ sostit. nella espress. preced. ricavo

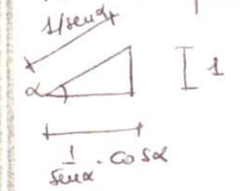
$$\tau\frac{\operatorname{sen}^2\alpha}{\cos\alpha} + \tau\cos\alpha = \sigma\operatorname{sen}\alpha$$

$$\tau = \sigma\operatorname{sen}\alpha\cos\alpha = \frac{\sigma}{2}\operatorname{sen}2\alpha$$

$$\tau_{max} \Rightarrow \operatorname{sen}2\alpha = 1 \Rightarrow 2\alpha = \frac{\pi}{2} \Rightarrow \alpha = \frac{\pi}{4}$$

N.B. $\tau = 0$ per $\alpha = 0$ e $\alpha = \frac{\pi}{2}$

Più in generale



scriveremo

$$\begin{cases}[\sigma_n\operatorname{sen}\alpha + \tau\cos\alpha]\frac{1\cdot c}{\cos\alpha} = \sigma_1\cdot 1\cdot c \\ [\sigma_n\cos\alpha - \tau\operatorname{sen}\alpha]\frac{1\cdot c}{\operatorname{sen}\alpha} = \sigma_2\, c\cdot\frac{\cos\alpha}{\operatorname{sen}\alpha}\end{cases}$$

$$\sigma_n\cos\alpha - \tau\operatorname{sen}\alpha = \sigma_2\cos\alpha$$

$$\sigma_n = \tau\frac{\operatorname{sen}\alpha}{\cos\alpha} + \sigma_2$$

$$\tau = (\sigma_1 - \sigma_2)\operatorname{sen}\alpha\cos\alpha$$

mine a general state of stress, and which do not depend on the particular coordinate system chosen to describe it.

9.4.1 Yield criteria for ductile materials

Because experimental evidence suggests that the yielding of ductile materials is due to shear stresses causing slipping between layers of the material, we can anticipate that appropriate yield criteria will be based on shear stresses. Two such criteria are in common use.

Maximum shear stress (Tresca) criterion

This criterion is based on the idea that a ductile material will yield under a general state of stress when the absolute maximum shear stress is equal to the absolute maximum shear stress at the yield point in a simple tension test. It is often known as Tresca's criterion, after the French engineer Henri Tresca, (1814–1885).

As we saw in Example 9.7a, in a state of simple uniaxial tension with an applied stress of σ, the absolute maximum shear stress is $\sigma/2$. Consequently, the maximum shear stress yield criterion can be expressed symbolically as

$$\tau_{max} = \frac{\sigma_Y}{2} \tag{9.47}$$

In a general three-dimensional state of stress, the absolute maximum shear stress is given in terms of the maximum and minimum principal stresses, σ_1 and σ_3, by equation (9.26) as

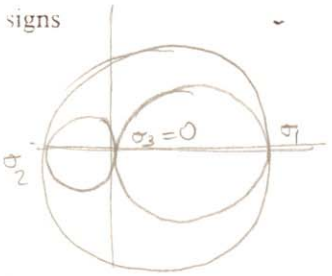
$$\tau_{max} = \frac{\sigma_1 - \sigma_3}{2} \tag{9.48}$$

On the other hand, in a state of plane stress with principal stresses σ_1 and σ_2, the absolute maximum is given by equation (9.28) as the largest of

$$\tau_{max} = \left|\frac{\sigma_1 - \sigma_2}{2}\right| \quad \text{or} \quad \left|\frac{\sigma_1}{2}\right| \quad \text{or} \quad \left|\frac{\sigma_2}{2}\right| \tag{9.49}$$

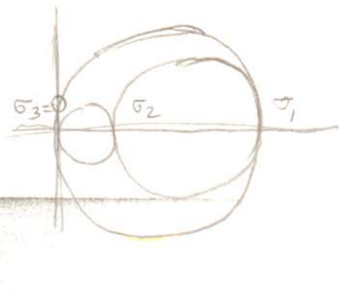
Therefore, if σ_1 and σ_2 are of opposite signs

$$\sigma_1\sigma_2 < 0 \quad \text{and} \quad |\sigma_1 - \sigma_2| = \sigma_Y \tag{9.50a}$$



and if they are of the same sign

$$\sigma_1\sigma_2 > 0 \quad \text{and} \quad |\sigma_1| = \sigma_Y \quad \text{or} \quad |\sigma_2| = \sigma_Y \tag{9.50b}$$



These relationships are plotted with the principal stresses as the axes in Fig. 9.42. Lines CD and AF in the second and fourth quadrants are given by equation (9.50a), while AB, BC, DE and EF in the first and third quadrants are given by equations (9.50b). The closed hexagon ABCDEF is known as the *yield locus*. In other words, if a two-dimensional state of stress has principal stresses which when plotted define a point within the hexagon, the material has not yielded; if it is outside, yielding has occurred. We can identify the various points marked on the locus with different types of stress states. Point A represents simple tension in one direction, while C corresponds to simple tension at right angles to this. Similarly, points D and F represent simple compression, and points B and E equal biaxial tension and compression, respectively. Also, points P and Q represent states of simple shear (Example 9.6 showed that such states have principal stresses which are equal in magnitude to the applied shear stress, τ, but opposite in sign). We note that a consequence of assuming that yielding depends only on shear stresses is that the yield stress in simple compression must be equal in magnitude to that in simple tension.

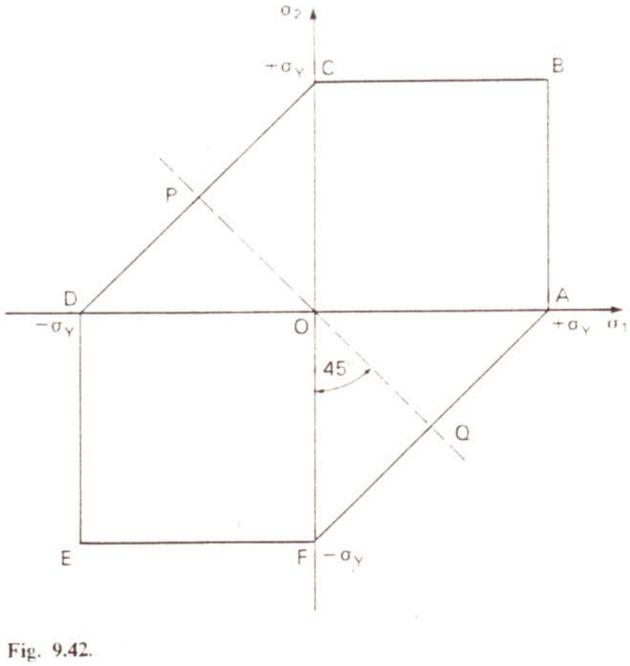


Fig. 9.42.

Shear strain energy (Von Mises) criterion

This criterion is often presented as being based on the idea that yielding of a ductile material under a general state of stress will occur when the density of shear strain energy is equal to the density of shear strain energy at the yield point in a simple tension test. It is often referred to as von Mises' criterion, after the German–American mathematician Richard von Mises (1883–1953). An equivalent, but much more straightforward, way of deriving it is to base

Nella trave lo stato tensionale è $\begin{bmatrix} 0 & 0 & \tau_{zx} \\ 0 & 0 & \tau_{zy} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix}$ o in un opport. riferimento $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \tau \\ 0 & \tau & \sigma \end{bmatrix}$ Le direz. princ. dell tensore bidim. $\begin{bmatrix} 0 & \tau \\ \tau & \sigma \end{bmatrix}$ sono

yielding not on just the absolute maximum shear stress in the material (as in the Tresca criterion), but on the root mean square maximum shear stress, thereby taking into account the shear stresses on planes at right angles to that of the absolute maximum.

Using equation (9.25) to define the maximum shear stresses associated with the three principal planes, the root mean square maximum shear stress for a complex three-dimensional state of stress is

$$\tau_m = \sqrt{\left\{\frac{1}{3}\left[\left(\frac{\sigma_1-\sigma_2}{2}\right)^2+\left(\frac{\sigma_2-\sigma_3}{2}\right)^2+\left(\frac{\sigma_3-\sigma_1}{2}\right)^2\right]\right\}} \tag{9.51}$$

In simple uniaxial tension, with $\sigma_1=\sigma_Y$, $\sigma_2=0$ and $\sigma_3=0$, this becomes

$$\tau'_m = \frac{\sigma_Y}{\sqrt{6}}$$

and we obtain the yield criterion by equating τ_m and τ'_m to give

$$(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2 = 2\sigma_Y^2 \tag{9.52}$$

Under plane stress conditions, with $\sigma_3 = 0$, this becomes

$$\sigma_1^2+\sigma_2^2-\sigma_1\sigma_2 = \sigma_Y^2 \tag{9.53}$$

Another way of expressing the same result is to define a von Mises *equivalent stress* (sometimes referred to as an *effective stress*) as

$$\sigma_e = \sqrt{(\sigma_1^2+\sigma_2^2-\sigma_1\sigma_2)} \tag{9.54}$$

and take yielding to occur when this normal stress is equal to the measured yield stress in simple tension. In other words, the equivalent stress is the stress in uniaxial tension which is equivalent to the complex state of stress according to the von Mises criterion of yielding. The equivalent stress is therefore a useful parameter with which to characterize a state of stress, which is why it is calculated in programs ROSETTE and ANAL2D in Sections 9.2.6 and 9.3.

If we plot the curve defined by equation (9.53) with axes of σ_1 and σ_2, we obtain Fig. 9.43. The shape of the yield locus is an ellipse, with the major and minor axes along the biaxial tension/compression line TT and pure shear line SS, respectively. Figure 9.44 shows the Tresca and von Mises yield loci, Figs 9.42 and 9.43, plotted on the same axes. We see that the points A, B, C, D, E and F at the corners of the Tresca locus also lie on the von Mises locus. This must be the case at points A and C, because both yield criteria use simple

$\det\begin{bmatrix} -\lambda & \tau \\ \tau & \sigma-\lambda \end{bmatrix} = 0$

$-\lambda(\sigma-\lambda)-\tau^2=0 \iff -\lambda\sigma+\lambda^2-\tau^2=0$

$\lambda_{1/2} = \frac{\sigma \mp \sqrt{\sigma^2+4\tau^2}}{2} = \frac{\sigma}{2} \mp \sqrt{\left(\frac{\sigma}{2}\right)^2+\tau^2}$

Sostituendo questi valori in $\sigma_1^2+\sigma_2^2-\sigma_1\sigma_2=\sigma_Y^2$ si ottiene ponendo $a=\frac{\sigma}{2}$ $b=\sqrt{\left(\frac{\sigma}{2}\right)^2+\tau^2}$

$\lambda_1=a+b$ $\lambda_2=a-b$

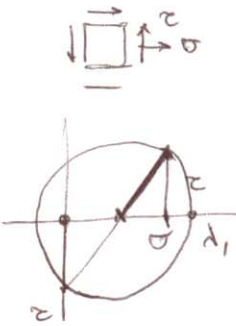
$(a^2+b^2+2ab)+(a^2+b^2-2ab)-(a^2-b^2)=\sigma_Y^2$

$a^2+3b^2=\sigma_Y^2$

$\frac{\sigma^2}{4}+3\left(\frac{\sigma^2}{4}+\tau^2\right)=\sigma_Y^2$

$\boxed{\sigma^2+3\tau^2=\sigma_Y^2}$

N.B. Le tensioni princ. si ottengono anche dal cerchio di Mohr



$\lambda_{1/2} = \frac{\sigma}{2} \pm \text{raggio} = \frac{\sigma}{2} \pm \sqrt{\frac{\sigma^2}{4}+\tau^2}$

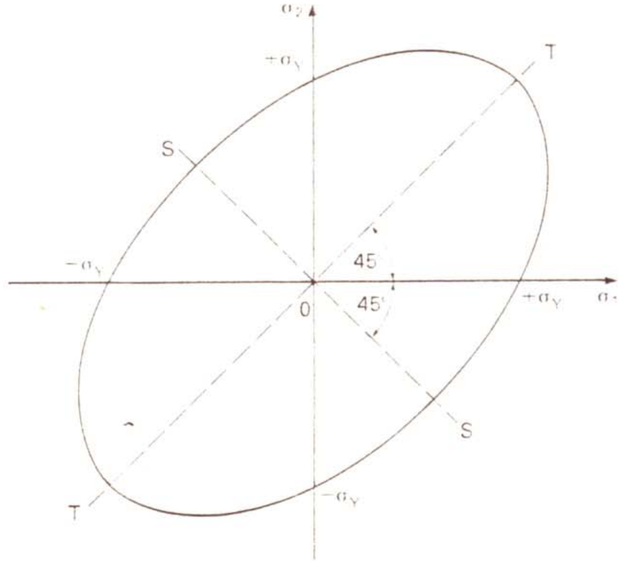


Fig. 9.43.

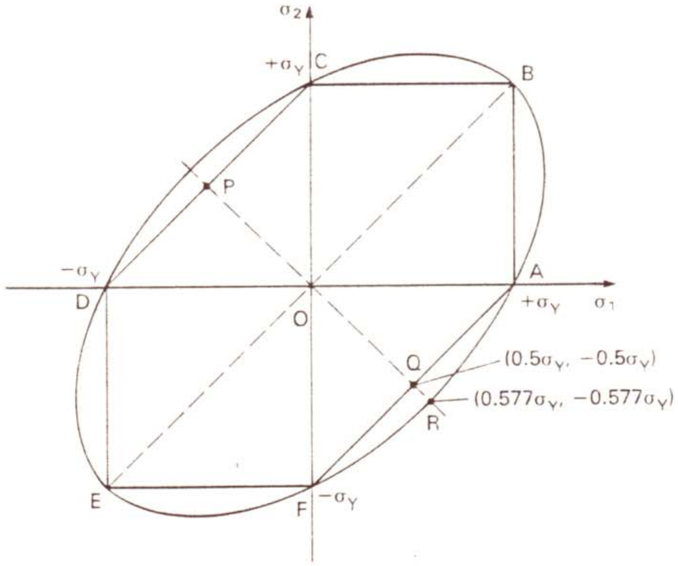


Fig. 9.44.

tension as a reference condition. We have seen already that the assumption that yielding is a phenomenon controlled by shearing means that the yield stress in simple compression is equal in magnitude to that in simple tension, which explains why the loci are coincident at points D and F. They are also coincident at points B and E representing biaxial tension and compression.

The most significant differences between the two criteria occur under pure shear conditions. For example, point Q on the Tresca locus is defined