An elementary course on the

Mathematical Structure of Classical Dynamics

II. Celestial and Terrestrial Dynamics

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Preface

After having illutrated (at an early stage) the *mathematical* structure of Classical Dynamics, [1] we shall now turn to its original fields of *physical* investigations, i.e. *celestial* and *terrestrial* dynamics.

On the one hand, these fields will provide the physical contents of the whole mathematical scheme previously described.

On the other hand, the problems of celestial and terrestrial dynamics, once framed into the above scheme, will be able to be given a thorough mathematical discussion.

From the *philosophical* point of view, celestial and terrestrial dynamics are no longer disjoint areas of knowledge since *Newtonian synthesis*, which recognized their common core in *gravitation*.

So celestial dynamics (chap. 1) will be presented as the dynamics of an isolated system consisting of two *gravitationally interacting* celestial bodies in a Galileian reference space (any one of a class of reference spaces distinguished by Newton's laws of gravitational dynamics by virtue of Galilei's principle of relativity). If one of the bodies is so massive as to be indifferent to the gravitational attraction of the other body and to keep stationary in a Galileian reference space (e.g. the Sun and a planet, or a planet and a test particle), then the above problem is just reduced to the dynamics of the lighter body. In any case, Newton's law of gravitational dynamics can be reduced to a differential equation, whose discussion will lead us so far as to deduce the historical Kepler's laws (adopting both the model of a point-like body and that of a 'small' rigid body).

Then terrestrial dynamics (chap. 2) will be presented as the dynamics, in the terrestrial reference space (where the Earth is stationary), of a 'light' body subject (at least) to *terrestrial gravity*, resulting from the gravitational attraction of the Earth and the dynamical effects which are due to the motion of rotation of the Earth with respect to a Galileian space. In such a context, Newton's law for the dynamics of an unconstrained system and d'Alembert's law for the dynamics of a constrained system will be discussed in quite a number of applications (including both the model of a point-like body and that of a 'small' rigid body).

An Appendix (chap. 3) will provide the main kinematic tools for the study of the dynamical effects which are to be expected from a change of reference space.

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[1] See I. Historical and Analytical Dynamics (which, in the sequel, will be referred to as HAD).

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Chapter 1

Celestial dynamics

In a large scale conception of the universe, the basic interaction between any two bodies is *gravitation*. The most elementary mathematical models describing gravitational dynamics, such as it arises from Newton's ideas, will be discussed.

1.1 Two-body problem

From an empirical point of view, we shall consider an 'isolated' system consisting of two point-like 'celestial' bodies.

'Celestial' means that we are referring to any two stars, planets, satellites or whatever (i.e. any two bodies in the universe).

'Isolated' means that we are taking account of no 'external action' on the system (as though there did exist no other body in the universe).

As to the 'gravitational interaction' between two celestial bodies, the basic assumption is that there exists at least one reference space – called *Galileian reference space* – where such a *δύναμις* , and the consequent *dynamically* possible motions of the bodies, can be described by the mathematical model which will now be illustrated.

1.1.1 Newtonian gravitation

The mathematical model of the above two-body gravitational problem is the following 'celestial' mechanical system.

Gravitational dynamics

The Galileian reference space is conceived as a 3-dimensional Euclidean affine space $\mathcal{E}_{\mathcal{G}}$ (modelled on an vector space $E_{\mathcal{G}}$, equipped with a scalar product

.

and an orientation $\mathrm{Or}(E_{\mathcal{G}})$).

For an ordered system of two point-like bodies, an *admissible* configuration (or position) in the above reference space is defined by assigning an ordered couple of distinct points of $\mathcal{E}_{\mathcal{G}}$ and therefore its *configuration space* is the open submanifold

$$Q := \{(\mathrm{p}_1, \mathrm{p}_2) \in \mathcal{E}_{\mathcal{G}} \times \mathcal{E}_{\mathcal{G}} \mid d(\mathrm{p}_1, \mathrm{p}_2) > 0\}$$

of the 6-dimensional Euclidean affine space $\mathcal{E}_{\mathcal{G}} \times \mathcal{E}_{\mathcal{G}}$ (modelled on $E_{\mathcal{G}} \times E_{\mathcal{G}}$).

If

$$m_1 > 0 \ , \quad m_2 > 0$$

denote the inertial masses of the bodies and

$$\mathrm{F}_1 : Q \to E_{\mathcal{G}} \ , \quad \mathrm{F}_2 : Q \to E_{\mathcal{G}}$$

denote their interaction, then (in accordance with the principle of action and reaction) *Newton's law of gravitational attraction* requires, for all $(\mathrm{p}_1, \mathrm{p}_2) \in Q$,

$$\mathrm{F}_1(\mathrm{p}_1, \mathrm{p}_2) := -h \, \frac{m_1 m_2}{|\mathrm{p}_1 - \mathrm{p}_2|^2} \frac{\mathrm{p}_1 - \mathrm{p}_2}{|\mathrm{p}_1 - \mathrm{p}_2|}$$

$$\mathrm{F}_2(\mathrm{p}_1, \mathrm{p}_2) := -h \, \frac{m_2 m_1}{|\mathrm{p}_2 - \mathrm{p}_1|^2} \frac{\mathrm{p}_2 - \mathrm{p}_1}{|\mathrm{p}_2 - \mathrm{p}_1|}$$

(h being a suitable positive constant).

According to Newton's law of dynamics, [1] the dynamically possible motions (DPMs) of the *celestial mechanical system*

$$\mathcal{C} := (Q, (m_1, m_2), (\mathrm{F}_1, \mathrm{F}_2))$$

(with 6 degrees of freedom) are the smooth motions

$$\gamma : I \subset \mathbb{R} \to \mathcal{E}_{\mathcal{G}} \times \mathcal{E}_{\mathcal{G}} : t \mapsto (\mathrm{p}_1(t), \mathrm{p}_2(t))$$

satisfying the admissibility condition [2]

$$(\mathrm{p}_1(t), \mathrm{p}_2(t)) \in Q \tag{0}$$

and the equations

$$m_1 \, \ddot{\mathrm{p}}_1(t) = -h \, \frac{m_1 m_2}{|\mathrm{p}_1(t) - \mathrm{p}_2(t)|^2} \frac{\mathrm{p}_1(t) - \mathrm{p}_2(t)}{|\mathrm{p}_1(t) - \mathrm{p}_2(t)|} \tag{1}$$

---

[1] See HAD, section 2.1.2.

[2] "$\forall t \in I$" will generally be understood.

$$m_2\,\ddot{\mathrm{p}}_2(t) = -h\,\frac{m_2 m_1}{|\mathrm{p}_2(t)-\mathrm{p}_1(t)|^2}\,\frac{\mathrm{p}_2(t)-\mathrm{p}_1(t)}{|\mathrm{p}_2(t)-\mathrm{p}_1(t)|} \tag{2}$$

Remark that, when the distance $|\mathrm{p}_1 - \mathrm{p}_2|$ between the bodies tends to infinity, the mutual gravitational attraction $\mathrm{F}_1 = -\mathrm{F}_2$, whose norm is inversely proportional to the square of the distance, rapidly tends to zero (in physical terms, when the two bodies are very far from each other, each one of them can just be regarded as an isolated body). So, in a region of Q characterized by suitably high values of the distance, equations (1)(2) are well approximated, in accordance with Newton's law of inertia, [3] by

$$\ddot{\mathrm{p}}_1(t) = 0$$

$$\ddot{\mathrm{p}}_2(t) = 0$$

Galileian or inertial reference spaces

As a consequence, the *Galileian reference spaces*, i.e. the reference spaces where the dynamics of gravitationally interacting bodies is governed by Newton's law (0)(1)(2), [4] can as well be distinguished as those reference spaces where the the dynamics of isolated bodies is governed by Newton's law of inertia and are therefore also called *inertial reference spaces.*

1.1.2 Galilei's principle of relativity

We started from the physical assumption that there exists a reference space $\mathcal{E}_{\mathcal{G}}$ where the dynamics of two celestial bodies is governed by Newton's law (0)(1)(2). That naturally poses the problem of whether the above law is invariant under a reference transformation or not.

Reference transformations

Let α be a transformation from the given Galileian reference space $\mathcal{E}_{\mathcal{G}}$ to any other reference space $\tilde{\mathcal{E}}_3$, i.e. a time-dependent, orientation-preserving, affine isometry [5]

$$\alpha_t : \tilde{\mathcal{E}}_3 \to \mathcal{E}_{\mathcal{G}} : \tilde{\mathrm{p}} \mapsto \alpha_t(\tilde{\mathrm{p}}) := \mathrm{o}_t + A_t(\tilde{\mathrm{p}} - \tilde{\mathrm{o}})\,, \quad \forall\, t \in \mathbb{R}$$

---

[3] See HAD, section 2.1.3.

[4] The name is due to Galilei's principle of relativity, owing to which any two Galileian reference spaces are related to each other by a 'Galileian transformation', i.e. a uniform and rectilinear motion of translation (see the next section 1.1.2).

[5] See Appendix, section 3.1.

A smooth motion

$$\gamma : I \subset \mathbb{R} \to \mathcal{E}_{\mathcal{G}} \times \mathcal{E}_{\mathcal{G}} : t \mapsto (\mathrm{p}_1(t), \mathrm{p}_2(t))$$

of the two bodies in $\mathcal{E}_{\mathcal{G}}$, would be described in $\tilde{\mathcal{E}}_3$ by

$$\tilde{\gamma} : I \subset \mathbb{R} \to \tilde{\mathcal{E}}_3 \times \tilde{\mathcal{E}}_3 : t \mapsto (\tilde{\mathrm{p}}_1(t), \tilde{\mathrm{p}}_2(t))$$

with [6]

$$\tilde{\mathrm{p}}_1(t) := \alpha_t^{-1}(\mathrm{p}_1(t)) , \quad \tilde{\mathrm{p}}_2(t) := \alpha_t^{-1}(\mathrm{p}_2(t))$$

that is,

$$\mathrm{p}_1(t) = \mathrm{o}_t + A_t(\tilde{\mathrm{p}}_1(t) - \tilde{\mathrm{o}}) , \quad \mathrm{p}_2(t) = \mathrm{o}_t + A_t(\tilde{\mathrm{p}}_2(t) - \tilde{\mathrm{o}})$$

On the one hand, as

$$\mathrm{p}_1(t) - \mathrm{p}_2(t) = A_t(\tilde{\mathrm{p}}_1(t) - \tilde{\mathrm{p}}_2(t))$$

(where A_t , linear part of α_t , is an isometry), α leaves Newton's law of gravitational attraction invariant, i.e.

$$A_t^{-1}\left(-h\,\frac{m_1 m_2}{|\mathrm{p}_1(t) - \mathrm{p}_2(t)|^2}\frac{\mathrm{p}_1(t) - \mathrm{p}_2(t)}{|\mathrm{p}_1(t) - \mathrm{p}_2(t)|}\right) = -h\,\frac{m_1 m_2}{|\tilde{\mathrm{p}}_1(t) - \tilde{\mathrm{p}}_2(t)|^2}\frac{\tilde{\mathrm{p}}_1(t) - \tilde{\mathrm{p}}_2(t)}{|\tilde{\mathrm{p}}_1(t) - \tilde{\mathrm{p}}_2(t)|}$$

(the same holds for $\mathrm{p}_2(t) - \mathrm{p}_1(t)$).

On the other hand, α transforms the accelerations through the more compex law [7]

$$\ddot{\tilde{\mathrm{p}}}_1(t) = A_t^{-1}(\ddot{\mathrm{p}}_1(t)) - \tilde{\mathrm{a}}(t, \tilde{\mathrm{p}}_1(t)) - 2\,\tilde{\omega}(t) \wedge \dot{\tilde{\mathrm{p}}}_1(t)$$

(the same holds for $\ddot{\tilde{\mathrm{p}}}_2(t)$).

As a consequence, γ satisfies (0)(1)(2), iff $\tilde{\gamma}$ satisfies the condition of type (0) expressed by [8]

$$(\tilde{\mathrm{p}}_1(t), \tilde{\mathrm{p}}_2(t)) \in \tilde{Q}$$

and the equations

$$m_1\,\ddot{\tilde{\mathrm{p}}}_1(t) = -h\,\frac{m_1 m_2}{|\tilde{\mathrm{p}}_1(t) - \tilde{\mathrm{p}}_2(t)|^2}\frac{\tilde{\mathrm{p}}_1(t) - \tilde{\mathrm{p}}_2(t)}{|\tilde{\mathrm{p}}_1(t) - \tilde{\mathrm{p}}_2(t)|} - m_1\tilde{\mathrm{a}}(t, \tilde{\mathrm{p}}_1(t)) - 2\,m_1\,\tilde{\omega}(t) \wedge \dot{\tilde{\mathrm{p}}}_1(t)$$

$$m_2\,\ddot{\tilde{\mathrm{p}}}_2(t) = -h\,\frac{m_2 m_1}{|\tilde{\mathrm{p}}_2(t) - \tilde{\mathrm{p}}_1(t)|^2}\frac{\tilde{\mathrm{p}}_2(t) - \tilde{\mathrm{p}}_1(t)}{|\tilde{\mathrm{p}}_2(t) - \tilde{\mathrm{p}}_1(t)|} - m_2\tilde{\mathrm{a}}(t, \tilde{\mathrm{p}}_2(t)) - 2\,m_2\,\tilde{\omega}(t) \wedge \dot{\tilde{\mathrm{p}}}_2(t)$$

whose right hand sides exhibit, as well as the Newtonian forces of gravitational attraction, additional forces due to the rigid motion α of $\tilde{\mathcal{E}}_3$ in $\mathcal{E}_{\mathcal{G}}$.

So *Newton's law* (0)(1)(2) *of gravitational dynamics, is not invariant under any reference transformation.*

---

[6] See Appendix, section 3.2, *Composition of motions*

[7] See Appendix, section 3.2, *Composition of accelerations.*

[8] Put $\tilde{Q} := \{(\tilde{\mathrm{p}}_1, \tilde{\mathrm{p}}_2) \in \tilde{\mathcal{E}}_3 \times \tilde{\mathcal{E}}_3 \mid d(\tilde{\mathrm{p}}_1, \tilde{\mathrm{p}}_2) > 0\}$.

Galileian transformations

However, the above two equations take the form (1)(2)

$$m_1\,\ddot{\tilde{\mathrm{p}}}_1(t) = -h\,\frac{m_1 m_2}{|\tilde{\mathrm{p}}_1(t)-\tilde{\mathrm{p}}_2(t)|^2}\,\frac{\tilde{\mathrm{p}}_1(t)-\tilde{\mathrm{p}}_2(t)}{|\tilde{\mathrm{p}}_1(t)-\tilde{\mathrm{p}}_2(t)|}$$

$$m_2\,\ddot{\tilde{\mathrm{p}}}_2(t) = -h\,\frac{m_2 m_1}{|\tilde{\mathrm{p}}_2(t)-\tilde{\mathrm{p}}_1(t)|^2}\,\frac{\tilde{\mathrm{p}}_2(t)-\tilde{\mathrm{p}}_1(t)}{|\tilde{\mathrm{p}}_2(t)-\tilde{\mathrm{p}}_1(t)|}$$

if, and (owing to determinism) only if, the vector field

$$\tilde{\Gamma}(t,\tilde{\mathrm{p}},\tilde{\mathrm{v}}) := \tilde{\mathrm{a}}(t,\tilde{\mathrm{p}}) + 2\,\tilde{\omega}(t)\wedge\tilde{\mathrm{v}}$$

vanishes for all $(t,\tilde{\mathrm{p}},\tilde{\mathrm{v}}) \in \mathbb{R}\times\tilde{\mathcal{E}}_3\times\tilde{E}_3$. [9]

As is known, $\tilde{\Gamma}$ vanishes identically, iff α is a Galileian transformation (corresponding to a uniform and rectilinear motion of translation). [10]

So *Newton's law* (0)(1)(2) *of gravitational dynamics is invariant only under Galileian transformations.*

As a consequence, any two Galileian reference spaces are related to each other by a Galileian transformation, and hence:

"No Galileian transformation from any given Galileian reference space to a new Galileian reference space, will ever be detected by therein observing the motions of gravitationally interacting bodies" (*Galilei's principle of relativity* in celestial dynamics).

1.2 One-body problem

The above gravitational two-body problem (with 6 degrees of freedom) can be reduced to a 'one-body problem' (with 3 degrees of freedom), as follows.

1.2.1 Methods of reduction

Two methods of reduction, both leading to the same result, will be illustrated: one is a purely mathematical device, arising from a (vector) first integral of Newton's law; the other is a mathematical approximation of Newton's law, exhibiting a physical meaning of its own.

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[9] $\tilde{E}_3$ denotes the vector space on which $\tilde{\mathcal{E}}_3$ is modelled.

[10] See Appendix, section 3.2, *Acceleration supplies and Galileian transformations.*

Centre of mass and radius vector

Let us introduce a ‘change of variables’, namely the diffeomorphism

$$\varphi : Q \to M : (\mathrm{p}_1, \mathrm{p}_2) \mapsto (\mathrm{c}, \mathrm{p})$$

of Q onto $M := \mathcal{E}_{\mathcal{G}} \times (\mathcal{E}_{\mathcal{G}} - \{\mathrm{o}\})$ defined, for any $(\mathrm{p}_1, \mathrm{p}_2) \in Q$, by the *centre of mass* c of (m_1, m_2) at $(\mathrm{p}_1, \mathrm{p}_2)$ and the end-point p of the *radius vector* $\mathrm{p}_1 - \mathrm{p}_2$ attached at an arbitrarily chosen ‘origin’ $\mathrm{o} \in \mathcal{E}_{\mathcal{G}}$, i.e.

$$\begin{aligned} \mathrm{c} &= \mathrm{o} + \frac{1}{m_1 + m_2}\,(m_1(\mathrm{p}_1 - \mathrm{o}) + m_2(\mathrm{p}_2 - \mathrm{o})) \\ \mathrm{p} &= \mathrm{o} + (\mathrm{p}_1 - \mathrm{p}_2) \end{aligned}$$

The inverse diffeomorphism

$$\varphi^{-1} : M \to Q : (\mathrm{c}, \mathrm{p}) \mapsto (\mathrm{p}_1, \mathrm{p}_2)$$

is easily seen to be given by

$$\begin{aligned} \mathrm{p}_1 &= \mathrm{c} + \frac{m_2}{m_1 + m_2}\,(\mathrm{p} - \mathrm{o}) \\ \mathrm{p}_2 &= \mathrm{c} - \frac{m_1}{m_1 + m_2}\,(\mathrm{p} - \mathrm{o}) \end{aligned}$$

Then let us introduce two ‘fictitious’ mechanical systems (with 3 degrees of freedom each).

The first is

$$\mathcal{C}_o := \Big(\mathcal{E}_{\mathcal{G}}, \mathrm{m}_\mathrm{o}, \mathrm{F}_\mathrm{o}\Big)$$

where

$$\mathrm{m}_\mathrm{o} := m_1 + m_2$$

and

$$\mathrm{F}_\mathrm{o} = 0$$

The DPMs of $\mathcal{C}_o$ are the solutions

$$\kappa_o : I \to \mathcal{E}_{\mathcal{G}} : t \mapsto \mathrm{c}(t)$$

of equation

$$\mathrm{m}_\mathrm{o}\ddot{\mathrm{c}}(t) = 0$$

The second is

$$\mathcal{C}_1 := \Big(\mathcal{E}_{\mathcal{G}} - \{\mathrm{o}\}, m, \mathrm{F}\Big)$$

where

$$m := \frac{m_1 m_2}{\mathrm{m}_\mathrm{o}}$$

and

$$\mathrm{F} := m\,\mathrm{G}$$

with

$$\mathrm{G} : \mathcal{E}_{\mathcal{G}} - \{\mathrm{o}\} \to E_{\mathcal{G}} : \mathrm{p} \mapsto \mathrm{G}(\mathrm{p}) := -h\,\frac{\mathrm{m_o}}{|\mathrm{p}-\mathrm{o}|^2}\,\frac{\mathrm{p}-\mathrm{o}}{|\mathrm{p}-\mathrm{o}|}$$

The DPMs of $\mathcal{C}$ are the solutions

$$\kappa : I \to \mathcal{E}_{\mathcal{G}} - \{\mathrm{o}\} : t \mapsto \mathrm{p}(t)$$

of equation

$$m\,\ddot{\mathrm{p}}(t) = m\,\mathrm{G}(\mathrm{p}(t))$$

Any such couple (κ_o, κ) will be said to be a DPM of $(\mathcal{C}_o, \mathcal{C}_1)$.

Proposition 1 *φ bijectively takes the* DPM*s of $\mathcal{C}$ onto the* DPM*s of $(\mathcal{C}_o, \mathcal{C}_1)$.*

Proof Let $(\mathrm{p}_1(t), \mathrm{p}_2(t)) \in Q$ and $(\mathrm{c}(t), \mathrm{p}(t)) \in M$ be C^∞ differentiable curves which correspond to each other through φ and φ^{-1}, i.e.

$$\begin{aligned}
\mathrm{c}(t) &= \mathrm{o} + \frac{1}{\mathrm{m_o}}\,(m_1(\mathrm{p}_1(t)-\mathrm{o}) + m_2(\mathrm{p}_2(t)-\mathrm{o})) \\
\mathrm{p}(t) &= \mathrm{o} + (\mathrm{p}_1(t) - \mathrm{p}_2(t))
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{p}_1(t) &= \mathrm{c}(t) + \frac{m_2}{\mathrm{m_o}}\,(\mathrm{p}(t)-\mathrm{o}) \\
\mathrm{p}_2(t) &= \mathrm{c}(t) - \frac{m_1}{\mathrm{m_o}}\,(\mathrm{p}(t)-\mathrm{o})
\end{aligned}$$

From transformation φ it follows that, if $(\mathrm{p}_1(t), \mathrm{p}_2(t))$ is a DPM of $\mathcal{C}$, then

$$\begin{aligned}
\mathrm{m_o}\ddot{\mathrm{c}}(t) &= m_1\,\ddot{\mathrm{p}}_1(t) + m_2\,\ddot{\mathrm{p}}_2(t) \\
&= 0
\end{aligned}$$

(which corresponds to the fact that the *linear momentum* $\Lambda : TQ \to E_{\mathcal{G}} : (\mathrm{p}_1, \mathrm{p}_2, \mathrm{v}_1, \mathrm{v}_2) \mapsto \Lambda(\mathrm{p}_1, \mathrm{p}_2, \mathrm{v}_1, \mathrm{v}_2) := m_1\mathrm{v}_1 + m_2\mathrm{v}_2$ is a first integral of Newton's equation (0), (1), (2)) and

$$\begin{aligned}
m\,\ddot{\mathrm{p}}(t) &= m\,(\ddot{\mathrm{p}}_1(t) - \ddot{\mathrm{p}}_2(t)) \\
&= m\left(-h\,\frac{\mathrm{m_o}}{|\mathrm{p}(t)-\mathrm{o}|^2}\,\frac{\mathrm{p}(t)-\mathrm{o}}{|\mathrm{p}(t)-\mathrm{o}|}\right) \\
&= m\,\mathrm{G}(\mathrm{p}(t))
\end{aligned}$$

that is, $(\mathrm{c}(t), \mathrm{p}(t))$ is a DPM of $(\mathcal{C}_o, \mathcal{C}_1)$.

Conversely, from transformation φ^{-1} it follows that, if $(\mathrm{c}(t), \mathrm{p}(t))$ is a DPM of $(\mathcal{C}_o, \mathcal{C}_1)$, then

$$\begin{aligned} m_1 \ddot{\mathrm{p}}_1(t) &:= m_1 \left(\ddot{\mathrm{c}}(t) + \frac{m_2}{\mathrm{m_o}} \, \ddot{\mathrm{p}}(t) \right) \\ &= \frac{m_1 m_2}{\mathrm{m_o}} \left(-h \, \frac{\mathrm{m_o}}{|\mathrm{p}(t) - \mathrm{o}|^2} \, \frac{\mathrm{p}(t) - \mathrm{o}}{|\mathrm{p}(t) - \mathrm{o}|} \right) \\ &= -h \, \frac{m_1 m_2}{|\mathrm{p}_1(t) - \mathrm{p}_2(t)|^2} \, \frac{\mathrm{p}_1(t) - \mathrm{p}_2(t)}{|\mathrm{p}_1(t) - \mathrm{p}_2(t)|} \end{aligned}$$

and

$$\begin{aligned} m_2 \ddot{\mathrm{p}}_2(t) &:= m_2 \left(\ddot{\mathrm{c}}(t) - \frac{m_1}{\mathrm{m_o}} \, \ddot{\mathrm{p}}(t) \right) \\ &= -\frac{m_1 m_2}{\mathrm{m_o}} \left(-h \, \frac{\mathrm{m_o}}{|\mathrm{p}(t) - \mathrm{o}|^2} \, \frac{\mathrm{p}(t) - \mathrm{o}}{|\mathrm{p}(t) - \mathrm{o}|} \right) \\ &= -h \, \frac{m_1 m_2}{|\mathrm{p}_1(t) - \mathrm{p}_2(t)|^2} \, \frac{\mathrm{p}_2(t) - \mathrm{p}_1(t)}{|\mathrm{p}_1(t) - \mathrm{p}_2(t)|} \end{aligned}$$

i.e. $(\mathrm{p}_1(t), \mathrm{p}_2(t))$ is a DPM of $\mathcal{C}$. □

Owing to the above proposition, the problem of determining the DPMs of $\mathcal{C}$ splits into the two 'separated' problems of determining the DPMs of $\mathcal{C}_o$ (i.e. of the centre of mass) and the DPMs of $\mathcal{C}_1$ (i.e. of the radius vector).

As the DPMs of $\mathcal{C}_o$ are all Newtonian inertial motions, [11] our problem is 'reduced' to that of determining only the DPMs of $\mathcal{C}_1$. [12]

Test particle in a gravitational field

We shall now consider the case of one of the two bodies being much more 'massive' than the other (e.g. the Sun and a planet, or a planet and a 'test particle'), say

$$m_1 \ll m_2$$

In such a case, the body of mass

$$\mathrm{m_o} := m_2$$

---

[11] See HAD, section 2.1.3.

[12] The DPMs of $\mathcal{C}_1$ will thoroughly be discussed in the next section 1.2.2, from *First integral of angular momentum* onwards.

will be called a *gravitational source* and the body of mass

$$m := m_1$$

will be called a *test particle.*

The above inequality implies, along any DPM of the two bodies, i.e. any smooth motion $(\mathrm{p}_1(t), \mathrm{p}_2(t)) \in Q$ satisfying (1)(2), that

$$\frac{|\ddot{\mathrm{p}}_2(t)|}{|\ddot{\mathrm{p}}_1(t)|} = \frac{m_1}{m_2} \approx 0$$

Therefore, in the above order of approximation, equation (2) can be replaced by

$$\ddot{\mathrm{p}}_2(t) = 0$$

(it all happens as though the 'massive' gravitational source were indifferent to the attraction of the 'light' test particle).

As a consequence, the gravitational source may stay at rest in the given Galileian reference space, [13] i.e.

$$\mathrm{p}_2(t) = \mathrm{o} \in \mathcal{E}_{\mathcal{G}} \,, \quad \forall\, t \in \mathbb{R}$$

Then a smooth motion

$$\mathrm{p}(t) := \mathrm{p}_1(t) \in \mathcal{E}_{\mathcal{G}} - \{\mathrm{o}\} \tag{0'}$$

of the test particle is possible under the gravitational attraction of the source, iff it is a solution of equation (1), that is,

$$m\, \ddot{\mathrm{p}}(t) = m\, \mathrm{G}(\mathrm{p}(t)) \tag{1'}$$

with

$$\mathrm{G} : \mathcal{E}_{\mathcal{G}} - \{\mathrm{o}\} \to E_{\mathcal{G}} : \mathrm{p} \mapsto \mathrm{G}(\mathrm{p}) := -h\, \frac{\mathrm{m_o}}{|\mathrm{p}-\mathrm{o}|^2}\, \frac{\mathrm{p}-\mathrm{o}}{|\mathrm{p}-\mathrm{o}|}$$

Conditions $(0)'(1)'$ characterize the DPMs of the mechanical system

$$\mathcal{C}_1 := \Big(\mathcal{E}_{\mathcal{G}} - \{\mathrm{o}\}, m, \mathrm{F}\Big)$$

---

[13] The gravitational source will anyway appear at rest in a suitable Galileian reference space, since any 'non-static' solution of equation $\ddot{\mathrm{p}}_2(t) = 0$, say $\mathrm{p}_2(t) = \mathrm{o} + (t - t_o)\, \mathrm{v}_o \in \mathcal{E}_{\mathcal{G}}$ with $\mathrm{v}_o \neq 0$, can be 'reduced to equilibrium', i.e. to $\tilde{\mathrm{p}}_2(t) = \tilde{\mathrm{o}} \in \tilde{\mathcal{E}}_3$, through the Galileian transformation $\alpha_t : \tilde{\mathrm{p}} \in \tilde{\mathcal{E}}_3 \mapsto \alpha_t(\tilde{\mathrm{p}}) := \big(\mathrm{o} + (t - t_o)\, \mathrm{v}_o\big) + A\,(\tilde{\mathrm{p}} - \tilde{\mathrm{o}}) \in \mathcal{E}_{\mathcal{G}}$ (for any choice of $A : \tilde{E}_3 \to E_{\mathcal{G}}$).

where

$$\mathrm{F} := m\,\mathrm{G}$$

So $\mathcal{C}_1$ is the mechanical system which corresponds to the 'one-body problem' of a test particle of mass m *freely falling* in the *gravitational field* G generated in a Galileian reference space $\mathcal{E}_{\mathcal{G}}$ by a source of mass $\mathrm{m_o}$ stationary at a point $\mathrm{o} \in \mathcal{E}_{\mathcal{G}}$. [14]

The *free falls* of the test particle m in the gravitational field G –i.e. the DPMs of $\mathcal{C}_1$ – are then the smooth motions

$$\kappa : t \in I \mapsto \mathrm{p}(t) \in \mathcal{E}_{\mathcal{G}} - \{\mathrm{o}\}$$

satisfying $(1')$, that is,

$$\ddot{\mathrm{p}}(t) = \mathrm{G}(\mathrm{p}(t)) \qquad (1')$$

In a region of $\mathcal{E}_{\mathcal{G}} - \{\mathrm{o}\}$ characterized by suitably high values of the distance from the position of the gravitational source, the free falls do not effectively differ from the inertial motions, since equation $(1)'$ is well approximated – in such a limit case– by

$$\ddot{\mathrm{p}}(t) = 0$$

1.2.2 Free falls

The above 'celestial' one-body problem and its solutions will now be thorougly discussed.

Universality of free falls

If the DPMs of mechanical system $\mathcal{C}_1 = (\mathcal{E}_{\mathcal{G}} - \{\mathrm{o}\}, m, m\mathrm{G})$ are thought of as 'free falls' of a test particle of mass m in the gravitational field G , then remark that they do *not* differ from the free falls of a test particle of unit mass $m = 1$, i.e. the DPMs of $\mathcal{C}_1^1 = (\mathcal{E}_{\mathcal{G}} - \{\mathrm{o}\}, 1, \mathrm{G})$. Hence the principle of *universality of free falls*, according to which "the free falls in a given gravitational field are the same for all the test particles". [15]

---

[14] In the order of approximation allowed by $m_1 \ll m_2$, the above mechanical system $\mathcal{C}_1$ does not differ from the homonymous system of the previous subsection, since

$$m_1 + m_2 \approx m_2 = \mathrm{m_o} \quad \text{and then} \quad \frac{m_1 m_2}{m_1 + m_2} \approx m_1 = m$$

[15] If the approximation $\mathrm{m_o} \approx m + \mathrm{m_o}$ is let to drop, the above 'universality' drops as well. Indeed, in a given Galileian space $\mathcal{E}_{\mathcal{G}}$, let $(\mathrm{p}(t), \mathrm{o}(t)) \in Q$ be a solution of equations (0)(1)(2) (therein putting $\mathrm{m_o} := m_2$ and $m := m_1$). Then consider the reference transformation α_t :

Gravitational v. inertial mass

Newton's law of gravitational attraction could be given an alternative formulation, by conceiving the attraction exerted by a gravitational source on a test particle as proportional to their *gravitational masses* (sort of 'gravitational charges') $\mathrm{M_o} > 0$ and $M > 0$, respectively, rather than to their inertial masses $\mathrm{m_o}$ and m, i.e. (the source being stationary at $\mathrm{o} \in \mathcal{E}_{\mathcal{G}}$)

$$\mathrm{F(p)} := -h \, \frac{M\mathrm{M_o}}{|\mathrm{p}-\mathrm{o}|^2} \, \frac{\mathrm{p}-\mathrm{o}}{|\mathrm{p}-\mathrm{o}|}$$

for all $\mathrm{p} \in \mathcal{E}_{\mathcal{G}} - \{\mathrm{o}\}$. [16]

With such an F in $\mathcal{C}_1$, the DPMs of $\mathcal{C}_1$ would be the solutions in $\mathcal{E}_{\mathcal{G}} - \{\mathrm{o}\}$ of equation

$$m \, \ddot{\mathrm{p}}(t) = -h \, \frac{M\mathrm{M_o}}{|\mathrm{p}(t)-\mathrm{o}|^2} \, \frac{\mathrm{p}(t)-\mathrm{o}}{|\mathrm{p}(t)-\mathrm{o}|}$$

Then, once chosen the same body as a unit of both gravitational and inertial mass, the principle of universality would require the above equation to be equivalent, for all m and M, to equation

$$\ddot{\mathrm{p}}(t) = -h \, \frac{\mathrm{M_o}}{|\mathrm{p}(t)-\mathrm{o}|^2} \, \frac{\mathrm{p}(t)-\mathrm{o}}{|\mathrm{p}(t)-\mathrm{o}|}$$

corresponding to the unit mass. Clearly that is true, iff

$$m = M$$

So the above formulation of the one-body problem predicts the universality of free falls, iff the *identity of inertial and gravitational mass* is assumed to hold (as we have implicitly done *ab initio*).

$\tilde{\mathrm{p}} \in \tilde{\mathcal{E}}_3 \mapsto \alpha_t(\tilde{\mathrm{p}}) := \mathrm{o}(t) + A(\tilde{\mathrm{p}} - \tilde{\mathrm{o}}) \in \mathcal{E}_{\mathcal{G}}$ (for any choice of $\tilde{\mathrm{o}} \in \tilde{\mathcal{E}}_{\mathcal{G}}$ and $A : \tilde{E}_3 \to E_{\mathcal{G}}$). Such a non-Galileian transformation reduces $\mathrm{o}(t)$ to 'equilibrium', i.e. $\tilde{\mathrm{o}}(t) := \alpha_t^{-1}(\mathrm{o}(t)) = \tilde{\mathrm{o}}$ and transforms $\mathrm{p}(t)$ into a smooth motion $\tilde{\mathrm{p}}(t) := \alpha_t^{-1}(\mathrm{p}(t)) \in \tilde{\mathcal{E}}_3 - \{\tilde{\mathrm{o}}\}$ satisfying the 'non-universal' equation

$$\ddot{\tilde{\mathrm{p}}}(t) = -h \, \frac{m + \mathrm{m_o}}{|\tilde{\mathrm{p}}(t)-\tilde{\mathrm{o}}|^2} \, \frac{\tilde{\mathrm{p}}(t)-\tilde{\mathrm{o}}}{|\tilde{\mathrm{p}}(t)-\tilde{\mathrm{o}}|}$$

[16] If

$$\mathrm{F_1(p)} = -h \, \frac{\mathrm{M_o}}{|\mathrm{p}-\mathrm{o}|^2} \, \frac{\mathrm{p}-\mathrm{o}}{|\mathrm{p}-\mathrm{o}|}$$

is the gravitaional attraction exerted on a test particle chosen as unit of gravitational mass, the above law implies the following 'operational' definition for the gravitational mass M of any test particle:

$$M = \frac{|\mathrm{F(p)}|}{|\mathrm{F_1(p)}|}$$

Geometrical v. dynamical gravitation

$\mathcal{C}_1^1$ exhibits the classical 'dynamical description' of gravitation, meant as a force field G whose effect is that of perturbing the inertial motions, characterized by vanishing accelerations, giving rise to free falls with non-vanishing accelerations.

Such a model coexists with a 'geometrical description' of gravitation, as will now be shown.

Recall [17] that, on the one hand, the inertial motions of $\mathcal{C}_1^1$ are the geodesic curves of $(\mathcal{E}_{\mathcal{G}} - \{\mathrm{o}\}, \mathsf{K})$, where K is the kinetic energy of a unit mass

$$\mathsf{K} : (\mathcal{E}_{\mathcal{G}} - \{\mathrm{o}\}) \times E_{\mathcal{G}} \to \mathbb{R} : (\mathrm{p}, \mathrm{v}) \mapsto \mathsf{K}(\mathrm{p}, \mathrm{v}) := \frac{1}{2}\, \mathrm{v} \cdot \mathrm{v}$$

i.e. the quadratic form – semi-square of norm – characterizing the 'flat' Euclidean geometry of $\mathcal{E}_{\mathcal{G}}$, whose geodesic orbits – excluding those that degenerate into singletons – are all rectilinear.

Also recall that, on the other hand, the DPMs of $\mathcal{C}_1^1$ –i.e. the free falls– are the geodesic curves of $(\mathcal{E}_{\mathcal{G}} - \{\mathrm{o}\}, \mathsf{L})$, where L is the Lagrangian function

$$\mathsf{L} : (\mathcal{E}_{\mathcal{G}} - \{\mathrm{o}\}) \times E_{\mathcal{G}} \to \mathbb{R} : (\mathrm{p}, \mathrm{v}) \mapsto \mathsf{L}(\mathrm{p}, \mathrm{v}) := \mathsf{K}(\mathrm{p}, \mathrm{v}) - \mathsf{V}(\mathrm{p})$$

and

$$\mathsf{V} : \mathcal{E}_{\mathcal{G}} - \{\mathrm{o}\} \to \mathbb{R} : \mathrm{p} \mapsto \mathsf{V}(\mathrm{p}) := -h\, \frac{\mathrm{m}_o}{|\mathrm{p} - \mathrm{o}|}$$

is the potential energy of the gravitational field. [18]

So gravitation can be described as a *potential field* $-\mathsf{V}$, whose effect is that of transforming the 'flat' Euclidean geometry K of $\mathcal{E}_{\mathcal{G}} - \{\mathrm{o}$, with generally 'rectilinear' geodesic orbits (the inertial motions in Newton's conception), into the 'non-flat' Lagrangian geometry $\mathsf{L} := \mathsf{K} - \mathsf{V}$, with generally 'curved' geodesics orbits (new 'inertial motions', coinciding with the free falls).

---

[17] See HAD, section 3.1..2.

[18] Remark that we can write $\mathsf{V} = -h\, \mathrm{m}_o\, |\psi|^{-1}$, where ψ is the restriction to $\mathcal{E}_{\mathcal{G}} - \{\mathrm{o}\}$ of the affine isomorphism $-\,\mathrm{o} : \mathrm{p} \in \mathcal{E}_{\mathcal{G}} \mapsto \psi(\mathrm{p}) := \mathrm{p} - \mathrm{o} \in E_{\mathcal{G}}$, whose linear part is $\mathrm{id}_{E_{\mathcal{G}}}$; then, owing to the rules of differentiation, we have

$$-d_{\mathrm{p}} V = -\frac{h\, \mathrm{m}_o}{|\psi(\mathrm{p})|^2}\, d_{\mathrm{p}}|\psi| = -\frac{h\, \mathrm{m}_o}{|\psi(\mathrm{p})|^2}\, \frac{\psi(\mathrm{p})}{|\psi(\mathrm{p})|} \cdot d_{\mathrm{p}}\psi = \mathrm{G}(\mathrm{p}) \cdot \mathrm{id}_{E_3} = \mathrm{G}(\mathrm{p})\cdot$$

which shows that V is the potential energy of G.

First integral of angular momentum

A qualitative analysis of the DPMs of $\mathcal{C}_1^1$ will now be carried out.

Our analysis starts from the following remark.

The gravitational field G is a *central vector field* with centre at $\mathrm{o} \in \mathcal{E}_\mathcal{G}$, i.e.[19]

$$(\mathrm{p}-\mathrm{o}) \wedge \mathrm{G}(\mathrm{p}) = 0\,, \quad \forall\, \mathrm{p} \in \mathcal{E}_\mathcal{G} - \{\mathrm{o}\}$$

As a consequence, the *angular momentum* (of a unit mass)

$$\Omega : (\mathcal{E}_\mathcal{G} - \{\mathrm{o}\}) \times E_\mathcal{G} \to E_\mathcal{G} : (\mathrm{p},\mathrm{v}) \mapsto \Omega(\mathrm{p},\mathrm{v}) := (\mathrm{p}-\mathrm{o}) \wedge \mathrm{v}$$

is a first integral of equation $(1')$, as will now be shown.

Let

$$\kappa : I \to \mathcal{E}_\mathcal{G} - \{\mathrm{o}\} \ : t \mapsto \mathrm{p} = \mathrm{p}(t)$$

be the maximal solution of equation

$$\ddot{\mathrm{p}} = \mathrm{G}(\mathrm{p}) \tag{$1'$}$$

determined by the initial conditions, at a time $t_o \in I$,

$$\mathrm{p}(t_o) = \mathrm{p}_o\,, \quad \dot{\mathrm{p}}(t_o) = \mathrm{v}_o \tag{$\circ$}$$

for any choice of the Cauchy data $(\mathrm{p}_o, \mathrm{v}_o) \in (\mathcal{E}_\mathcal{G} - \{\mathrm{o}\}) \times E_\mathcal{G}$.

Proposition 2 *Ω keeps constant along the tangent lift $\dot{\kappa}$ of the maximal free fall κ, i.e.*

$$\Omega \circ \dot{\kappa} = \Omega(\mathrm{p}_o, \mathrm{v}_o)$$

Proof: By evaluating Ω along $\dot{\kappa} = (\mathrm{p}, \dot{\mathrm{p}})$, we obtain a vector-valued function of time which satisfies, owing to $(1')$, the normal differential equation

$$\begin{aligned} \frac{d}{dt}(\Omega \circ \dot{\kappa}) &= \frac{d}{dt}((\mathrm{p}-\mathrm{o}) \wedge \dot{\mathrm{p}}) \\ &= (\mathrm{p}-\mathrm{o}) \wedge \ddot{\mathrm{p}} + \dot{\mathrm{p}} \wedge \dot{\mathrm{p}} \\ &= (\mathrm{p}-\mathrm{o}) \wedge \ddot{\mathrm{p}} \\ &= (\mathrm{p}-\mathrm{o}) \wedge \mathrm{G}(\mathrm{p}) \\ &= 0 \end{aligned}$$

---

[19] For the *wedge product* $\wedge$, see Appendix, section 3.1, footnote [5].

and the initial condition

$$(\Omega \circ \dot{\kappa})(t_o) = \Omega(\mathrm{p}_o, \mathrm{v}_o)$$

Hence, owing to determinism,

$$\Omega \circ \dot{\kappa} = \Omega(\mathrm{p}_o, \mathrm{v}_o)$$

That proves our claim. □

The first integral of angular momentum will prove to be an important tool for the qualitative analysis of the free falls.

We shall distinguish two cases:

1. *Radial Cauchy data*

   v_o is null or has the same direction as the radius vector $\mathrm{p}_o - \mathrm{o}$, i.e.

   $$\Omega(\mathrm{p}_o, \mathrm{v}_o) := (\mathrm{p}_o - \mathrm{o}) \wedge \mathrm{v}_o = 0$$

2. *Non-radial Cauchy data*

   v_o is non-null and does not have the same direction as the radius vector $\mathrm{p}_o - \mathrm{o}$, i.e.

   $$\Omega(\mathrm{p}_o, \mathrm{v}_o) := (\mathrm{p}_o - \mathrm{o}) \wedge \mathrm{v}_o \neq 0$$

Radial free falls

Let $\kappa : t \in I \mapsto \mathrm{p} = \mathrm{p}(t) \in \mathcal{E}_{\mathcal{G}} - \{\mathrm{o}\}$ be the maximal solution of Cauchy problem $(1')(\circ)$ with radial Cauchy data.

Owing to the first integral of angular momentum, the above maximal free fall exhibits the following property :

Proposition 3 *κ is a* radial free fall

$$\Omega \circ \dot{\kappa} = 0$$

whose orbit lies on the open halph-line [20]

$$\mathrm{Im}(\kappa) \subset \mathcal{S}^+ := \{\mathrm{p} \in \mathcal{E}_{\mathcal{G}} \mid \mathrm{p} = \mathrm{o} + \rho\, \mathrm{u}_o\,,\ \rho \in \mathbb{R}^+\}$$

where

$$\mathrm{u}_o := \frac{\mathrm{p}_o - \mathrm{o}}{|\mathrm{p}_o - \mathrm{o}|}$$

---

[20] Put $\mathbb{R}^+ := (0, +\infty)$

Proof: Along κ, put

$$\rho := |\mathrm{p} - \mathrm{o}| > 0\,, \quad \mathrm{u} := \frac{1}{\rho}\,(\mathrm{p} - \mathrm{o})$$

The radial character of κ, i.e. $\Omega \circ \dot{\kappa} = (\mathrm{p} - \mathrm{o}) \wedge \dot{\mathrm{p}} = (\mathrm{p}_o - \mathrm{o}) \wedge \mathrm{v}_o = 0$, can be expressed by

$$\rho\, \mathrm{u} \wedge (\rho\, \dot{\mathrm{u}} + \dot{\rho}\, \mathrm{u}) = 0$$

$$\rho^2\,(\mathrm{u} \wedge \dot{\mathrm{u}}) = 0$$

$$\mathrm{u} \wedge \dot{\mathrm{u}} = 0$$

$$\dot{\mathrm{u}} = \lambda\, \mathrm{u}$$

As $\mathrm{u} \cdot \mathrm{u} = 1$, we obtain

$$\begin{aligned} \lambda &= \dot{\mathrm{u}} \cdot \mathrm{u} \\ &= \frac{1}{2}(\frac{d}{dt}\,\mathrm{u} \cdot \mathrm{u}) \\ &= 0 \end{aligned}$$

and then

$$\dot{\mathrm{u}} = 0$$

whence

$$\mathrm{u} = \mathrm{u}(t_o) = \mathrm{u}_o$$

That means, along κ,

$$\mathrm{p}(t) = \mathrm{o} + \rho(t)\,\mathrm{u}_o\,, \quad \rho(t) > 0$$

which proves our claim. □

Remark that $\mathcal{S}^+$ admits the global chart

$$\xi : \mathbb{R}^+ \to \mathcal{S}^+ : \rho \mapsto \mathrm{p} = \mathrm{o} + \rho\,\mathrm{u}_o$$

As a consequence, a smooth motion $\mathrm{p} = \mathrm{p}(t) \in \mathcal{S}^+$ can be described in terms of its global coordinate expression $\rho = \rho(t) := \xi^{-1}(\mathrm{p}(t)) \in \mathbb{R}^+$, i.e.

$$\mathrm{p}(t) = \xi(\rho(t)) = \mathrm{o} + \rho(t)\,\mathrm{u}_o$$

whence

$$\dot{\mathrm{p}} = d_\rho \xi\,(\dot{\rho}) = \dot{\rho}\,\mathrm{u}_o$$

and

$$\ddot{\mathrm{p}} = \ddot{\rho}\,\mathrm{u}_o$$

Proposition 4 *A smooth motion* $\mathrm{p} = \mathrm{p}(t) \in \mathcal{S}^+$ *is the maximal solution of Cauchy problem*

$$\ddot{\mathrm{p}} = \mathrm{G}(\mathrm{p}) \tag{$1'$}$$

$$\mathrm{p}(t_o) = \mathrm{p}_o \,, \;\; \dot{\mathrm{p}}(t_o) = \mathrm{v}_o \tag{$\circ$}$$

with

$$\mathrm{p}_o = \xi(\rho_o) \in \mathcal{S}^+ \,, \;\; \mathrm{v}_o = d_{\rho_o}\xi\,(v_o) \in T_{\mathrm{p}_o}\mathcal{S}^+$$

iff it is the image by ξ *of the maximal solution* $\rho = \rho(t) \in \mathbb{R}^+$ *of Cauchy problem*

$$\ddot{\rho} = -\frac{dV}{d\rho} \tag{$1'$}_\rho$$

$$\rho(t_o) = \rho_o \,, \;\; \dot{\rho}(t_o) = v_o \tag{$\circ$}_\rho$$

where

$$V(\rho) := -h\,\frac{\mathrm{m_o}}{\rho}$$

is the coordinate expression in ξ *of the gravitational potential energy* $\boldsymbol{V}(\mathrm{p}) := -h\,\frac{\mathrm{m_o}}{|\mathrm{p}-\mathrm{o}|}$ *restricted to* $\mathcal{S}^+$ *.*

Proof: Just remark that, in the global chart ξ, Cauchy problem $(1')(\circ)$ reads [21]

$$\ddot{\rho}\,\mathrm{u}_o = -\frac{dV}{d\rho}\,\mathrm{u}_o$$

$$\mathrm{o} + \rho(t_o)\,\mathrm{u}_o = \mathrm{o} + \rho_o\,\mathrm{u}_o \,, \;\; \dot{\rho}(t_o)\,\mathrm{u}_o = v_o\,\mathrm{u}_o$$

whence our claim. □

Recall that the second-order Cauchy problem $(1')_\rho(\circ)_\rho$ in the unknown $\rho = \rho(t) \in \mathbb{R}^+$, can be reformulated as a first-order Cauchy problem in the unknown $(\rho, p) = (\rho(t), p(t)) \in \mathbb{R}^+ \times \mathbb{R}$ by putting [22]

$$\dot{\rho} = p \,, \;\; \dot{p} = -\frac{dV}{d\rho}$$

$$\rho(t_o) = \rho_o \,, \;\; p(t_o) = p_o := v_o$$

Now remark that the above first-order equations are the Hamilton equations

$$\dot{\rho} = \frac{\partial H}{\partial p} \,, \;\; \dot{p} = -\frac{\partial H}{\partial \rho}$$

---

[21] Recall the coordinate expression of $\ddot{\mathrm{p}}$ and $\mathrm{G}(\mathrm{p}) = -h\,\frac{\mathrm{m_o}}{|\mathrm{p}-\mathrm{o}|^2}\,\frac{\mathrm{p}-\mathrm{o}}{|\mathrm{p}-\mathrm{o}|} = -h\,\frac{\mathrm{m_o}}{\rho^2}\,\mathrm{u}_o$.

[22] See HAD, section 4.6.2, *First-order reformulation.*

associated with the Hamiltonian function

$$H : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R} : (\rho, p) \mapsto H(\rho, p) := \frac{1}{2}\,p^2 + V(\rho)$$

As H is a first integral of Hamilton equations, we have, along the maximal solution $(\rho(t), p(t)) \in \mathbb{R}^+ \times \mathbb{R}$ of the first-order Cauchy problem,

$$(\rho(t), p(t)) \in H^{-1}(c_o) := \Big\{(\rho, p) \in \mathbb{R}^+ \times \mathbb{R} \mid \frac{1}{2}\,p^2 + V(\rho) = c_o\Big\}$$

with

$$\begin{aligned} c_o &:= H(\rho_o, p_o) \\ &= \frac{1}{2}\,p_o^2 + V(\rho_o) \\ &= \frac{1}{2}\,v_o^2 - h\,\frac{\mathrm{m_o}}{\rho_o} \\ &= \frac{1}{2}\,|\mathrm{v}_o|^2 - h\,\frac{\mathrm{m_o}}{|\mathrm{p}_o - \mathrm{o}|} \end{aligned}$$

A qualitative analysis of the radial free fall $\mathrm{p}(t)$ corresponding to such a solution, can then be carried out through the geometric-kinematic discussion of the level line $H^{-1}(c_o) \subset \mathbb{R}^+ \times \mathbb{R}$, described by the algebraic equation

$$p = \pm\,\sqrt{2(c_o - V(\rho))}$$

time-oriented by the tangent Hamiltonian vector field

$$\Gamma_H(\rho, p) := \left(\frac{\partial H}{\partial p}\Big|_{(\rho,p)}, -\frac{\partial H}{\partial \rho}\Big|_{(\rho,p)}\right) = \left(p,\, -\frac{dV}{d\rho}\Big|_{\rho}\right)$$

and travelled according to the timetable

$$t(\rho_1) - t(\rho_o) = \pm\int_{\rho_o}^{\rho_1} \frac{1}{\sqrt{2(c_o - V)}}\,d\rho$$

Portrait 1 Determine the graph of V and then draw the level line $H^{-1}(c_o)$ [23] in two typical cases:

(i) Low energy: $c_o = E < 0$, i.e

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[23] See HAD, section 3.2.2, *Example.*

$$|\mathrm{v}_o| < \sqrt{\frac{2h\mathrm{m_o}}{|\mathrm{p}_o - \mathrm{o}|}}$$

$H^{-1}(E)$ is a time-oriented, connected line, contained in the strip

$$\left\{(\rho, p) \in \mathbb{R}^+ \times \mathbb{R} \mid \rho \le \rho_E := -\frac{h\mathrm{m_o}}{E}\right\}$$

Such a line shows that the radial free fall $\mathrm{p}(t)$ first proceeds with decreasing scalar velocity $|\dot{\mathrm{p}}| = |\dot{\rho}| = |p|$ from the centre of attraction towards a point at a distance ρ_E , where it comes to a stop (owing to the low energy), and then symmetrically goes back with increasing scalar velocity towards the centre of attraction .

A discussion of the integral timetable for $\rho \longrightarrow 0$ and $\rho \longrightarrow \rho_E$ would lead us to recognize a collapse onto the centre of attraction , and then incompleteness, both in the past and in the future.

(ii) High energy: $c_o = E' \ge 0$, i.e.

$$|\mathrm{v}_o| \ge \sqrt{\frac{2h\mathrm{m_o}}{|\mathrm{p}_o - \mathrm{o}|}}$$

$H^{-1}(E') = C^1_{E'} \cup C^2_{E'}$ is a time-oriented, disconnected line, contained in the whole half-plane $\mathbb{R}^+ \times \mathbb{R}$ ($C^1_{E'}$ denotes the branch characterized by $p > 0$ and $C^2_{E'}$ the branch characterized by $p < 0$).

Such a line, together with the integral timetable, shows that the radial free fall $\mathrm{p}(t)$ proceeds as follows.

Along $C^1_{E'}$, where $p > 0$ (*centrifugal* velocity), $\mathrm{p}(t)$ comes out of the centre of attraction (incompleteness in the past) and goes on towards infinite distance, since its scalar velocity, though decreasing, keeps high enough (owing to the high energy) to escape attraction.

Along $C^2_{E'}$, where $p < 0$ (*centripetal* velocity), $\mathrm{p}(t)$ comes from infinite distance and goes on with increasing scalar velocity as far as to collapse onto the centre of attraction (incompleteness in the future). □

Non-radial free falls

Let $\kappa : t \in I \mapsto \mathrm{p} = \mathrm{p}(t) \in \mathcal{E}_{\mathcal{G}} - \{\mathrm{o}\}$ be the maximal solution of Cauchy problem $(1')(\circ)$ with non-radial Cauchy data.

Owing to the first integral of angular momentum, the above maximal free fall exhibits the following property :

Proposition 5 *κ is a* non-radial free fall

$$\Omega \circ \dot{\kappa} \neq 0$$

whose orbit lies on the holed plane

$$\mathrm{Im}(\kappa) \subset \mathcal{A}_{\mathrm{o}} := \mathcal{A} - \{\mathrm{o}\}$$

where

$$\mathcal{A} := \mathrm{o} + \mathrm{Span}\,(\mathrm{p}_o - \mathrm{o}, \mathrm{v}_o)$$

Proof: Along κ, the non-radial character $\Omega \circ \dot{\kappa} = (\mathrm{p} - \mathrm{o}) \wedge \dot{\mathrm{p}} = (\mathrm{p}_o - \mathrm{o}) \wedge \mathrm{v}_o \neq 0$ implies

$$(\mathrm{p} - \mathrm{o}) \cdot ((\mathrm{p}_o - \mathrm{o}) \wedge \mathrm{v}_o) = (\mathrm{p} - \mathrm{o}) \cdot ((\mathrm{p} - \mathrm{o}) \wedge \dot{\mathrm{p}}) = 0$$

that is,

$$\mathrm{p} - \mathrm{o} \in \mathrm{Span}^{\perp}((\mathrm{p}_o - \mathrm{o}) \wedge \mathrm{v}_o) = \mathrm{Span}\,(\mathrm{p}_o - \mathrm{o}, \mathrm{v}_o)$$

which proves our claim. □

On $\mathcal{A}_{\mathrm{o}}$ we now introduce a system of *polar coordinates*

$$\xi : \mathbb{R}^+ \times \mathbb{R} \to \mathcal{A}_{\mathrm{o}} : (\rho, \theta) \mapsto \mathrm{p} = \mathrm{o} + (\rho \cos\theta)\mathrm{e}_1 + (\rho \sin\theta)\mathrm{e}_2$$

whence $\rho = |\mathrm{p} - \mathrm{o}|$ and $\theta \equiv \angle(\mathrm{p} - \mathrm{o}, \mathrm{e}_1)$ $(mod\ 2\pi)$. [24] We inform that ξ is a covering map, [25] owing to which a smooth motion $\mathrm{p} = \mathrm{p}(t) \in \mathcal{A}_{\mathrm{o}}$ admits a smooth 'global coordinate expression' $(\rho, \theta) = (\rho(t), \theta(t)) \in \mathbb{R}^+ \times \mathbb{R}$, i.e.

$$\begin{aligned} \mathrm{p} &= \xi(\rho, \theta) \\ &= \mathrm{o} + (\rho \cos\theta)\mathrm{e}_1 + (\rho \sin\theta)\mathrm{e}_2 \end{aligned}$$

whence

$$\begin{aligned} \dot{\mathrm{p}} &= d_{(\rho,\theta)}\xi(\dot{\rho}, \dot{\theta}) \\ &= (\dot{\rho} \cos\theta - \rho\,\dot{\theta} \sin\theta)\mathrm{e}_1 + (\dot{\rho} \sin\theta + \rho\,\dot{\theta} \cos\theta)\mathrm{e}_2 \end{aligned}$$

and

$$\begin{aligned} \ddot{\mathrm{p}} &= ((\ddot{\rho} - \dot{\theta}^2\rho) \cos\theta - (2\dot{\rho}\,\dot{\theta} + \rho\,\ddot{\theta}) \sin\theta)\mathrm{e}_1 \\ &+ ((\ddot{\rho} - \dot{\theta}^2\rho) \sin\theta + (2\dot{\rho}\,\dot{\theta} + \rho\,\ddot{\theta}) \cos\theta)\mathrm{e}_2 \end{aligned}$$

---

[24] Here $(\mathrm{e}_1, \mathrm{e}_2)$ denotes an orthonormal basis of $\mathrm{Span}\,(\mathrm{p}_o - \mathrm{o}, \mathrm{v}_o)$ and, in the sequel, $(\mathrm{e}_1, \mathrm{e}_2, \mathrm{e}_3)$ will denote the orthonormal basis of $E_{\mathcal{G}}$, belonging to $\mathrm{Or}(E_{\mathcal{G}})$, obtained via completion from $(\mathrm{e}_1, \mathrm{e}_2)$.

[25] A *covering map* onto an n-dimensional manifold Q, is a kind of C^∞ surjective mapping $\xi : W \subset \mathbb{R}^n \to Q$ which defines –via restrictions– a whole atlas of local charts on Q, with the property that any C^∞ curve $\gamma : I \to Q$ admits some C^∞ lift $\alpha : I \to W$, i.e. $\gamma = \xi \circ \alpha$.

Proposition 6 *A smooth motion* $\mathrm{p} = \mathrm{p}(t) \in \mathcal{A}_{\mathrm{o}}$ *is the maximal solution of Cauchy problem*

$$\ddot{\mathrm{p}} = \mathrm{G}(\mathrm{p}) \tag{$1'$}$$

$$\mathrm{p}(t_o) = \mathrm{p}_o \,, \quad \dot{\mathrm{p}}(t_o) = \mathrm{v}_o \tag{$\circ$}$$

with

$$\mathrm{p}_o = \xi(\rho_o, \theta_o) \in \mathcal{A}_{\mathrm{o}} \,, \quad \mathrm{v}_o = d_{(\rho_o,\theta_o)}\xi\,(v_o, \nu_o) \in T_{\mathrm{p}_o}\mathcal{A}_{\mathrm{o}}$$

iff it is the image by ξ *of* $(\rho, \theta) = (\rho(t), \theta(t)) \in \mathbb{R}^+ \times \mathbb{R}$*, where* $\rho = \rho(t)$ *is the maximal solution of Cauchy problem*

$$\ddot{\rho} = -\frac{dV_{\mu_o}}{d\rho} \tag{$1'$}_\rho$$

$$\rho(t_o) = \rho_o \,, \quad \dot{\rho}(t_o) = v_o \tag{$\circ$}_\rho$$

with

$$\mu_o := \rho_o^2 \nu_o = ((\mathrm{p}_o - \mathrm{o}) \wedge \mathrm{v}_o) \cdot \mathrm{e}_3 \neq 0$$

$$V_{\mu_o} : \mathbb{R}^+ \to \mathbb{R} : \rho \mapsto V_{\mu_o}(\rho) := -h\,\frac{\mathrm{m_o}}{\rho} + \frac{\mu_o^2}{2\rho^2}$$

and $\theta = \theta(t)$ *is given by the quadrature*

$$\theta(t) \equiv \theta_o + \int_{t_o}^{t} \frac{\mu_o}{\rho^2(\tau)}\, d\tau \ \ (mod\ 2\pi) \tag{$1'$}_\theta$$

satisfying the initial conditions

$$\theta(t_o) \equiv \theta_o \ (mod\ 2\pi), \quad \dot{\theta}(t_o) = \nu_o \tag{$\circ$}_\theta$$

Proof: Check, by direct calculations, that initial conditions $(\circ)$ can be expressed – through ξ – in the form $(\circ)_\rho(\circ)_\theta$.

Then remark that, in the above covering map, equation $(1')$ can be expressed – through ξ – in the form [26]

$$(\ddot{\rho} - \dot{\theta}^2\rho)\cos\theta - (2\dot{\rho}\dot{\theta} + \rho\ddot{\theta})\sin\theta = -h\,\frac{\mathrm{m_o}}{\rho^2}\cos\theta$$

$$(\ddot{\rho} - \dot{\theta}^2\rho)\sin\theta + (2\dot{\rho}\dot{\theta} + \rho\ddot{\theta})\cos\theta = -h\,\frac{\mathrm{m_o}}{\rho^2}\sin\theta$$

---

[26] Recall the coordinate expression of $\ddot{\mathrm{p}}$ and

$$\mathrm{G}(\mathrm{p}) = -h\,\frac{\mathrm{m_o}}{|\mathrm{p}-\mathrm{o}|^2}\,\frac{\mathrm{p}-\mathrm{o}}{|\mathrm{p}-\mathrm{o}|} = \left(-h\,\frac{\mathrm{m_o}}{\rho^2}\cos\theta\right)\mathrm{e}_1 + \left(-h\,\frac{\mathrm{m_o}}{\rho^2}\sin\theta\right)\mathrm{e}_2$$

that is,

$$\left(\ddot\rho - \dot\theta^2\rho + h\,\frac{\mathrm{m_o}}{\rho^2}\right)\cos\theta - \left(\frac{1}{\rho}\,\frac{d}{dt}\,(\rho^2\dot\theta)\right)\sin\theta = 0$$
$$\left(\ddot\rho - \dot\theta^2\rho + h\,\frac{\mathrm{m_o}}{\rho^2}\right)\sin\theta + \left(\frac{1}{\rho}\,\frac{d}{dt}\,(\rho^2\dot\theta)\right)\cos\theta = 0$$

which is equivalent to

$$\ddot\rho - \dot\theta^2\rho + h\,\frac{\mathrm{m_o}}{\rho^2} = 0$$
$$\frac{d}{dt}\,(\rho^2\dot\theta) = 0$$

or, owing to $(\circ)_\rho(\circ)_\theta$,

$$\ddot\rho = -h\,\frac{\mathrm{m_o}}{\rho^2} + \frac{\mu_o^2}{\rho^3}$$
$$\rho^2\dot\theta = \mu_o$$

Hence our claim. □

Recall that the second-order Cauchy problem $(1')_\rho(\circ)_\rho$ in the unknown $\rho = \rho(t) \in \mathbb{R}^+$, can be reformulated as a first-order Cauchy problem in the unknown $(\rho, p) = (\rho(t), p(t)) \in \mathbb{R}^+ \times \mathbb{R}$ by putting [27]

$$\dot\rho = p\,,\quad \dot p = -\frac{dV_{\mu_o}}{d\rho}$$

$$\rho(t_o) = \rho_o\,,\quad p(t_o) = p_o := v_o$$

Now remark that the above first-order equations are the Hamilton equations

$$\dot\rho = \frac{\partial H}{\partial p}\,,\quad \dot p = -\frac{\partial H}{\partial \rho}$$

associated with the Hamiltonian function

$$H : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R} : (\rho, p) \mapsto H(\rho, p) := \frac{1}{2}\,p^2 + V_{\mu_o}(\rho)$$

As H is a first integral of Hamilton equations, we have, along the maximal solution $(\rho(t), p(t)) \in \mathbb{R}^+ \times \mathbb{R}$ of the first-order Cauchy problem,

$$(\rho(t), p(t)) \in H^{-1}(c_o) := \left\{(\rho, p) \in \mathbb{R}^+ \times \mathbb{R} \mid \frac{1}{2}\,p^2 + V_{\mu_o}(\rho) = c_o\right\}$$

---

[27] See HAD, section 4.6.2, *First-order reformulation.*

with

$$\begin{aligned} c_o &:= H(\rho_o, p_o) \\ &= \frac{1}{2}\, p_o^2 + V_{\mu_o}(\rho_o) \\ &= \frac{1}{2}\, p_o^2 + \frac{\mu_o^2}{2\rho_o^2} - h\,\frac{\mathrm{m_o}}{\rho_o} \\ &= \frac{1}{2}(v_o^2 + \rho_o^2\,\nu_o^2) - h\,\frac{\mathrm{m_o}}{\rho_o} \\ &= \frac{1}{2}\,|\mathrm{v}_o|^2 - h\,\frac{\mathrm{m_o}}{|\mathrm{p}_o - \mathrm{o}|} \end{aligned}$$

A qualitative analysis of the non-radial free fall $\mathrm{p}(t)$ corresponding to such a solution, can then be carried out through the geometric-kinematic discussion of the level line $H^{-1}(c_o) \subset \mathbb{R}^+ \times \mathbb{R}$, described by the algebraic equation

$$p = \pm\sqrt{2(c_o - V_{\mu_o}(\rho))}$$

time-oriented by the tangent Hamiltonian vector field

$$\Gamma_H(\rho, p) := \left(\frac{\partial H}{\partial p}\Big|_{(\rho,p)}, -\frac{\partial H}{\partial \rho}\Big|_{(\rho,p)} \right) = \left(p,\, -\frac{dV_{\mu_o}}{d\rho}\Big|_{\rho} \right)$$

and travelled according to the timetable

$$t(\rho_1) - t(\rho_o) = \pm \int_{\rho_o}^{\rho_1} \frac{1}{\sqrt{2(c_o - V_{\mu_o})}}\, d\rho$$

Portrait 2 Determine the graph of V_{μ_o} [28] and then draw the level line $H^{-1}(c_o)$ [29] in two typical cases:

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[28] From

$$\frac{dV_{\mu_o}}{d\rho} = \frac{1}{\rho^3}(h\mathrm{m_o}\rho - \mu_o^2) \lesseqqgtr 0 \Longleftrightarrow \rho \lesseqqgtr \rho_s := \frac{\mu_o^2}{h\,\mathrm{m_o}}$$

it follows that V_{μ_o} admits a minimum at the singular point ρ_s, i.e.

$$V_{\mu_o}(\rho) > E_s := V_{\mu_o}(\rho_s) = -\frac{1}{2}\left(\frac{h\mathrm{m_o}}{\mu_o}\right)^2 < 0\,, \quad \forall\, \rho \neq \rho_s$$

[29] See HAD, section 3.2.2, *Example.*

(i) Low energy: $c_o = E < 0$ (*Keplerian case*), i.e.

$$|\mathrm{v}_o| < \sqrt{\frac{2h\mathrm{m_o}}{|\mathrm{p}_o - \mathrm{o}|}}$$

For $E = E_s$, [30] $H^{-1}(E_s)$ degenerates into the singleton $\{(\rho_s, 0)\}$. In such a case, one has $\rho(t) = \rho_s$ and then $\theta(t) = \theta_o + \frac{\mu_o}{\rho_s^2}(t - t_o)$ for all $t \in \mathbb{R}$. As a consequence, the non-radial free fall $\mathrm{p}(t)$ is a complete, uniform, circular motion with centre at o.

For $E > E_s$, $H^{-1}(E)$ is a loop, contained in the strip

$$\{(\rho, p) \in \mathbb{R}^+ \times \mathbb{R} \mid \rho_{min} \le \rho \le \rho_{max}\}$$

(ρ_{min} and ρ_{max} being the points where V_{μ_o} takes the value E).

In such a case, $\rho(t)$ eternally oscillates between ρ_{min} and ρ_{max} and then $\mathrm{p}(t)$ is a complete motion describing a bounded orbit (namely an ellipse, as will be shown in the next Proposition).

Along the ellipse, $\mathrm{p}(t)$ −under the effect of the gravitational attraction− travels towards o with an increasing 'radial velocity' $|\dot\rho(t)| = |p(t)|$, which however, at due time, starts decreasing and then vanishes at all at the distance ρ_{min} (since, in the right hand side of equation $(1')_\rho$, the 'repulsive barrier' $\frac{\mu_o^2}{\rho^3(t)}$ prevails, for small values of $\rho(t)$, on the gravitational attraction $-h\,\frac{m_o}{\rho^2(t)}$). After that, $\mathrm{p}(t)$ −under the effect of the repulsive barrier− travels far away from o with an increasing radial velocity, which however, at due time, starts decreasing (since the attraction prevails, for high values of $\rho(t)$, on the repulsion) up to vanishing at the distance ρ_{max} (owing to the low energy) and yielding again to gravitational attraction.

(ii) High energy: $c_o = E' \ge 0$, i.e

$$|\mathrm{v}_o| \ge \sqrt{\frac{2h\mathrm{m_o}}{|\mathrm{p}_o - \mathrm{o}|}}$$

$H^{-1}(E')$ is a connected line, contained in the strip

$$\{(\rho, p) \in \mathbb{R}^+ \times \mathbb{R} \mid \rho' \le \rho < +\infty\}$$

(ρ' being the point where V_{μ_o} takes the value E').

---

[30] Remark that, for $E < E_s$, $H^{-1}(E)$ is empty.

In such a case, $\rho(t) \to +\infty$ when $t \to \pm\infty$ and then $\mathrm{p}(t)$ is a complete motion describing an unbounded orbit (namely a parabola or a branch of hyperbola, as will be shown in the next Proposition).

Along such an orbit, under the alternating prevalence of the gravitational attraction and the repulsive barrier, $\mathrm{p}(t)$ travels from infinite distance towards o with an increasing radial velocity which, at due time, starts decreasing and then vanishes at the distance ρ'. After that $\mathrm{p}(t)$ travels far away from o towards infinite distance, with a radial velocity which, at due time, again starts decreasing, but still keeps high enough (owing to the high energy) to escape from gravitational attraction. ☐

Three more pieces of information on non-radial free falls will now be deduced from Proposition 6

The first piece of information concerns the orbit of a non-radial free fall.

Proposition 7 *The orbit of a non-radial free fall is a conic section with a focus at the centre* o *of attraction and it is an ellipse or a parabola or a branch of hyperbola according to whether*

$$|\mathrm{v}_o| \lesseqgtr \sqrt{\frac{2hm_o}{|\mathrm{p}_o - \mathrm{o}|}}$$

Proof: The orbit of the maximal solution of Cauchy problem $(1')(\circ)$ in Proposition 6, is described, in polar coordinates, by the function

$$\rho = \rho(\theta)$$

with

$$\rho(\theta) := \rho(t(\theta))$$

$t(\theta)$ being the inverse of $\theta(t)$. [31]

Owing to $(1')_\theta$, we have

---

[31] The invertibility of $\theta(t)$ is guaranted by $(1')_\theta$, i.e.

$$\dot\theta = \frac{\mu_o}{\rho^2} \neq 0$$

$$\begin{aligned}\ddot\rho &= \dot\theta\,\frac{d\dot\rho}{d\theta} = \dot\theta\,\frac{d}{d\theta}\left(\dot\theta\,\frac{d\rho}{d\theta}\right) = \frac{\mu_o}{\rho^2}\frac{d}{d\theta}\left(\frac{\mu_o}{\rho^2}\frac{d\rho}{d\theta}\right) = \frac{\mu_o^2}{\rho^2}\frac{d}{d\theta}\left(\frac{1}{\rho^2}\frac{d\rho}{d\theta}\right)\\ &= \frac{\mu_o^2}{\rho^2}\frac{d}{d\theta}\left(-\frac{d}{d\rho}\left(\frac{1}{\rho}\right)\frac{d\rho}{d\theta}\right) = -\frac{\mu_o^2}{\rho^2}\frac{d}{d\theta}\left(\frac{d}{d\theta}\left(\frac{1}{\rho}\right)\right)\\ &= -\frac{\mu_o^2}{\rho^2}\frac{d^2}{d\theta^2}\left(\frac{1}{\rho}\right)\end{aligned}$$

Then, owing to $(1')_\rho$,

$$\begin{aligned}-\frac{\mu_o^2}{\rho^2}\frac{d^2}{d\theta^2}\left(\frac{1}{\rho}\right) &= \frac{\mu_o^2}{\rho^3} - h\frac{\mathrm{m_o}}{\rho^2}\\ \frac{d^2}{d\theta^2}\left(\frac{1}{\rho}\right) &= -\frac{1}{\rho} + h\frac{\mathrm{m_o}}{\mu_o^2}\\ \frac{d^2}{d\theta^2}\left(\frac{1}{\rho}\right) &= -\frac{1}{\rho} + \frac{1}{q_o}\\ \frac{d^2}{d\theta^2}\left(\frac{1}{\rho} - \frac{1}{q_o}\right) &= -\left(\frac{1}{\rho} - \frac{1}{q_o}\right)\\ \frac{d^2}{d\theta^2}\left(\frac{1}{\rho} - \frac{1}{q_o}\right) + \left(\frac{1}{\rho} - \frac{1}{q_o}\right) &= 0\end{aligned}$$

with

$$q_o := \frac{\mu_o^2}{h\mathrm{m_o}}$$

So

$$x(\theta) := \frac{1}{\rho(\theta)} - \frac{1}{q_o}$$

is the maximal solution of the 'harmonic oscillator'

$$\frac{d^2x}{d\theta^2} + x = 0$$

satisfying, owing to $(\circ)_\rho(\circ)_\theta$, the initial conditions

$$x(\theta_o) = \frac{1}{\rho_o} - \frac{1}{q_o}$$

and

$$x'(\theta_o) := \frac{dx}{d\theta}\Big|_{\theta_o} = \frac{d\rho^{-1}}{d\theta}\Big|_{\theta_o} = -\frac{1}{\rho_o^2}\frac{d\rho}{d\theta}\Big|_{\theta_o} = -\frac{1}{\rho_o^2}\frac{d\rho}{dt}\Big|_{t_o}\frac{dt}{d\theta}\Big|_{\theta_o} = -\frac{1}{\rho_o^2}\frac{v_o}{\nu_o}$$

Such a solution is easily seen to be given by

$$\frac{1}{\rho} - \frac{1}{q_o} = \frac{e_o}{q_o}\,\cos(\theta + \varphi_o)$$

(defined on the largest open interval, containing θ_o, where inequality

$$\frac{1}{q_o} + \frac{e_o}{q_o}\,\cos(\theta + \varphi_o) > 0$$

is satisfied), the constants

$$e_o \geq 0$$

and

$$\varphi_o \in (-\theta_o - \pi, -\theta_o + \pi]$$

being determined by the above assigned values of $(x(\theta_o), x'(\theta_o))$, since initial conditions

$$x(\theta_o) = \frac{e_o}{q_o}\,\cos(\theta_o + \varphi_o)\ , \quad x'(\theta_o) = -\frac{e_o}{q_o}\,\sin(\theta_o + \varphi_o)$$

imply

$$\frac{e_o^2}{q_o^2} = x^{\,2}(\theta_o) + x'^{\,2}(\theta_o)$$

and, if $e_o > 0$,

$$\theta_o + \varphi_o = \pm\,\arccos\left(\frac{q_o x(\theta_o)}{e_o}\right)$$

(the sign + or - being to be chosen, according to whether $x'(\theta_o) \leq 0$ or $x'(\theta_o) > 0$).

Hence we obtain

$$\rho = \frac{q_o}{1 + e_o \cos(\theta + \varphi_o)}$$

which, as is known from geometry, is the equation, in polar coordinates, of a conic section with a focus at the point o, namely an ellipse if $e_o < 1$ (in particular, a circle if $e_o = 0$), a parabola if $e_o = 1$ or a branch of hyperbola if $e_o > 1$.

Through simple calculations on the above value of e_o^2 , we finally obtain

$$e_o \lesseqgtr 1 \Longleftrightarrow |\mathrm{v}_o|^2 \lesseqgtr \frac{2h\mathrm{m_o}}{\rho_o}$$

which completes our proof. □

The second piece of information concerns the areal velocity of a non-radial free fall. [32]

Proposition 8 *The areal velocity* $A(t)$ *–with respect to the centre* o *of attraction– of a non-radial free fall, is a non-null constant.*

Proof: The areal velocity of the maximal solution of Cauchy problem $(1')(\circ)$ in Proposition 6, is given, owing to $(1')_\theta$, by

$$A(t) = \frac{\mu_o}{2}$$

which proves our claim. □

The third piece of information concerns a *Keplerian motion*, i.e. a non-radial free fall with elliptic orbit.

---

[32] Consider a smooth motion $t \in I \subset \mathbb{R} \mapsto \mathrm{p}(t) \in \mathcal{A}_\mathrm{o} - \{\mathrm{o}\}$ and, for any $t, t+a \in I$, put

$$\mathit{area}\,(\mathrm{o}, \mathrm{p}(t), \mathrm{p}(t+a)) := \frac{1}{2}\big((\mathrm{p}(t) - \mathrm{o}) \wedge (\mathrm{p}(t+a) - \mathrm{o})\big) \cdot \mathrm{e}_3$$

Check that, if the points are not aligned, the above quantity is the area of the corresponding triangle or its opposite, according to whether $\big(\mathrm{p}(t)-\mathrm{o},\ \mathrm{p}(t+a)-\mathrm{o},\ \mathrm{e}_3\big) \in \Omega$ or not.

For 'small' values of $|a|$, $\mathit{area}\,(\mathrm{o}, \mathrm{p}(t), \mathrm{p}(t+a))$ can approximately be thought of as the 'oriented area' swept by the radius vector $(\mathrm{p}(t) - \mathrm{o})$ after a (positive or negative) time interval a ; as a consequence, when $a \to 0$, the limit

$$A(t) := \mathit{lim}_{a\to 0} \frac{\mathit{area}\,(\mathrm{o}, \mathrm{p}(t), \mathrm{p}(t+a))}{a} = \frac{1}{2}\big((\mathrm{p}(t) - \mathrm{o}) \wedge \dot{\mathrm{p}}(t)\big) \cdot \mathrm{e}_3$$

is called *areal velocity* , with respect to o , at time t .

In polar coordinates, it is expressed by

$$A(t) = \frac{1}{2}\rho^2(t)\dot{\theta}(t)$$

Preliminarly remark that the *period* of time taken by a Keplerian motion $\mathrm{p}(t)$ to make a *revolution*, i.e. to travel its elliptic orbit, is

$$T = \frac{2\pi ab}{\mu_o}$$

(a and $b < a$ being the semiaxes of the ellipse), since the area πab of the ellipse is swept by the radius vector $(\mathrm{p}(t) - \mathrm{o})$ with constant areal velocity $\frac{\mu_o}{2}$.

Proposition 9 *The ratio* $\frac{T^2}{a^3}$ *is the same for all the Keplerian motions.*

Proof: Recall that, for any given Keplerian motion, the parameter q_o of its elliptic orbit is given by

$$q_o = \frac{\mu_o^2}{h\mathrm{m_o}}$$

and that the period of revolution is given by

$$T = \frac{2\pi ab}{\mu_o}$$

Moreover recall that, as is known from geometry, the parameter q_o and the couple of semiaxes (a, b) of an ellipse are related to each other by

$$q_o = \frac{b^2}{a}$$

Hence

$$\frac{\mu_0^2}{h\mathrm{m_o}} = \frac{b^2}{a} = \frac{1}{a}\left(\frac{\mu_o T}{2\pi a}\right)^2 = \frac{\mu_o^2 T^2}{4\pi^2 a^3}$$

that is,

$$\frac{T^2}{a^3} = \frac{4\pi^2}{h\mathrm{m_o}}$$

The above result shows that the ratio $\frac{T^2}{a^3}$ does not depend on the Keplerian motion taken into consideration. □

Kepler's laws

In the *Copernican reference space* $\mathcal{E}_{\mathcal{C}}$, i.e. the Galileian reference space where the Sun is stationary, we can apply the above results to any single planet, meant as a test particle freely falling in the gravitational field of the Sun.

Thus, if we start from the fact that, at any 'initial' time, the 'actual' motions of the planets exhibit non-radial initial conditions satisfying the 'low energy' condition, then we deduce that the above motions are all maximal solutions of the universal equation $(1')$ satisfying –owing to Propositions 7, 8 and 9– the historical Kepler's laws:

1. The orbit described by a planet round the Sun is an ellipse, with a focus in the Sun.
2. The areal velocity of a planet with respect to the Sun is constant.
3. The ratio $\frac{T^2}{a^3}$ is the same for all of the planets.

1.2.3 Rigid body

We shall now adopt the model of a 'small' 3-dimensional rigid body in celestial dynamics.

Newton and Euler's equations

Let $\mathcal{R} = (Q, m, \mathrm{F})$ be a mechanical system corresponding to the one-body problem of a 'small' 3-dimensional rigid body freely falling in the gravitational field G generated by a source of mass $\mathrm{m_o}$ stationary at a 'distant' point o of a Galileian reference space $\mathcal{E}_{\mathcal{G}}$. At any admissible configuration $\mathrm{p} = (\mathrm{p}_i)_{i=1,\dots,\nu} \in Q$, with centre of mass $\mathrm{c} \in \mathcal{E}_{\mathcal{G}}$, [33] in the approximation $\mathrm{p}_i - \mathrm{o} = (\mathrm{p}_i - \mathrm{c}) + (\mathrm{c} - \mathrm{o}) \approx \mathrm{c} - \mathrm{o}$ due to hypothesis $|\mathrm{p}_i - \mathrm{c}| \ll |\mathrm{c} - \mathrm{o}|$ ('small' body and 'distant' source), we have

$$\begin{aligned} \mathrm{F}_i(\mathrm{p}) &= m_i\,\mathrm{G}(\mathrm{p}_i) \\ &= -\frac{m_i\,\mathrm{m_o}}{|\mathrm{p}_i - \mathrm{o}|^2}\frac{\mathrm{p}_i - \mathrm{o}}{|\mathrm{p}_i - \mathrm{o}|} \\ &\approx -\frac{m_i\,\mathrm{m_o}}{|\mathrm{c} - \mathrm{o}|^2}\frac{\mathrm{c} - \mathrm{o}}{|\mathrm{c} - \mathrm{o}|} \\ &= m_i\,\mathrm{G}(\mathrm{c}) \end{aligned}$$

Hence

$$\begin{aligned} \mathrm{R}^{\mathrm{F}}(\mathrm{p}) &= \textstyle\sum_{i=1}^{\nu} \mathrm{F}_i(\mathrm{p}) \\ &\approx \textstyle\sum_{i=1}^{\nu} m_i\,\mathrm{G}(\mathrm{c}) \\ &= (\textstyle\sum_{i=1}^{\nu} m_i)\mathrm{G}(\mathrm{c}) \\ &= m_o\,\mathrm{G}(\mathrm{c})\,, \quad m_o := \textstyle\sum_{i=1}^{\nu} m_i \end{aligned}$$

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[33] See HAD, section 2.3.1.

and

$$\begin{aligned}\mathrm{T}^{\mathrm{F}}(\mathrm{p}) &= \textstyle\sum_{i=1}^{\nu}(\mathrm{p}_i - \mathrm{c}) \wedge \mathrm{F}_i(\mathrm{p}) \\ &\approx \textstyle\sum_{i=1}^{\nu}(\mathrm{p}_i - \mathrm{c}) \wedge m_i\, \mathrm{G}(\mathrm{c}) \\ &= \textstyle\sum_{i=1}^{\nu} m_i\,(\mathrm{p}_i - \mathrm{c}) \wedge \mathrm{G}(\mathrm{c}) \\ &= (\textstyle\sum_{i=1}^{\nu} m_i\,(\mathrm{p}_i - \mathrm{c})) \wedge \mathrm{G}(\mathrm{c}) \\ &= 0\end{aligned}$$

As a consequence, [34] the DPMs of $\mathcal{R}$ are the admissible motions characterized by the motions $\mathrm{c} = \mathrm{c}(t)$ of the centre of mass and the motions $A = A(t)$ around the centre of mass satisfying Newton equation

$$\ddot{\mathrm{c}} = \mathrm{G} \circ \mathrm{c}$$

and Euler geodesic equation

$$\frac{d}{dt}\,(\mathbf{I}_A \omega) = 0$$

respectively.

Keplerian motions and regular precessions

In the Copernican reference space $\mathcal{E}_{\mathcal{C}}$, we can apply the above results to any single planet (e.g. the Earth), meant as a rigid body freely falling in the gravitational field of the Sun.

Thus, if we start from the fact that the actual motion in $\mathcal{E}_{\mathcal{C}}$ of the centre of mass of a planet exhibits – at any initial time – non-radial initial conditions satisfying the 'low energy' condition, we deduce that the above motion is a maximal solution of Newton equation satisfying Kepler's laws. [35]

Moreover, if we assume that a planet has a gyroscopic structure around an axis passing through the centre of mass (e.g. the terrestrial axis in the case of the Earth), we deduce that its motion around the centre of mass is a regular precession with a proper uniform rotation around the gyroscopic axis or, if it exhibits – at any initial time – an initial angular velocity with the direction of the gyroscopic axis, just a uniform rotation around that axis. [36]

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[34] See HAD, sections 2.3.4 and 2.3.5.

[35] See section 1.2.2, *Kepler's laws.*

[36] See HAD, section 2.3.5, *Euler geodesic equation.*

Chapter 2

Terrestrial dynamics

Terrestrial dynamics is basically concerned with the possible motions, with respect to the Earth, of a test particle (or a 'small' rigid body) subject, at least, to *terrestrial gravity*, resulting from the gravitational attraction of the Earth and the 'gravitational supplies' that are due to the motion of rotation of the Earth with respect to a Galileian reference space.

2.1 One-body problem

If the couple (Earth, test particle) is regarded as an 'isolated' system, then [1] there exists a Galileian reference space $\mathcal{E}_{\mathcal{G}}$ (modelled on $E_{\mathcal{G}}$) where the centre of Earth – meant as a source of gravitation carrying the whole *terrestrial mass* $m_{_T}$ – is at rest (say at $\mathrm{o} \in \mathcal{E}_{\mathcal{G}}$) and the free falls of the test particle of mass m are characterized as the DPMs of

$$\mathcal{C}_1 := \Big(\mathcal{E}_{\mathcal{G}} - \{\mathrm{o}\}, m, \mathrm{F}\Big)$$

where

$$\mathrm{F} := m\,\mathrm{G}$$

and

$$\mathrm{G} : \mathcal{E}_{\mathcal{G}} - \{\mathrm{o}\} \to E_{\mathcal{G}} : \mathrm{p} \mapsto \mathrm{G}(\mathrm{p}) := -h\,\frac{m_{_T}}{|\mathrm{p}-\mathrm{o}|^2}\frac{\mathrm{p}-\mathrm{o}}{|\mathrm{p}-\mathrm{o}|}$$

that is, as the smooth motions $\mathrm{p}(t)$ in $\mathcal{E}_{\mathcal{G}}$ satisfying the admissibility condition

$$\mathrm{p}(t) \in \mathcal{E}_{\mathcal{G}} - \{\mathrm{o}\}$$

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[1] See 'Test particle in a gravitational field' in section 1.2.1.

and the 'universal' equation

$$\begin{aligned}\ddot{\mathrm{p}}(t) &= \mathrm{G}(\mathrm{p}(t)) \\ &= -h\,\frac{m_{T}}{|\mathrm{p}(t)-\mathrm{o}|^{2}}\,\frac{\mathrm{p}(t)-\mathrm{o}}{|\mathrm{p}(t)-\mathrm{o}|}\end{aligned}$$

Such a one-body problem (as well as other related problems) will now be discussed in the *terrestrial reference space*, where the whole Earth is at rest.

2.1.1 Terrestrial gravity

In the terrestrial reference space, the focal point is that the gravitational field generated by the terrestrial mass is altered by some additional dynamical effects, that are due to the motion of rotation of the Earth with respect to the above Galileian reference space.

Rotation of the Earth

Let $\tilde{\mathcal{E}}_{T}$ (modelled on $\tilde{E}_{T}$ and equipped with the 'anti-clockwise' orientation) be the terrestrial reference space, where the whole Earth is at rest.

With respect to $\mathcal{E}_{\mathcal{G}}$, the motion of $\tilde{\mathcal{E}}_{T}$ can (approximately) be described as a uniform circular motion of rotation round the terrestrial axis (passing through the centre of the Earth $\tilde{\mathrm{o}} \in \tilde{\mathcal{E}}_{T}$, stationary at $\mathrm{o} \in \mathcal{E}_{\mathcal{G}}$, and oriented towards the North), say

$$\alpha_t : \tilde{\mathcal{E}}_{T} \to \mathcal{E}_{\mathcal{G}} : \tilde{\mathrm{p}} \mapsto \alpha_t(\tilde{\mathrm{p}}) := \mathrm{o} + A_t(\tilde{\mathrm{p}} - \tilde{\mathrm{o}})$$

where the derivative of A_t corresponds to the constant *terrestrial angular velocity* $\mathrm{w}(t) = \mathrm{w}_{T}$ (whose oriented direction is that of the terrestrial axis). [2]

A smooth motion $\tilde{\mathrm{p}}(t) \in \tilde{\mathcal{E}}_{T}$ will be said to be a *free fall* of the test particle in $\tilde{\mathcal{E}}_{T}$, iff

$$\mathrm{p}(t) := \alpha_t(\tilde{\mathrm{p}}(t)) = \mathrm{o} + A_t(\tilde{\mathrm{p}}(t) - \tilde{\mathrm{o}}) \in \mathcal{E}_{\mathcal{G}}$$

is a free fall in $\mathcal{E}_{\mathcal{G}}$.

Now recall the composition law of accelerations [3]

$$\ddot{\tilde{\mathrm{p}}}(t) = A_t^{-1}(\ddot{\mathrm{p}}(t)) + \tilde{w}_{T}^{2}(\tilde{\mathrm{p}}(t) - \tilde{\mathrm{p}}_{T}(t)) - 2\,\tilde{\mathrm{w}}_{T} \wedge \dot{\tilde{\mathrm{p}}}(t)$$

---

[2] See Appendix, 3.1.

[3] See Appendix, 3.2. For any $\tilde{\mathrm{p}} \in \tilde{\mathcal{E}}_{T}$, $\tilde{\mathrm{p}}_{T}$ denotes its orthogonal projection onto the terrestrial axis. Moreover we put $\tilde{w}_{T} := |\tilde{\mathrm{w}}_{T}|$.

and the transformation

$$A_t^{-1}\left(-h\,\frac{m_T}{|\mathrm{p}(t)-\mathrm{o}|^2}\frac{\mathrm{p}(t)-\mathrm{o}}{|\mathrm{p}(t)-\mathrm{o}|}\right)=-h\,\frac{m_T}{|\tilde{\mathrm{p}}(t)-\tilde{\mathrm{o}}|^2}\frac{\tilde{\mathrm{p}}(t)-\tilde{\mathrm{o}}}{|\tilde{\mathrm{p}}(t)-\tilde{\mathrm{o}}|}$$

From the law characterizing the free falls in $\mathcal{E}_{\mathcal{G}}$ and the above results, it follows that $\tilde{\mathrm{p}}(t)\in\tilde{\mathcal{E}}_T$ is a free fall of the test particle in $\tilde{\mathcal{E}}_T$, iff it satisfies the admissibility condition

$$\tilde{\mathrm{p}}(t)\in\tilde{\mathcal{E}}_T-\{\tilde{\mathrm{o}}\}$$

and the 'universal' equation

$$\ddot{\tilde{\mathrm{p}}}(t)=-h\,\frac{m_T}{|\tilde{\mathrm{p}}(t)-\tilde{\mathrm{o}}|^2}\frac{\tilde{\mathrm{p}}(t)-\tilde{\mathrm{o}}}{|\tilde{\mathrm{p}}(t)-\tilde{\mathrm{o}}|}+\tilde{w}_T^2(\tilde{\mathrm{p}}(t)-\tilde{\mathrm{p}}_T(t))-2\,\tilde{\mathrm{w}}_T\wedge\dot{\tilde{\mathrm{p}}}(t)$$

Terrestrial field of gravity

The terrestrial reference space will now be denoted by $\mathcal{E}_T$ (modelled on E_T and equipped with the 'anti-clockwise' orientation). [4]

Owing to the above results on the dynamical effects of the terrestrial rotation, the free falls in $\mathcal{E}_T$ of a test particle of mass m are the DPMs of mechanical system $\mathcal{T}_1:=\left(\mathcal{E}_T-\{\mathrm{o}\},m,\mathrm{F}_T\right)$, where $\mathrm{o}\in\mathcal{E}_T$ denotes the centre of the Earth and the *force of gravity* F_T is defined by

$$\mathrm{F}_T:=m\,\mathrm{G}_T$$

where [5]

$$\mathrm{G}_T:(\mathcal{E}_T-\{\mathrm{o}\})\times E_T\to E_T:(\mathrm{p},\mathrm{v})\mapsto\mathrm{G}_T(\mathrm{p},\mathrm{v}):=\mathrm{g}_T(\mathrm{p})-2\,\mathrm{w}_T\wedge\mathrm{v}$$

$$\mathrm{g}_T:\mathcal{E}_T-\{\mathrm{o}\}\to E_T:\mathrm{p}\mapsto\mathrm{g}_T(\mathrm{p}):=\mathrm{G}(\mathrm{p})+w_T^2(\mathrm{p}-\mathrm{p}_T)$$

$$\mathrm{G}:\mathcal{E}_T-\{\mathcal{O}\}\to E_T:\mathrm{p}\mapsto\mathrm{G}(\mathrm{p}):=-h\,\frac{m_T}{|\mathrm{p}-\mathrm{o}|^2}\frac{\mathrm{p}-\mathrm{o}}{|\mathrm{p}-\mathrm{o}|}$$

In a more 'realistic' model, the configuration space should be assumed to be $\mathcal{E}_T-\mathcal{O}$, where $\mathcal{O}$ (open ball centred at o) denotes the stationary position occupied by the whole 'spherical' Earth in $\mathcal{E}_T$.

---

[4] The 'tilde' will then disappear from all of the symbols adopted in the previous subsection.

[5] For any $\mathrm{p}\in\mathcal{E}_T$, p_T will denote its orthogonal projection onto the terrestrial axis. Moreover w_T will denote the constant terrestrial angular velocity (whose oriented direction is that of the terrestrial axis pointing to North) and $w_T:=|\mathrm{w}_T|$.

So

$$\mathcal{T}_1 := \Big(\mathcal{E}_T - \mathcal{O}, m, \mathrm{F}_T\Big)$$

corresponds to the 'one-body problem' of a test particle of mass m *freely falling* in the *terrestrial field of gravity* G_T, [6] resulting in $\mathcal{E}_T$ from the Newtonian gravitational field $\mathrm{G}(\mathrm{p})$ of the Earth (whose mass m_T is thought of as concentrated at the centre o) plus the *centrifugal gravitational supply* $w_T^2(\mathrm{p}-\mathrm{p}_T)$ and the *Coriolis' gravitational supply* $-2\,\mathrm{w}_T \wedge \mathrm{v}$, that are due the rotation of the Earth with respect to the Galileian reference space where the centre of the Earth is at rest.

The DPMs $\mathrm{p}(t) \in \mathcal{E}_T - \mathcal{O}$ of $\mathcal{T}_1$ are then characterized by the 'universal' equation

$$\ddot{\mathrm{p}}(t) = \mathrm{G}_T(\mathrm{p}(t), \dot{\mathrm{p}}(t)) \tag{1}$$

owing to which, the free falls of a test particle of mass m do *not* differ from those of a unit mass $m = 1$, i.e. the free falls in the terrestrial field of gravity are the same for all the test particles (*universality of free falls*).

Remark that the dynamical effects of the terrestrial rotation have been called 'gravitational supplies' and then thought of as being 'equivalent' to gravitational forces (*equivalence principle*), since they – added to the Newtonian gravitational field – preserve the universality of the free falls.

In a 'small' open region $Q_o \subset \mathcal{E}_T - \mathcal{O}$ 'near' the surface of Earth, the centrifugal gravitational supply can be neglected (owing to the 'smallness' of w_T) with respect to the 'strong' Newtonian gravitational field of the Earth, whose variations all over Q_o are in turn negligible. [7]

So the positional part g_T of the terrestrial field of gravity, can locally be approximated by

$$\mathrm{g}_T(\mathrm{p}) \approx \mathrm{g} = \text{const.} \neq 0$$

---

[6] Here G_T is thought of as restricted to $(\mathcal{E}_T - \{\mathcal{O}\}) \times E_T$.

[7] As w_T is 'small', say $w_T^2 \ll h\, m_T / r_T^3$ (r_T being the radius of the Earth), the gravitational field $\mathrm{G}(\mathrm{p})$ prevails on the centrifugal field $w_T^2(\mathrm{p}-\mathrm{p}_T)$ at any point $\mathrm{p} \in \mathcal{E}_T$ 'near' the surface of the Earth (where $|\mathrm{p}-\mathrm{o}| \approx r_T$), since

$$w_T^2 |\mathrm{p}-\mathrm{p}_T| \leq w_T^2 |\mathrm{p}-\mathrm{o}| \approx w_T^2 r_T \ll h \frac{m_T}{r_T^2} \approx |\mathrm{G}(\mathrm{p})|$$

Therefore, near the surface of the Earth, we have $\mathrm{g}_T(\mathrm{p}) \approx \mathrm{G}(\mathrm{p}) \approx -(h\, m_T / r_T^3)\,(\mathrm{p}-\mathrm{o}) \neq 0$, i.e. $\mathrm{g}_T \approx -(h\, m_T / r_T^3)\,\psi \neq 0$ (where $\psi : \mathrm{p} \in \mathcal{E}_T \mapsto \psi(\mathrm{p}) := \mathrm{p}-\mathrm{o} \in E_T$ is an affine mapping, with linear part id_{E_T}), and hence – for any 'small' displacement $\delta\mathrm{p} \in E_T$, say $|\delta\mathrm{p}| \ll r_T$ – the increment of g_T turns out to be negligible

$$d_{\mathrm{p}}\mathrm{g}_T(\delta\mathrm{p}) \approx -h \frac{m_T}{r_T^3}\, d_{\mathrm{p}}\psi(\delta\mathrm{p}) = -h \frac{m_T}{r_T^3}\, \delta\mathrm{p} \approx 0$$

for all $\mathrm{p} \in Q_o$.

As a consequence, in the above order of approximation, we shall put

$$\mathrm{G}_{T}(\mathrm{p},\mathrm{v}) = \mathrm{g} - 2\,\mathrm{w}_{T} \wedge \mathrm{v}$$

and then, for $\mathrm{p}(t) \in Q_o$, equation (1) will take the form

$$\ddot{\mathrm{p}}(t) = \mathrm{g} - 2\,\mathrm{w}_{T} \wedge \dot{\mathrm{p}}(t)$$

Moreover, since Coriolis' gravitational supply will be shown to be negligible along free falls of 'short' duration, [8] we can put

$$\mathrm{G}_{T}(\mathrm{p},\mathrm{v}) = \mathrm{g}$$

and then, for $\mathrm{p}(t) \in Q_o$, equation (1) will take the simpler form

$$\ddot{\mathrm{p}}(t) = \mathrm{g} \tag{1$'$}$$

which, for all practical purposes, is the usual 'local formulation' of Newton's law of free falls in the terrestrial field of gravity (owing to such a formulation, g is called *acceleration of gravity*).

Condition $(1)'$ characterizes the DPMs $\mathrm{p}(t) \in Q_o$ of $(Q_o, 1, \mathrm{g})$, which is a conservative mechanical system, since

$$\mathsf{V} : Q_o \to \mathbb{R} : \mathrm{p} \mapsto \mathsf{V}(\mathrm{p}) := -\mathrm{g} \cdot (\mathrm{p} - \mathrm{c})$$

(for any choice of $\mathrm{c} \in \mathcal{E}_T$) is the potential energy of g. [9]

So, the terrestrial field of gravity can locally be described as a *potential field* $-\mathsf{V}$, whose effect is that of transforming the 'flat' Euclidean geometry K of $\mathcal{E}_{T}$ with 'rectilinear' geodesics (the 'inertial motions' in Newton's conception) into a 'non-flat' Lagrangian geometry $\mathsf{L} := \mathsf{K} - \mathsf{V}$ with (generally) 'curved' geodesics (new 'inertial motions', coinciding with the free falls). [10]

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[8] See *Falls in the vacuum* (ii) in section 2.2.1.

[9] Remark that $\mathsf{V} = -\mathrm{g} \cdot \psi_{|Q_o}$ (where $\psi : \mathrm{p} \in \mathcal{E}_{T} \mapsto \psi(\mathrm{p}) := \mathrm{p} - \mathrm{c} \in E_{T}$ is an affine mapping, with linear part $\mathrm{id}_{E_{T}}$). Owing to Leibniz rule, we have

$$-d_{\mathrm{p}}\mathsf{V} = \mathrm{g} \cdot d_{\mathrm{p}}\psi = \mathrm{g} \cdot \mathrm{id}_{E_{T}} = \mathrm{g} \cdot$$

[10] See *Geometrical v. dynamical gravitation* in section 1.2.1.

Newton's principle in terrestrial dynamics

It is a matter of daily experience that additional forces may act or may be made to act on a test particle in $\mathcal{E}_{T}$, with the result of turning motions different from the free falls into dynamically possible motions.

Such a case corresponds to a mechanical system

$$\mathcal{T}_1^{(a)} := \Big(\mathcal{E}_{T} - \mathcal{O}, m, \mathrm{F}\Big)$$

where

$$\mathrm{F} := m\,\mathrm{G}_{T} + \mathrm{F}^{(a)}$$

is the sum of the force of gravity $\mathrm{F}_T = m\,\mathrm{G}_{T}$ and some 'additional' force $\mathrm{F}^{(a)}$, i.e.

$$\mathrm{F} : \mathbb{R} \times (\mathcal{E}_{T} - \{\mathrm{o}\}) \times E_{T} \to E_{T} : (t, \mathrm{p}, \mathrm{v}) \mapsto \mathrm{F}(t, \mathrm{p}, \mathrm{v}) = m\,\mathrm{G}_{T}(\mathrm{p}, \mathrm{v}) + \mathrm{F}^{(a)}(t, \mathrm{p}, \mathrm{v})$$

The DPMs of $\mathcal{T}_1^{(a)}$ are then the smooth motions $\mathrm{p}(t) \in \mathcal{E}_{T} - \mathcal{O}$ satisfying Newton's principle

$$m\,\ddot{\mathrm{p}}(t) = m\,\mathrm{G}_{T}(\mathrm{p}(t), \dot{\mathrm{p}}(t)) + \mathrm{F}^{(a)}(t, \mathrm{p}(t), \dot{\mathrm{p}}(t)) \tag{2}$$

A point $\mathrm{p}_* \in \mathcal{E}_{T} - \mathcal{O}$ is an *equilibrium configuration* of $\mathcal{T}_1^{(a)}$, iff equation (2) admits the *static* solution $\gamma_* : t \in \mathbb{R} \mapsto \mathrm{p}_*(t) = \mathrm{p}_* \in \mathcal{E}_{T} - \{\mathrm{o}\}$, i.e.

$$m\,\ddot{\mathrm{p}}_*(t) = m\,\mathrm{g}_{T}(\mathrm{p}_*(t)) - 2\,m\,\mathrm{w}_{T} \wedge \dot{\mathrm{p}}_*(t) + \mathrm{F}^{(a)}(t, \mathrm{p}_*(t), \dot{\mathrm{p}}_*(t))$$

that is to say,

$$\mathrm{F}^{(a)}(t, \mathrm{p}_*, 0) = -m\,\mathrm{g}_{T}(\mathrm{p}_*)$$

As a consequence, when $\mathrm{F}^{(a)} = 0$, there is no equilibrium configuration near the surface of the Earth (where $\mathrm{g}_{T} \neq 0$). In order to keep a test particle of mass m stationary at a given point, one has to apply an additional force $\mathrm{F}^{(a)}$ able to 'counterbalance', at that point, the *weight* force field acting on m, i.e.

$$\mathrm{w}_m = m\,\mathrm{g}_{T}$$

For $\mathrm{p}(t) \in Q_o$ (small open region near the surface of the Earth), equation (2) takes the simpler form

$$m\,\ddot{\mathrm{p}}(t) = m\,\mathrm{g} + \mathrm{F}^{(a)}(t, \mathrm{p}(t), \dot{\mathrm{p}}(t)) \tag*{(2)$'$}$$

where weight takes the constant value

$$\mathrm{w}_m = m\,\mathrm{g} \neq 0$$

The ‘universal’ direction $\mathrm{Span(g)}$ of the weight is called *vertical.* [11]

Check that the universality of the vertical direction – confirmed by *Eötvös*’ celebrated experiments on plumb lines – is one more consequence of the identity of inertial and gravitational mass. [12]

d’Alembert’s principle in terrestrial dynamics

The case of a test particle subject to any kind of holonomic constraint, corresponds to a mechanical system

$$\mathcal{T} := \Big(Q, m, \mathrm{F}\Big)$$

where

$$Q \subset \mathcal{E}_{T}$$

is the configuration space allowed by the constraints and

$$\mathrm{F} := m\,\mathrm{G}_{T} + \mathrm{F}^{(a)}$$

According to the general theoretical frame of classical dynamics, [13] the DPMs of $\mathcal{T}$ are the admissible motions $\mathrm{p}(t) \in Q$ satisfying Newton’s principle

$$m\,\ddot{\mathrm{p}}(t) = m\,\mathrm{G}_{T}(\mathrm{p}(t), \dot{\mathrm{p}}(t)) + \mathrm{F}^{(a)}(t, \mathrm{p}(t), \dot{\mathrm{p}}(t)) + \Phi(t)$$

with a right hand side encompassing the (unknown) constraint reaction, ‘orthogonal’ to Q [14]

$$\Phi(t) \in T^{\perp}_{\mathrm{p}(t)}Q$$

---

[11] The two orientations of $\mathrm{Span(g)}$ determined by g and $-\mathrm{g}$ are called *descending* and *ascending*, respectively. A straight line $\mathcal{S} = \mathrm{p} + S \subset \mathcal{E}_{T}$ is said to have a vertical or horizontal direction S, according to whether $S = \mathrm{Span\,(g)}$ or $S \subset \mathrm{Span}^{\perp}\mathrm{(g)}$, respectively. A plane $\mathcal{A} = \mathrm{p} + A \subset \mathcal{E}_{T}$ is then said to be horizontal or vertical, according to whether the orthogonal direction $S := A^{\perp}$ is vertical or horizontal, respectively.

[12] Notice that the force of gravity $\mathrm{F}_T(\mathrm{p}, \mathrm{v}) = m\,\mathrm{G}_{T}(\mathrm{p}, \mathrm{v}) = m\,\mathrm{g}_T(\mathrm{p}) - 2m\,\mathrm{w}_T \wedge \mathrm{v}$, acting on a test particle of inertial mass m and gravitational mass M, should be defined by putting $\mathrm{g}_T(\mathrm{p}) := \frac{M}{m}\,\mathrm{G}(\mathrm{p}) + w_T^2(\mathrm{p} - \mathrm{p}_T)$. As a consequence, the direction $\mathrm{Span}(\mathrm{g}_T(\mathrm{p}))$ of the weight force $m\,\mathrm{g}_T(\mathrm{p})$ at a point, turns out to be universal, iff so is the ratio $\frac{M}{m}$, e.g. (once chosen the same body as a unit of both inertial and gravitational mass) $\frac{M}{m} = 1$ for all the particles.

[13] See HAD, section 2.2.2.

[14] Possible constraint frictions, which admit a ‘tangent’ force field description, are meant to be embodied in $\mathrm{F}^{(a)}$.
Recall that $T^{\perp}_{\mathrm{p}}Q$ denotes the orthogonal complement of $T_{\mathrm{p}}Q$ in E_{T}.

In other words, the DPMs of $\mathcal{T}$ are the admissible motions

$$\mathrm{p}(t) \in Q$$

satisfying d'Alembert's principle

$$m\,\mathrm{G}_{\mathcal{T}}(\mathrm{p}(t), \dot{\mathrm{p}}(t)) + \mathrm{F}^{(a)}(t, \mathrm{p}(t), \dot{\mathrm{p}}(t)) - m\,\ddot{\mathrm{p}}(t) \in T^{\perp}_{\mathrm{p}(t)}Q \tag{3}$$

or, in the usual local formulation,

$$m\,\mathrm{g} + \mathrm{F}^{(a)}(t, \mathrm{p}(t), \dot{\mathrm{p}}(t)) - m\,\ddot{\mathrm{p}}(t) \in T^{\perp}_{\mathrm{p}(t)}Q \tag{3$'$}$$

Obvious is the extension of the above theory to a system consisting of $\nu > 1$ test particles. [15]

2.1.2 Galilei's principle of relativity

We shall now study the effects of a reference transformation [16]

$$\alpha_t : \tilde{\mathcal{E}}_3 \to \mathcal{E}_{\mathcal{T}} : \tilde{\mathrm{p}} \mapsto \alpha_t(\tilde{\mathrm{p}}) := \mathrm{o}_t + A_t(\tilde{\mathrm{p}} - \tilde{\mathrm{o}})$$

on 'local' terrestrial dynamics.

Free falls and Galileian transformations

Let us start with the free falls.

A smooth motion $\tilde{\mathrm{p}}(t) \in \tilde{\mathcal{E}}_3$ will be said to be a 'free fall' of a test particle in $\tilde{\mathcal{E}}_3$, if

$$\mathrm{p}(t) := \alpha_t(\tilde{\mathrm{p}}(t)) = \mathrm{o}_t + A_t(\tilde{\mathrm{p}}(t) - \tilde{\mathrm{o}})$$

is a free fall in $\mathcal{E}_{\mathcal{T}}$, i.e. (locally)

$$\ddot{\mathrm{p}}(t) = \mathrm{g} \tag{1$'$}$$

with

$$\mathrm{g} = \text{const.} \neq 0$$

---

[15] The DPMs of such a system are those of $\mathcal{T} := \Big(Q, m, \mathrm{F}\Big)$, where $Q \subset \mathcal{E}^{\nu}_{\mathcal{T}}$, $m = (m_1, \ldots, m_\nu) \in (\mathbb{R})^\nu$ and – for any $t \in \mathbb{R}$, $\mathrm{p} = (\mathrm{p}_1, \ldots, \mathrm{p}_\nu) \in Q$ and $\mathrm{v} = (\mathrm{v}_1, \ldots, \mathrm{v}_\nu) \in T_{\mathrm{p}}Q$ – $\mathrm{F}(t, \mathrm{p}, \mathrm{v}) := m\,\mathrm{G}_{\mathcal{T}}(\mathrm{p}, \mathrm{v}) + \mathrm{F}^{(a)}(t, \mathrm{p}, \mathrm{v})$ with $m\,\mathrm{G}_{\mathcal{T}}(\mathrm{p}, \mathrm{v}) := (m_1\,\mathrm{G}_{\mathcal{T}}(\mathrm{p}_1, \mathrm{v}_1), \ldots, m_\nu\,\mathrm{G}_{\mathcal{T}}(\mathrm{p}_\nu, \mathrm{v}_\nu))$, substituded, in the local formulation, by $m\mathrm{g} := (m_1\mathrm{g}, \ldots, m_\nu\mathrm{g})$.

[16] See Appendix 3.1.

Owing to the composition law of accelerations [17]

$$\ddot{\tilde{\mathrm{p}}}(t) = A_t^{-1}(\ddot{\mathrm{p}}(t)) - \tilde{\mathrm{a}}(t, \tilde{\mathrm{p}}(t)) - 2\,\tilde{\omega}(t) \wedge \dot{\tilde{\mathrm{p}}}(t)$$

$\tilde{\mathrm{p}}(t)$ is a free fall in $\tilde{\mathcal{E}}_3$, iff

$$\ddot{\tilde{\mathrm{p}}}(t) = \tilde{\mathrm{g}}(t) - \tilde{\mathrm{a}}(t, \tilde{\mathrm{p}}(t)) - 2\,\tilde{\omega}(t) \wedge \dot{\tilde{\mathrm{p}}}(t)$$

with

$$\tilde{\mathrm{g}}(t) := A_t^{-1}(\mathrm{g}) \neq 0$$

which shows that, in $\tilde{\mathcal{E}}_3$, the field of gravity is locally described by the vector field

$$\tilde{\mathrm{G}}(t, \tilde{\mathrm{p}}, \tilde{\mathrm{v}}) := \tilde{\mathrm{g}}(t) - \tilde{\mathrm{a}}(t, \tilde{\mathrm{p}}) - 2\,\tilde{\omega}(t) \wedge \tilde{\mathrm{v}}$$

resulting from the terrestrial gravity field $\tilde{\mathrm{g}}(t)$ plus the *gravity supplies* $-\tilde{\mathrm{a}}(t, \tilde{\mathrm{p}})$ and $-2\,\tilde{\omega}(t) \wedge \tilde{\mathrm{u}}$, which are due to the motion of $\tilde{\mathcal{E}}_3$ with respect to the Earth.

Notice now that the free falls in $\tilde{\mathcal{E}}_3$ are locally characterized by equation

$$\ddot{\tilde{\mathrm{p}}}(t) = \tilde{\mathrm{g}}(t)$$

iff $\tilde{\mathcal{E}}_3$ is related to $\mathcal{E}_T$ by a Galileian tranformation, and, in such a case,

$$\tilde{\mathrm{g}}(t) = \tilde{\mathrm{g}} = \text{const.} \neq 0$$

(indeed, the equation characterizing the free falls in $\tilde{\mathcal{E}}_3$ takes the above simple form, iff the vector field $\tilde{\mathrm{a}}(t, \tilde{\mathrm{p}}) + 2\,\tilde{\omega}(t) \wedge \tilde{\mathrm{v}}$ vanishes identically, that is, iff α_t is a Galileian transformation, i.e. $\dot{\mathrm{o}}_t = \text{const.}$ and $A_t = \text{const.}$; moreover, from $A_t = \text{const.}$ –i.e. $\omega(t) = 0$– it follows that [18] $\tilde{\mathrm{g}}(t) = \text{const.} \neq 0$).

So the free falls in a reference space $\tilde{\mathcal{E}}_3$ related to $\mathcal{E}_T$ by a Galileian transformation, are locally characterized by an equation which does not differs from equation $(1)'$ locally characterizing the free falls in $\mathcal{E}_T$.

---

[17] See Appendix 3.2.

[18] From $\mathrm{g} = A_t(\tilde{\mathrm{g}}(t))$, we obtain (by derivation)

$$0 = A_t(\dot{\tilde{\mathrm{g}}}(t)) + \dot{A}_t(\tilde{\mathrm{g}}(t)) = A_t(\dot{\tilde{\mathrm{g}}}(t)) + (\dot{A}_t \circ A_t^{-1})(\mathrm{g}) = A_t(\dot{\tilde{\mathrm{g}}}(t)) + \omega(t) \wedge \mathrm{g}$$

whence

$$\dot{\tilde{\mathrm{g}}}(t) = 0 \iff \omega(t) \wedge \mathrm{g} = 0$$

As a consequence, we have $\tilde{\mathrm{g}}(t) = \tilde{\mathrm{g}} = \text{const.}$, iff, for all $t \in \mathbb{R}$, $\omega(t)$ is null or vertical.

d'Alembert's principle and Galileian transformations

Such a result can be extended to a more general situation, as follows.

Let $\tilde{\mathcal{E}}_3$ still be a reference space related to $\mathcal{E}_{\scriptscriptstyle T}$ by a Galileian transformation.

Owing to the above result, in $\tilde{\mathcal{E}}_3$ the local field of gravity only consists of the terrestrial field, i.e. $\tilde{\mathrm{g}} = \text{const.} \neq 0$.

Then consider a test particle (of a given mass m) subject in $\tilde{\mathcal{E}}_3$ to the same constraints and the same additional forces as those imposed to a test particle (of equal mass) in $\mathcal{E}_{\scriptscriptstyle T}$ (whose DPMs are those of $\mathcal{T}$ [19]), that is to say,

$$\tilde{\mathcal{T}} := \left(\tilde{Q}, m, \tilde{\mathrm{F}}\right)$$

where $\tilde{Q} \subset \tilde{\mathcal{E}}_3$ is characterized by the same equalities/inequalities as those characterizing $Q \subset \mathcal{E}_{\scriptscriptstyle T}$, and $\tilde{\mathrm{F}} := m\tilde{\mathrm{g}} + \tilde{\mathrm{F}}^{(a)}$, $\tilde{\mathrm{F}}^{(a)}$ being the same vector-valued mapping as that describing $\mathrm{F}^{(a)}$.

The DPMs of $\tilde{\mathcal{T}}$ are characterized by conditions

$$\tilde{\mathrm{p}}(t) \in \tilde{Q}\ , \quad m\,\tilde{\mathrm{g}} + \tilde{\mathrm{F}}^{(a)}(t, \tilde{\mathrm{p}}(t), \dot{\tilde{\mathrm{p}}}(t)) - m\,\ddot{\tilde{\mathrm{p}}}(t) \in T^{\perp}_{\tilde{\mathrm{p}}(t)}\tilde{Q}$$

which, under the above hypotheses, do not differ from those characterizing the DPMs of $\mathcal{T}$.

Hence: "No Galileian transformation from the terrestrial reference space to a new reference space, will ever be detected by therein observing the local motions of test particles" (*Galilei's principle of relativity* in terrestrial dynamics).

2.2 Miscellaneous problems

Now we shall discuss quite a number of dynamical problems, concerning test particles subject to the combined action of different kinds of constraints and forces in the terrestrial (or a moving) reference space.

2.2.1 Unconstrained body

We start with the study of the dynamically possible 'falls', with respect to the terrestrial reference space $\mathcal{E}_{\scriptscriptstyle T}$, of an unconstrained test particle both in the vacuum and in the air. The effects of some non-Galileian reference transformation will also be analysed.

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[19] See section 2.1.1, *d'Alembert's principle in terrestrial dynamics.*

In any case, we shall only think of ‘local’ falls occurring in ‘small’ regions of $\mathcal{E}_T$ near the surface of the Earth, where the positional part of the terrestrial field of gravity can be regarded as a non-null constant vector field g (however, from a purely mathematical point of view, g will be treated as a constant vector field defined all over $\mathcal{E}_T$).

Falls in the vacuum

(i) For a test particle in $\mathcal{E}_T$, the dynamically possible local falls in the vacuum – or local free falls – are the DPMs of the mechanical system $\mathcal{T}_1 = (\mathcal{E}_T, m, m\,\mathrm{G}_T)$, where, neglecting the dynamical effects of the terrestrial rotation, we put $\mathrm{G}_T(\mathrm{p},\mathrm{v}) = \mathrm{g}$.

The DPMs of $\mathcal{T}_1$ are characterized by Newton's law

$$\ddot{\mathrm{p}}(t) = \mathrm{g} \tag{1$'$}$$

to which we add initial conditions

$$\mathrm{p}(t_o) = \mathrm{p}_o\ , \quad \dot{\mathrm{p}}(t_o) = \mathrm{v}_o \tag{$\diamond$}$$

for any $(t_o, \mathrm{p}_o, \mathrm{v}_o) \in \mathbb{R} \times \mathcal{E}_T \times E_T$.

The maximal solution of Cauchy problem $(1)'(\diamond)$ is immediately seen to be given, for all $t \in \mathbb{R}$, by

$$\mathrm{p}(t) := \mathrm{p}_o + \mathrm{v}_o(t - t_o) + \frac{1}{2}\,\mathrm{g}\,(t - t_o)^2$$

For a qualitative analysis of $\mathrm{p}(t)$ and its derivative

$$\dot{\mathrm{p}}(t) = \mathrm{v}_o + \mathrm{g}\,(t - t_o)$$

we shall distinguish two cases.

First consider the case of null or vertical initial velocity, i.e.

$$\mathrm{v}_o \in \mathrm{Span}\,(\mathrm{g})$$

That amounts to saying $\mathrm{v}_o = v_{oz}\mathrm{u}$, where $\mathrm{u} := -\frac{\mathrm{g}}{g}$ (with $g := |\mathrm{g}|$) and then $v_{oz} = \mathrm{v}_o \cdot \mathrm{u}$.

In such a case, the above solution reads

$$\mathrm{p}(t) = \mathrm{p}_o + z(t)\,\mathrm{u}$$

with

$$z(t) := v_{oz}(t - t_o) - \frac{1}{2}\,g\,(t - t_o)^2$$

and its derivative reads

$$\dot{\mathrm{p}}(t) = \dot{z}(t)\mathrm{u}$$

with

$$\dot{z}(t) := v_{oz} - g\,(t - t_o)$$

So $\mathrm{p}(t)$ is a *rectilinear* motion along the *vertical* straight line

$$\mathcal{S} := \mathrm{p}_o + \mathrm{Span}\,(\mathrm{u}) = \mathrm{p}_o + \mathrm{Span}\,(\mathrm{g})$$

whose velocity only vanishes at time

$$t^* := t_o + \frac{v_{oz}}{g}$$

For all $t \neq t^*$, the ascending or descending character of $\mathrm{p}(t)$, i.e. the increasing or decreasing rate of height $z(t)$, can be deduced from the sign of $\dot{z}(t)$, given by

$$\dot{z}(t) \gtrless 0 \Longleftrightarrow t \lessgtr t^*$$

Moreover, for all $t \neq t^*$, the decelerated or accelerated character of $\mathrm{p}(t)$, i.e. the decreasing or increasing rate of scalar velocity

$$\dot{\mathrm{p}}^2(t) = \dot{z}^2(t) = v_{oz}^2 - 2v_{oz}g(t - t_o) + g^2(t - t_o)^2$$

(whose limit for $t \to \mp\infty$ is $+\infty$), can be deduced from the sign of

$$\frac{d\dot{\mathrm{p}}^2}{dt} = \frac{d\dot{z}^2}{dt} = 2g(g(t - t_o) - v_{oz})$$

given by

$$\frac{d\dot{z}^2}{dt} \lessgtr 0 \Longleftrightarrow t \lessgtr t^*$$

So, along a *vertical fall*, a test particle first moves upwards with a decreasing scalar velocity, comes to a stop at a time t^* and then falls downwards with an increasing and unbounded scalar velocity.

Then consider the case of non-null and non-vertical initial velocity, i.e.

$$\mathrm{v}_o \notin \mathrm{Span}\,(\mathrm{g})$$

That amounts to saying $\mathrm{v}_o = v_{ox}\mathrm{e} + v_{oz}\mathrm{u}$ with $v_{ox} \neq 0$, where $\mathrm{e} \in$ (Span $(\mathrm{g}, \mathrm{v}_o)$ with $|\mathrm{e}| = 1$ and $\mathrm{e} \cdot \mathrm{u} = 0$).

In such a case, the above solution reads

$$\mathrm{p}(t) = \mathrm{p}_o + x(t)\mathrm{e} + z(t)\mathrm{u}$$

with

$$x(t) := v_{ox}(t - t_o)\ , \quad z(t) := v_{oz}(t - t_o) - \frac{1}{2}\, g\, (t - t_o)^2$$

and its derivative reads

$$\dot{\mathrm{p}}(t) = \dot{x}(t)\mathrm{e} + \dot{z}(t)\mathrm{u}$$

with

$$\dot{x}(t) = v_{ox}, \quad \dot{z}(t) = v_{oz}\, g\, (t - t_o)$$

So $\mathrm{p}(t)$ is a *parabolic* motion on the *vertical* plane

$$\mathcal{A} := \mathrm{p}_o + \mathrm{Span}\,(\mathrm{e}, \mathrm{u}) = \mathrm{p}_o + \mathrm{Span}\,(\mathrm{v}_o, \mathrm{g})$$

whose orbit is the unbounded, connected, non-degenerate conic section of equation [20] and whose velocity has a vertical component which only vanishes at time

$$t^* := t_o + \frac{v_{oz}}{g}$$

For all $t \neq t^*$, the ascending or descending character of $\mathrm{p}(t)$, i.e. the increasing or decreasing rate of height $z(t)$, can be deduced from the sign of $\dot{z}(t)$, given by

$$\dot{z}(t) \gtrless 0 \Longleftrightarrow t \lessgtr t^*$$

Moreover, for all $t \neq t^*$, the decelerated or accelerated character of $\mathrm{p}(t)$, i.e. the decreasing or increasing rate of scalar velocity

$$\dot{\mathrm{p}}^2(t) = \dot{x}^2(t) + \dot{z}^2(t) = v_{ox}^2 + v_{oz}^2 - 2v_{oz}g(t - t_o) + g^2(t - t_o)^2$$

(whose limit for $t \to \mp\infty$ is $+\infty$), can be deduced from the sign of

$$\frac{d\dot{\mathrm{p}}^2}{dt} = \frac{d\dot{z}^2}{dt} = 2g(g(t - t_o) - v_{oz})$$

---

[20] The algebraic equation of the orbit can easily be obtained from its parametric equations

$$x = v_{ox}(t - t_o)\ , \quad z = v_{oz}(t - t_o) - \frac{1}{2}g(t - t_o)^2$$

by substituting the value $t - t_o = \frac{x}{v_{ox}}$, deduced from the first equation, into the second one. Check that the parabola meets the horizontal axis $\mathrm{p}_o + \mathrm{Span}\,(\mathrm{e})$ at the points $\mathrm{p}_1 := \mathrm{p}_o$ and $\mathrm{p}_2 := \mathrm{p}_o + \frac{2v_{ox}v_{oz}}{g}\mathrm{e}$.
The *range* of the parabola – i.e. the distance r between p_1 and p_2 – is expressed, in function on the norm $v_o := |\mathrm{v}_o| > 0$ and the *slope* $\alpha := \angle(\mathrm{v}_o, \mathrm{e}) \in [0, \frac{\pi}{2})$ of v_o (owing to which $v_{ox} = v_o \cos\alpha$ and $v_{oz} = v_o \sin\alpha$), by

$$r := \left| \frac{2v_{ox}v_{oz}}{g} \right| = \frac{v_o^2}{g} 2\cos\alpha \sin\alpha = \frac{v_o^2}{g} \sin 2\alpha$$

So, for any given initial scalar velocity v_o, the range r of the parabola takes its maximum value at a slope of $\alpha = \frac{\pi}{4}$.

given by

$$\frac{d\dot{z}^2}{dt} \lesseqgtr 0 \Longleftrightarrow t \lesseqgtr t^*$$

So, also along a *parabolic fall*, a test particle first moves upwards with a decreasing scalar velocity till a time t^* and then falls downwards with an increasing and unbounded scalar velocity.

(ii) Now we want to appreciate the effect of the terrestrial rotation on the local free falls, i.e. we put $\mathrm{G}_{_T}(\mathrm{p}, \mathrm{v}) = \mathrm{g} - 2\mathrm{w}_{_T} \wedge \mathrm{v}$.

Equation (1)' is then substituted by

$$\ddot{\mathrm{p}}(t) = \mathrm{g} - 2\mathrm{w}_{_T} \wedge \dot{\mathrm{p}}(t) \tag{1$''$}$$

to which we add, for the sake of simplicity, initial conditions of the type

$$\mathrm{p}(t_o) = \mathrm{p}_o\ , \quad \dot{\mathrm{p}}(t_o) = 0 \tag{$\circ$}$$

for any $(t_o, \mathrm{p}_o) \in \mathbb{R} \times \mathcal{E}_{_T}$.

A solution $\mathrm{p}(t)$ of Cauchy problem $(1)''(\circ)$ can approximately be expressed by

$$\begin{aligned}
\mathrm{p}(t) \quad &\approx \quad \mathrm{p}(t_o) + \dot{\mathrm{p}}(t_o)\,(t-t_o) + \frac{1}{2!}\,\ddot{\mathrm{p}}(t_o)\,(t-t_o)^2 + \frac{1}{3!}\,\dddot{\mathrm{p}}(t_o)\,(t-t_o)^3 \\
&= \quad \mathrm{p}_o + \frac{1}{2}\,\mathrm{g}\,(t-t_o)^2 + \frac{1}{3}\,(\mathrm{g} \wedge \mathrm{w}_{_T})\,(t-t_o)^3 \\
&= \quad \mathrm{p}_1(t) + \frac{1}{3}\,(\mathrm{g} \wedge \mathrm{w}_{_T})\,(t-t_o)^3
\end{aligned}$$

where the vertical fall $\mathrm{p}_1(t) := \mathrm{p}_o + \frac{1}{2}\,\mathrm{g}\,(t-t_o)^2$ is a solution of Cauchy problem $(1)'(\circ)$.

So, at the above order of approximation, the expected effect of the terrestrial rotation is the *deviation*

$$\mathrm{p}(t) - \mathrm{p}_1(t) \approx \frac{1}{3}\,(\mathrm{g} \wedge \mathrm{w}_{_T})\,(t-t_o)^3$$

Wherever $\mathrm{g} \wedge \mathrm{w}_{_T} \neq 0$ [21] and for all $t > t_o$, the above deviation is non-null and has the same *horizontal* direction and *eastern* orientation of $\mathrm{g} \wedge \mathrm{w}_{_T}$.

As the length of such a deviation is proportional to $(t-t_o)^3$, it rapidly decreases when $t - t_o \to 0$ (i.e. when one considers a fall of very short duration), whereas it rapidly increases and becomes appreciable when $t - t_o \to +\infty$ (i.e. when one considers a fall of longer duration).

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[21] $\mathrm{g} \wedge \mathrm{w}_{_T} = 0$ only ‘near’ the Poles of the Earth.

Falls in the air

For a test particle in $\mathcal{E}_T$, the dynamically possible local falls in the air are the DPMs of the mechanical system $\mathcal{T}_1^{(a)} = (\mathcal{E}_T, m, m\,\mathrm{G}_T + \mathrm{F}^{(a)})$, where the terrestrial field of gravity is assumed to be expressed by the constant vector field $\mathrm{G}_T = \mathrm{g}$, and the 'wind resistance' is assumed to be expressed (at least within a suitably small range of scalar velocities) by the dissipative force field

$$\mathrm{F}^{(a)} : E_T \to E_T : \mathrm{v} \mapsto \mathrm{F}^{(a)}(\mathrm{v}) := -k|\mathrm{v}|\mathrm{v} \;, \quad k > 0$$

which vanishes at $\mathrm{v} = 0$ and, at any $\mathrm{v} \neq 0$, has the same direction as v, the opposite orientation and a norm proportional to $|\mathrm{v}|^2$. In such a case, Newton's law (2) –characterizing the DPMs of $\mathcal{T}_1^{(a)}$– takes the form (2)', namely

$$\ddot{\mathrm{p}}(t) = \mathrm{g} - \frac{k}{m}\,|\dot{\mathrm{p}}(t)|\dot{\mathrm{p}}(t) \tag{2$'$}$$

to which we again add, for the sake of simplicity, initial conditions of the type

$$\mathrm{p}(t_o) = \mathrm{p}_o \;, \quad \dot{\mathrm{p}}(t_o) = 0 \tag{$\circ$}$$

for any $(t_o, \mathrm{p}_o) \in \mathbb{R} \times \mathcal{E}_T$.

The maximal solution of Cauchy problem $(2)'(\circ)$ is easily seen to be the vertical fall [22]

$$\mathrm{p}(t) := \mathrm{p}_o + z(t)\mathrm{u}$$

where $z(t)$ denotes the maximal solution of Cauchy problem consisting of equation

$$\ddot{z}(t) = -g - \varepsilon\frac{k}{m}\dot{z}^2(t)$$

(with $\varepsilon = 1$ or $\varepsilon = -1$ according to whether $\dot{z}(t) > 0$ or $\dot{z}(t) < 0$) and initial conditions

$$z(t_o) = 0 \;, \quad \dot{z}(t_o) = 0$$

In fact the above $\mathrm{p}(t)$, whose first derivative $\dot{\mathrm{p}}(t) = \dot{z}(t)\mathrm{u}$ has norm $|\dot{\mathrm{p}}(t)| = |\dot{z}(t)| = \varepsilon\,\dot{z}(t)$, satisfies both equation

$$\begin{aligned}
\ddot{\mathrm{p}}(t) &= \ddot{z}(t)\mathrm{u} \\
&= \left(-g - \varepsilon\frac{k}{m}\dot{z}^2(t)\right)\mathrm{u} \\
&= -g\mathrm{u} - \frac{k}{m}(\varepsilon\dot{z}(t))\dot{z}(t)\mathrm{u} \\
&= \mathrm{g} - \frac{k}{m}\,|\dot{\mathrm{p}}(t)|\dot{\mathrm{p}}(t)
\end{aligned}$$

---

[22] Recall that $\mathrm{u} := -\frac{\mathrm{g}}{g}$ with $g := |\mathrm{g}|$.

and initial conditions

$$\mathrm{p}(t_o) = \mathrm{p}_o + z(t_o)\mathrm{u} = \mathrm{p}_o \ , \quad \dot{\mathrm{p}}(t_o) = \dot{z}(t_o)\mathrm{u} = 0$$

Now remark that, from $\ddot{z}(t_o) = -g < 0$, it follows –by continuity– that (at least in an open neighbourhood of t_o)

$$\ddot{z}(t) < 0$$

and then $\dot{z}(t)$ behaves as a decreasing function of time. Hence, owing to $\dot{z}(t_o) = 0$,

$$\dot{z}(t) \gtreqless 0 \Longleftrightarrow t \lesseqgtr t_o$$

and, owing to $\dot{\mathrm{p}}^2(t) = \dot{z}^2(t)$,

$$\frac{d\dot{\mathrm{p}}^2}{dt} = 2\dot{z}(t)\ddot{z}(t) \lesseqgtr 0 \Longleftrightarrow t \lesseqgtr t_o$$

So the vertical fall $\mathrm{p}(t)$ exhibits the same ascending/descending and decelerated/accelerated characters as a vertical fall in the vacuum.

In order to show the effect of the wind resistance on the descending fall, we shall explicitly determine $\dot{z}(t)$ for $t \geq t_o$, i.e. the solution of the problem [23]

$$\frac{d\dot{z}}{dt} = -g\left(1 - \frac{\dot{z}^2}{V^2}\right) < 0 \ , \quad \dot{z}(t_o) = 0$$

with

$$V := \sqrt{\frac{mg}{k}}$$

As $\dot{z}(t)$ is an increasing (and then invertible) function, we can consider the inverse function $t(\dot{z})$, whose derivative is

$$\begin{aligned} \frac{dt}{d\dot{z}} &= \left(\frac{d\dot{z}}{dt}\right)^{-1} \\ &= -\frac{1}{g\left(1 - \frac{\dot{z}^2}{V^2}\right)} \\ &= -\frac{1}{2g}\left(\frac{1}{1 + \frac{\dot{z}}{V}} + \frac{1}{1 - \frac{\dot{z}}{V}}\right) \end{aligned}$$

and satisfies

$$t(0) = t_o$$

---

[23] Recall that, for $t > t_o$, we have $\dot{z}(t) < 0$ and then $\varepsilon = -1$.

By a quadrature, we obtain

$$\begin{aligned} t &= t_o - \frac{1}{2g}\int_0^{\dot z}\left(\frac{1}{1+\frac{\dot z}{V}} + \frac{1}{1-\frac{\dot z}{V}}\right) d\dot z \\ &= t_o - \frac{1}{2g}\int_0^{\dot z}\frac{d}{d\dot z}\left(V\log\left(1+\frac{\dot z}{V}\right) - V\log\left(1-\frac{\dot z}{V}\right)\right) d\dot z \\ &= t_o - \frac{V}{2g}\int_0^{\dot z}\frac{d}{d\dot z}\log\left(\frac{1+\frac{\dot z}{V}}{1-\frac{\dot z}{V}}\right) d\dot z \\ &= t_o - \frac{V}{2g}\log\left(\frac{1+\frac{\dot z}{V}}{1-\frac{\dot z}{V}}\right) \end{aligned}$$

that is,

$$\begin{aligned} &\tfrac{2g}{V}(t-t_o) = -\log\left(\tfrac{1+\frac{\dot z}{V}}{1-\frac{\dot z}{V}}\right) \\ &e^{\frac{2g}{V}(t-t_o)} = \tfrac{1-\frac{\dot z}{V}}{1+\frac{\dot z}{V}} \\ &e^{\frac{2g}{V}(t-t_o)} + \tfrac{\dot z}{V}e^{\frac{2g}{V}(t-t_o)} = 1 - \tfrac{\dot z}{V} \\ &e^{\frac{2g}{V}(t-t_o)} - 1 = -\tfrac{\dot z}{V}\left(e^{\frac{2g}{V}(t-t_o)} + 1\right) \\ &-\tfrac{\dot z}{V} = \tfrac{e^{\frac{2g}{V}(t-t_o)}-1}{e^{\frac{2g}{V}(t-t_o)}+1} = 1 - \tfrac{2}{e^{\frac{2g}{V}(t-t_o)}+1} \\ &\dot z = -V\left(1 - \tfrac{2}{e^{\frac{2g}{V}(t-t_o)}+1}\right) \end{aligned}$$

Hence, for $t > t_o$,

$$|\dot{\mathrm{p}}(t)| = |\dot z(t)| = V\left(1 - \frac{2}{e^{\frac{2g}{V}(t-t_o)}+1}\right) < V$$

As a consequence, a test particle descends in the air with a scalar velocity, which – although increasing in time– is *upperly bounded* by a critical value V (whose square is inversely proportional to k). [24]

Falls in translating spaces

Let $\alpha_t : \tilde{\mathrm{p}} \in \tilde{\mathcal{E}}_3 \mapsto \alpha_t(\tilde{\mathrm{p}}) = \mathrm{o}_t + A_t(\tilde{\mathrm{p}} - \tilde{\mathrm{o}}) \in \mathcal{E}_T$ be a reference transformation from the terrestrial space $\mathcal{E}_T$ to a 'moving' space $\tilde{\mathcal{E}}_3$, where the local free falls

---

[24] A parachute, for instance, is planned so as to provide a suitably high value of the wind resistance coefficient k and then obtain a suitably small value of the critical velocity V.

of a test particle are characterized by equation [25]

$$\ddot{\tilde{\mathrm{p}}}(t) = \tilde{\mathrm{g}}(t) - \tilde{\mathrm{a}}(t, \tilde{\mathrm{p}}(t)) - 2\,\tilde{\omega}(t) \wedge \dot{\tilde{\mathrm{p}}}(t)$$

We shall discuss the above equation only under the hypothesis of $\tilde{\mathcal{E}}_3$ being a translating space

$$A_t = A = \text{const.}$$

with a non-null, constant, vertical acceleration

$$\ddot{\mathrm{o}}_t = a\,\frac{\mathrm{g}}{g}\ , \quad a := \text{const.} \neq 0\ , \quad g := |\mathrm{g}| = \text{const.} > 0$$

That implies $\tilde{\mathrm{g}}(t) := A_t^{-1}(\mathrm{g}) = A^{-1}(\mathrm{g}) =: \tilde{\mathrm{g}} = \text{const.}$ (whence $|\tilde{\mathrm{g}}| = g$), $\tilde{\mathrm{a}}(t, \tilde{\mathrm{p}}(t)) = a\,\frac{\tilde{\mathrm{g}}}{g}$ and $\tilde{\omega}(t) = 0$, and then the above equation reads

$$\ddot{\tilde{\mathrm{p}}}(t) = \tilde{\mathrm{G}} \qquad (\tilde{1})'$$

with

$$\tilde{\mathrm{G}} := \tilde{\mathrm{g}} - a\,\frac{\tilde{\mathrm{g}}}{g} = (g - a)\,\frac{\tilde{\mathrm{g}}}{g}$$

1) First we consider an ascending acceleration, i.e. $a < 0$.

As $g - a > 0$, the constant field of gravity $\tilde{\mathrm{G}}$ is a descending vertical force, with norm $|\tilde{\mathrm{G}}| = g - a > g$ (in $\tilde{\mathcal{E}}_3$, the weight of a test particle just appears to increase in amount). Equation $(\tilde{1})'$ will therefore have solutions whose behaviour is the same as that of the motions obeying equation (1)'.

2) Now we consider a descending acceleration, i.e. $a > 0$.

(i) Let $a < g$.

As $g - a > 0$, the constant field of gravity $\tilde{\mathrm{G}}$ is still a descending vertical force, but its norm is $|\tilde{\mathrm{G}}| = g - a < g$ (in $\tilde{\mathcal{E}}_3$, the weight of a test particle just appears to decrease in amount). Equation $(\tilde{1})'$ will therefore have solutions whose behaviour is the same as that of the motions obeying equation (1)'.

(ii) Let $a = g$.

In such a case, $\tilde{\mathrm{G}} = 0$ (in the *freely falling space* $\tilde{\mathcal{E}}_3$, the weight of a test particle vanishes as if gravity did not exist at all). Equation $(\tilde{1})'$ will therefore have solutions whose behaviour is the same as that of motions obeying Newton's law of inertia.

(iii) Let $a > g$.

As $g - a < 0$, the constant field of gravity $\tilde{\mathrm{G}}$ is now an ascending vertical force (in $\tilde{\mathcal{E}}_3$, the weight of a test particle 'points upwards'). Equation $(\tilde{1})'$ will therefore have solutions whose behaviour is the 'reverse' of that of the motions obeying equation (1)'.

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[25] See 2.1.2.

2.2.2 Constrained body

We shall now pass on to the study of the possible motions of a constrained test particle in a number of dynamical situations. The configuration space of the particle will be a smooth curve or surface of the terrestrial (or a moving) reference space. [26] The terrestrial field of gravity will be regarded as a non-null constant vector field g on $\mathcal{E}_{_T}$.

1 degree of freedom

We shall first consider a test particle with one degree of freedom in the terrestrial (or a moving) reference space, that is to say, a mechanical system

$$\mathcal{T} := (Q, m, \mathrm{F})$$

whose configuration space Q is a smooth curve −i.e. a 1-dimensional, connected manifold− of the given reference space.

We inform that such a Q admits a global parametrization

$$Q = \operatorname{Im} \xi$$

provided by a global chart or a covering mapping [27]

$$\xi : W \subset \mathbb{R} \to Q : s \mapsto \mathrm{p} = \mathrm{p}(s)$$

whose derivative

$$\mathrm{u}(s) := \frac{d\mathrm{p}}{ds}\Big|_s \in T_{\mathrm{p}(s)}Q$$

has non-null constant norm

$$|\mathrm{u}(s)| = u = \text{const.} > 0$$

whence

$$T_{\mathrm{p}(s)}Q = \operatorname{Span}\left(\mathrm{u}(s)\right), \quad \frac{d\mathrm{u}}{ds} \cdot \mathrm{u} = \frac{1}{2}\,\frac{d\mathrm{u}^2}{ds} = 0$$

Moreover, any admissible motion $\mathrm{p}(t) \in Q$ admits some ‘global coordinate expression’ $s(t) \in W$ through ξ , i.e.

$$\mathrm{p}(t) = \mathrm{p}(s(t))$$

---

[26] Here ‘moving’ means ‘moving with respect to the Earth’.

[27] Recall that a *covering mapping* onto an n-dimensional manifold Q, is a kind of surjective mapping $\xi : W \subset \mathbb{R}^n \to Q$ which defines −via restrictions− a whole atlas of local charts on Q with the property that any C^∞ curve $\kappa : I \to Q$ admits some C^∞ ‘lift’ $\alpha : I \to W$ through ξ , i.e. $\kappa = \xi \circ \alpha$.

whence

$$\dot{\mathrm{p}}(t) = \dot{s}(t)\,\mathrm{u}(s(t))$$

$$\ddot{\mathrm{p}}(t) = \ddot{s}(t)\,\mathrm{u}(s(t)) + \dot{s}^2(t)\,\frac{d\mathrm{u}}{ds}\Big|_{s(t)}$$

and then

$$\ddot{\mathrm{p}}(t) \cdot \mathrm{u}(s(t)) = u^2\,\ddot{s}(t)$$

As a consequence, an admissible motion $\mathrm{p}(t) = \mathrm{p}(s(t)) \in Q$ is a DPM of $\mathcal{T}$, i.e. satisfies d'Alembert's principle

$$\mathrm{F}(t, \mathrm{p}(t), \dot{\mathrm{p}}(t)) - m\,\ddot{\mathrm{p}}(t) \;\in\; T^{\perp}_{\mathrm{p}(t)}Q = \mathrm{Span}^{\perp}(\mathrm{u}(s(t)))$$

or, equivalently,

$$m\,\ddot{\mathrm{p}}(t)) \cdot \mathrm{u}(s(t) = \mathrm{F}(t, \mathrm{p}(t), \dot{\mathrm{p}}(t)) \cdot \mathrm{u}(s(t)$$

iff it corresponds through ξ to a solution $s(t) \in W$ of the *global* Lagrange equation [28]

$$m\,\ddot{s}(t) = F_{\mathrm{u}}(t, s(t), \dot{s}(t)) \qquad (\dagger)$$

where we have put, for all $t \in \mathbb{R}$, $\mathrm{p} = \mathrm{p}(s) \in Q$ and $\mathrm{v} = v\,\mathrm{u}(s) \in T_{\mathrm{p}}Q$,

$$F_{\mathrm{u}}(t, s, v) := \frac{1}{u^2}\,(\mathrm{F}(t, \mathrm{p}, \mathrm{v}) \cdot \mathrm{u}(s))$$

(recall that $\mathrm{F}_{tang}(t, \mathrm{p}, \mathrm{v}) = F_{\mathrm{u}}(t, s, v)\,\mathrm{u}(s)$, i.e. the vector component of $\mathrm{F}(t, \mathrm{p}, \mathrm{v})$ tangent to Q, is the 'effective force' acting on the particle, the orthogonal component being dynamically uninfluential).

The examples in the sequel will take place on the two standard types of smooth curve, i.e. a straight line and a circle.

Straight line

Let Q be a straight line of the terrestrial reference space $\mathcal{E}_{T}$.

We choose an orthogonal Cartesian system $(\mathrm{o}, \mathrm{e}_1, \mathrm{e}_2, \mathrm{e}_3)$ in $\mathcal{E}_{T}$ s.t.

$$\begin{aligned} Q &= \{\mathrm{p} \in \mathcal{E}_{T} \mid (\mathrm{p} - \mathrm{o}) \cdot \mathrm{e}_1 = 0\,,\ (\mathrm{p} - \mathrm{o}) \cdot \mathrm{e}_2 = 0\} \\ &= \mathrm{o} + \mathrm{Span}\,(\mathrm{e}_3) \end{aligned}$$

and

$$\mathrm{g} = (g\cos\alpha)\,\mathrm{e}_1 - (g\sin\alpha)\,\mathrm{e}_3$$

---

[28] See II. 3.1.1.

with

$$g := |\mathrm{g}| \ , \quad \alpha := \angle(\mathrm{g}, \mathrm{e}_1) \in \left[0, \frac{\pi}{2}\right]$$

α is said to be the *slope* of Q. In particular, if $\alpha = 0$, Q is horizontal (since $\mathrm{g} \cdot \mathrm{e}_3 = 0$) and, if $\alpha = \frac{\pi}{2}$, Q is vertical (since $\mathrm{g} = -g\,\mathrm{e}_3$). For any $\alpha > 0$, e_3 points upwards (since $\mathrm{g} \cdot \mathrm{e}_3 < 0$, whence $\angle(\mathrm{g}, \mathrm{e}_3) > \frac{\pi}{2}$).

On Q we shall consider the global chart defined by

$$\xi : \mathbb{R} \to Q : z \mapsto \mathrm{p} = \mathrm{p}(z) := \mathrm{o} + z\,\mathrm{e}_3$$

whence

$$\mathrm{u} := \frac{d\mathrm{p}}{dz} = \mathrm{e}_3 \ , \quad u := |\mathrm{u}| = 1$$

Falls along a sloping axis

We start with

$$\mathrm{F} := m\,\mathrm{g}$$

In such a case, we have

$$F_{\mathrm{u}} = m\,\mathrm{g} \cdot \mathrm{e}_3 = -\,mg \sin\alpha$$

So an admissible motion

$$\mathrm{p}(t) = \mathrm{o} + z(t)\,\mathrm{e}_3 \in Q$$

is the maximal DPM of $\mathcal{T}$ satisfyying, at a time $t_o \in \mathbb{R}$, the initial conditions

$$\mathrm{p}(t_o) = \mathrm{p}_o := \mathrm{o} + z_o\,\mathrm{e}_3 \in Q \ , \quad \dot{\mathrm{p}}(t_o) = \mathrm{v}_o := v_o\,\mathrm{e}_3 \in T_{\mathrm{p}_o}Q = \mathrm{Span}(\mathrm{e}_3)$$

iff its coordinate expression

$$z(t) \in \mathbb{R}$$

satisfies Lagrange equation (†) taking the ‘universal’ form

$$\ddot{z}(t) = -g \sin\alpha \qquad (\dagger)_1$$

and the initial conditions

$$z(t_o) = z_o \ , \quad \dot{z}(t_o) = v_o$$

Hence, by quadratures, we obtain

$$\dot{z}(t) = v_o - (g \sin\alpha)(t - t_o)$$

and

$$z(t) = z_o + v_o(t - t_o) - \frac{1}{2}(g \sin\alpha)(t - t_o)^2$$

for all $t \in \mathbb{R}$,

The DPM is then given by

(i) for $\alpha = 0$ (horizontal axis),

$$\begin{aligned} \mathrm{p}(t) &= \mathrm{o} + (z_o + v_o(t - t_o))\,\mathrm{e}_3 \\ &= \mathrm{p}_o + \mathrm{v}_o(t - t_o) \end{aligned}$$

which is just a uniform motion if $\mathrm{v}_o \neq 0$, or degenerates into a state of rest if $\mathrm{v}_o = 0$ (the efffects of gravity disappear); [29]

(ii) for $\alpha \neq 0$ (inclined or vertical axis),

$$\begin{aligned} \mathrm{p}(t) &= \mathrm{o} + \left(z_o + v_o(t - t_o) - \frac{1}{2}(g \sin\alpha)(t - t_o)^2 \right) \mathrm{e}_3 \\ &= \mathrm{p}_o + \mathrm{v}_o(t - t_o) + \frac{1}{2}\,\mathrm{g}_3(t - t_o)^2 \end{aligned}$$

with

$$\mathrm{g}_3 := -(g \sin\alpha)\,\mathrm{e}_3$$

which exhibits the same features as a rectilinear fall in the vacuum. [30]

Friction

Now we put

$$\mathrm{F}(\mathrm{v}) := m\,\mathrm{g} + \mathrm{A}(\mathrm{v})$$

where $\mathrm{A}(\mathrm{v})$ is a force field – called *friction* and due to the roughness of the constraint devices – depending on velocity $\mathrm{v} \in \mathrm{Span}\,(\mathrm{e}_3)$ as follows:
for $\mathrm{v} = 0$, $\mathrm{A}(0)$ is *not* uniquely defined, but just obeys the *law of static friction*

$$\mathrm{A}(0) := A_o\,\mathrm{e}_3\,, \quad |A_o| \leq c_s(m\,\mathrm{g} \cdot \mathrm{e}_1)\,, \quad 0 < c_s < 1$$

---

[29] Any point $\mathrm{p}_o \in Q$ is therefore an equilibrium configuration, but unstable since, among the DPMs, there do not exist 'small' motions near p_o.

[30] Check that equation $(\dagger)_1$ is the Euler-Lagrange equation associated with the Lagrangian function $L(z, v) := \frac{1}{2}\,v^2 - V(z)$, where $\frac{1}{2}\,v^2 = \frac{1}{2}\,(v\,\mathrm{e}_3) \cdot (v\,\mathrm{e}_3)$ and $V(z) := (g \sin\alpha)\,z = -\mathrm{g} \cdot (\mathrm{p} - \mathrm{o})$ (for all $\mathrm{p} = \mathrm{o} + z\,\mathrm{e}_3 \in Q$ and $\mathrm{v} = v\,\mathrm{e}_3 \in T_{\mathrm{p}}Q$). As a consequence, the qualitative portrait of its solutions is also exhibited by the time-oriented level lines of the Hamiltonian function $H(z, p) := \frac{1}{2}\,p^2 + V(z)$ (with $p := \frac{\partial L}{\partial v} = v$).
See II. 3.2.2.

for $\mathrm{v} \neq 0$, $\mathrm{A}(\mathrm{v})$ is uniquely defined by the *law of dynamic friction*

$$\mathrm{A}(\mathrm{v}) := -A\,\frac{\mathrm{v}}{|\mathrm{v}|}\,, \quad A := c_d(m\,\mathrm{g}\cdot \mathrm{e}_1)\,, \quad 0 < c_d < c_s$$

The scalar value A of dynamic friction – which is a fraction of the normal component (with respect to Q) of the weight $m\,\mathrm{g}$ – is lower than the maximum value of static friction $|A_o|$. That is due to $0 < c_d < c_s < 1$, which also implies $0 < \varphi_d < \varphi_s < \frac{\pi}{4}$ for the *angles*

$$\varphi_s := \arctan c_s\,,\ \varphi_d := \arctan c_d$$

of static and dynamic friction.

Friction is a very anomalous force field, owing to its indeterminacy at $\mathrm{v} = 0$, which puts the whole problem beyond the limits of deterministic smooth dynamics, as will be shown.

On the one hand, d'Alembert's principle

$$(m\mathrm{g} + \mathrm{A}(\dot{\mathrm{p}}(\mathrm{t})) - m\,\ddot{\mathrm{p}}(t))\cdot \mathrm{e}_3 = 0$$

admits a *static* solution

$$\mathrm{p}(t) := \mathrm{o} \in Q$$

(defined on on any time interval), iff

$$(m\,\mathrm{g} + A_o\,\mathrm{e}_3)\cdot \mathrm{e}_3 = 0$$

i.e.

$$A_o = mg\sin\alpha$$

with the above non-negative value of A_o compatible with the law of static friction

$$mg\sin\alpha \leq c_s mg\cos\alpha$$

that is to say, iff

$$\alpha \leq \varphi_s$$

On the other hand, the above d'Alembert's principle admits a *non-static* solution

$$\mathrm{p}(t) = \mathrm{o} + z(t)\,\mathrm{e}_3 \in Q\ , \quad \dot{\mathrm{p}}(t) = \dot{z}(t)\,\mathrm{e}_3 \neq 0$$

iff

$$(m\mathrm{g} - c_d mg\cos\alpha\,\frac{\dot{z}(t)}{|\dot{z}(t)|}\,\mathrm{e}_3 - m\,\ddot{z}(t)\mathrm{e}_3)\cdot \mathrm{e}_3 = 0$$

that is,

$$m\ddot{z}(t) = -mg\sin\alpha - c_d mg\cos\alpha\,\frac{\dot{z}(t)}{|\dot{z}(t)|}$$

which amounts to saying

$$\ddot{z}(t) = g_\epsilon \tag{$\dagger$}_2$$

with

$$\epsilon = \pm 1 \Longleftrightarrow \dot{z}(t) \gtrless 0$$

and

$$\begin{aligned} g_\epsilon &= -g(\sin\alpha + \epsilon c_d \cos\alpha) \\ &= -g\left(\sin\alpha + \epsilon \frac{\sin\varphi_d}{\cos\varphi_d}\cos\alpha\right) \\ &= -g\,\frac{\sin\alpha\cos\varphi_d + \epsilon\sin\varphi_d\cos\alpha}{\cos\varphi_d} \\ &= -g\,\frac{\sin(\alpha + \epsilon\varphi_d)}{\cos\varphi_d} \end{aligned}$$

whence

$$g_1 < 0\ , \quad g_{-1} \gtreqless 0 \Longleftrightarrow \alpha \lesseqgtr \varphi_d$$

Now focus on the non-static solutions of d'Alembert's principle with coordinate expression

$$z(t) = \frac{1}{2}\, g_\epsilon\, t^2$$

which is a solution of equation $(\dagger)_2$ corresponding, [31]
for $\epsilon = 1$, to an ascending and decelerated motion defined for all $t \in (-\infty, 0)$, and,
for $\epsilon = -1$, to
a descending and decelerated motion, defined for all $t \in (-\infty, 0)$, if $\alpha < \varphi_d$, or
a descending and accelerated motion, defined for all $t \in (0, +\infty)$, if $\alpha > \varphi_d$
(in any case, $z(t)$ and $\dot{z}(t)$ tend to zero, when t tends to zero).

Some of the above static and non-static solutions of d'Alembert's principle can suitably be 'glued together', so as to build up 'maximal solutions' (with second derivatives generally discontinuous at the 'critical' time $t = 0$) satisfying the initial conditions $\mathrm{p}(0) = \mathrm{o}$ and $\dot{\mathrm{p}}(0) = 0$, i.e. $z(0) = 0$ and $\dot{z}(0) = 0$.

For instance, if

$$\alpha > \varphi_s$$

---

[31] In the case $\alpha = 0$, the terms 'ascending' ($\dot{z} > 0$) and 'descending' ($\dot{z} < 0$) will obviously loose their literal meaning.

the above Cauchy problem (which cannot admit a static solution) only admits the ‘ascending and then descending’ maximal solution

$$z(t) = \begin{cases} \frac{1}{2}\, g_1\, t^2 & \text{if } t \in (-\infty, 0) \\ 0 & \text{if } t = 0 \\ \frac{1}{2}\, g_{-1}\, t^2 & \text{if } t \in (0, +\infty) \end{cases}$$

whereas, if

$$\alpha \leq \varphi_d$$

the problem *fails to be deterministic*, since it admits the maximal static solution

$$z(t) = 0 \;, \quad \forall\, t \in (-\infty, +\infty)$$

as well as the ‘ascending and then static’ solution

$$z(t) = \begin{cases} \frac{1}{2}\, g_1\, t^2 & \text{if } t \in (-\infty, 0) \\ 0 & \text{if } t \in [0, +\infty) \end{cases}$$

and the ‘descending and then static’ solution

$$z(t) = \begin{cases} \frac{1}{2}\, g_{-1}\, t^2 & \text{if } t \in (-\infty, 0) \\ 0 & \text{if } t \in [0, +\infty) \end{cases}$$

Harmonic oscillator

Consider the case of an *elastic force* acting (as well as the weight) on the particle, i.e.

$$\mathrm{F}(\mathrm{p}) := m\, \mathrm{g} - k\, (\mathrm{p} - \mathrm{o})$$

with

$$k > 0 \;, \quad \mathrm{o} \in Q$$

Note that, at any point $\mathrm{p} = \mathrm{o} + z\mathrm{e}_3 \in Q$, the effective force $\mathrm{F}_{tang}(\mathrm{p}) = F_{\mathrm{u}}(z)\, \mathrm{u} = F_{\mathrm{u}}(z)\, \mathrm{e}_3$ exhibits an *attractive* character with respect to the *centre*

$$\mathrm{o}_* := \mathrm{o} + z_* \mathrm{e}_3$$

with

$$z_* := -\frac{mg \sin\alpha}{k} \leq 0$$

since

$$F_{\mathrm{u}}(z) = (m\mathrm{g} - k\, (\mathrm{p} - \mathrm{o})) \cdot \mathrm{e}_3 = -mg \sin\alpha - kz \gtreqless 0 \iff z \lesseqgtr z_*$$

An admissible motion

$$\mathrm{p}(t) = \mathrm{o} + z(t)\, \mathrm{e}_3 \in Q$$

is a DPM of $\mathcal{T}$, iff its coordinate expression

$$z(t) \in \mathbb{R}$$

satisfies Lagrange equation (†), which, in this case, reads

$$m\ddot{z}(t) = -mg\sin\alpha - kz(t) \tag{$\dagger$}_3$$

or

$$\ddot{z}(t) + \omega^2 z(t) = \omega^2 z_*$$

with

$$\omega := \sqrt{\frac{k}{m}}$$

(i) For $\alpha = 0$ (horizontal axis), the effect of gravity disappears, for equation $(\dagger)_3$ turns into the linear equation

$$\ddot{z}(t) + \omega^2 z(t) = 0$$

called *harmonic oscillator* (the attractive centre is now $\mathrm{o}_* = \mathrm{o}$, since $z_* = 0$).

We shall show, by direct calculations, that the typical maximal solution of the above equation is given, for all $t \in \mathbb{R}$, by

$$z(t) = r\cos(\omega t + \varphi)\ , \quad r \geq 0$$

Indeed, by derivation we obtain

$$\dot{z}(t) = -\omega r\sin(\omega t + \varphi)\ , \quad \ddot{z}(t) = -\omega^2 r\cos(\omega t + \varphi) = -\omega^2 z(t)$$

and, for any $(t_o, z_o, v_o) \in \mathbb{R}^3$, there exists a solution $z(t) = r\cos(\omega t + \varphi)$, with $r \geq 0$, satisfying initial conditions

$$z(t_o) = z_o\ , \quad \dot{z}(t_o) = v_o$$

i.e.

$$r\cos(\omega t_o + \varphi) = z_o\ , \quad r\sin(\omega t_o + \varphi) = -\frac{v_o}{\omega}$$

namely, the one determined by

$$r^2 = z_o^2 + \frac{v_o^2}{\omega^2}\ \geq\ 0$$

and –if $r > 0$, i.e. $(z_o, v_o) \neq (0,0)$,–

$$\omega t_o + \varphi \equiv \arccos\frac{z_o}{r}\ \ (\mathrm{mod}\, 2\pi)$$

where the sign plus or minus is to be chosen according to whether $v_o \leq 0$ or $v_o > 0$, respectively.

Initial conditions $(z_o, v_o) \neq (0,0)$ then determine a maximal solution $z(t)$ that is bounded

$$-r \leq z(t) \leq r$$

and periodic

$$z(t_2) = z(t_1) \iff t_2 \equiv t_1 \left(\operatorname{mod} \frac{2\pi}{\omega}\right)$$

and the corresponding DPM

$$\mathrm{p}(t) = \mathrm{o} + r\cos(\omega t + \varphi)\, \mathrm{e}_3$$

is said to be a *harmonic oscillation*, with *centre* at o and *amplitude* $r > 0$, whose *period* (resp. *frequency*) is $T := \frac{2\pi}{\omega}$ (resp. $\nu := \frac{\omega}{2\pi}$).

Initial conditions $(z_o, v_o) = (0,0)$ determine the maximal solution $z(t) = 0$, that is, the 'static' DPM $\mathrm{p}(t) = \mathrm{o}$.

Clearly, the centre o is the only equilibrium configuration and the equilibrium is stable. [32]

A qualitative analisys of such DPMs can be carried out as follows.

Remark that the harmonic oscillator is the Euler-Lagrange equation associated with the Lagrangian function $L(z,v) := \frac{1}{2}mv^2 - V(z)$, where [33]

$$V(z) := \frac{1}{2}kz^2$$

As a consequence, [34] a qualitative portrait of its solutions is exhibited by the time-oriented level lines $H^{-1}(c_o)$ of the Hamiltonian function $H(z,p) := \frac{1}{2m}p^2 + V(z)$ (with $p := \frac{\partial L}{\partial v} = mv$), described by equation

$$p = \pm\sqrt{2m(c_o - V(z))}$$

The portrait is the following:

---

[32] Check, by direct calculations, that the maximal solution of the harmonic oscillator with initial conditions $z(t_o) = z_o$ and $\dot{z}(t_o) = v_o$,, can as well be expressed by

$$z(t) = z_o \cos\omega(t - t_o) + \frac{v_o}{\omega}\sin\omega(t - t_o)$$

[33] $L(z,v)$ is the coordinate expression, for all $\mathrm{p} = \mathrm{o} + z\mathrm{e}_3 \in Q$ and $\mathrm{v} = v\,\mathrm{e}_3 \in T_{\mathrm{p}}Q$, of the Lagrangian function $L(\mathrm{p},\mathrm{v}) := \frac{1}{2}m\mathrm{v}\cdot\mathrm{v} - V(\mathrm{p})$, where $V(\mathrm{p}) := \frac{1}{2}k(\mathrm{p}-\mathrm{o})\cdot(\mathrm{p}-\mathrm{o})$ is the potential energy of the elastic force, since $-\,d_{\mathrm{p}}V = -\,k(\mathrm{p}-\mathrm{o})\cdot$

[34] See II. 3.2.2.

Portrait 3 In Fig.3 you will find – as is suggested by the graph of V – the level lines $H^{-1}(c_o)$ in two typical cases.

Let $c_o = 0$. [35]

$H^{-1}(0)$ is the singleton $\{(0,0)\}$, which corresponds to the state of rest at the equilibrium configuration o, whose stability could be checked through Dirichlet's criterion or directly follows from the portrait itself (as is shown below).

Let $c_o = E > 0$.

$H^{-1}(E)$ is an ellipse, which corresponds to harmonic oscillations (arbitrarily 'small', if so is E) with centre o and amplitude $r = \sqrt{\frac{2E}{k}}$, along which the particle alternates phases of accelerated centripetal motion with phases of decelerated centrifugal motion. □

(ii) For $\alpha \neq 0$ (inclined or vertical axis), we can still write equation $(\dagger)_3$ in the linear form

$$\ddot{\zeta}(t) + \omega^2 \zeta(t) = 0 \qquad (\dagger)_3$$

by putting

$$\zeta(t) := z(t) - z_*$$

(in the present case $z_* < 0$).

Then, to any maximal solution

$$\zeta(t) = r\cos(\omega t + \varphi)$$

there corresponds the DPM

$$\begin{aligned} \mathrm{p}(t) &= \mathrm{o} + z(t)\,\mathrm{e}_3 \\ &= \mathrm{o} + (z_* + \zeta(t))\,\mathrm{e}_3 \\ &= \mathrm{o} + (z_* + r\cos(\omega t + \varphi))\,\mathrm{e}_3 \\ &= \mathrm{o}_* + (r\cos(\omega t + \varphi))\,\mathrm{e}_3 \end{aligned}$$

So the only effect of gravity is that of moving the centre of the harmonic oscillations down to $\mathrm{o}_* = \mathrm{o} + z_*\,\mathrm{e}_3$.

Beats

We shall now add a *time-dependent, sinusoidal force* tangent to Q to the weight and the elastic force, i.e.

$$\mathrm{F}(t, \mathrm{p}) := m\,\mathrm{g} - k\,(\mathrm{p} - \mathrm{o}) + (a\sin\omega_1 t)\,\mathrm{e}_3$$

---

[35] For $c_o < 0$, $H^{-1}(c_o)$ is empty.

with

$$k > 0 \,, \quad \mathrm{o} \in Q \,, \quad a > 0 \,, \quad 0 < \omega_1 \approx \sqrt{\frac{k}{m}}$$

An admissible motion

$$\mathrm{p}(t) = \mathrm{o} + z(t)\, \mathrm{e}_3 \in Q$$

is a DPM of $\mathcal{T}$, iff its coordinate expression

$$z(t) \in \mathbb{R}$$

satisfies Lagrange equation (†), which, in this case, reads

$$m\ddot{z}(t) = -mg \sin\alpha - kz(t) + a \sin\omega_1 t \qquad (\dagger)_4$$

or

$$\ddot{\zeta}(t) + \omega^2 \zeta(t) = \frac{a}{m} \sin\omega_1 t \qquad (\dagger)_4$$

by putting

$$\zeta(t) := z(t) - z_* \,, \quad z_* := -\frac{mg \sin\alpha}{k} \,, \quad \omega := \sqrt{\frac{k}{m}}$$

One could check, by direct calculations, that the typical maximal solution of the above equation splits into a sum

$$\zeta(t) = \zeta_o(t) + \zeta_1(t)$$

where

$$\zeta_o(t) = r \cos(\omega t + \varphi)$$

is the typical maximal solution of the harmonic oscillator $(\dagger)_3$ and

$$\zeta_1(t) = \frac{a}{m\omega} \, \frac{\omega \sin\omega_1 t - \omega_1 \sin\omega t}{(\omega + \omega_1)(\omega - \omega_1)}$$

is a particular maximal solution of equation $(\dagger)_4$.

Remark that, from $\omega_1 = \frac{\omega_1 + \omega}{2} + \frac{\omega_1 - \omega}{2}$ and $\omega = \frac{\omega_1 + \omega}{2} - \frac{\omega_1 - \omega}{2}$, we infer

$$\begin{aligned}
\frac{\omega \sin\omega_1 t - \omega_1 \sin\omega t}{(\omega + \omega_1)(\omega - \omega_1)} \quad &= \quad \frac{1}{\omega + \omega_1} \, \sin\frac{\omega_1 + \omega}{2} t \, \cos\frac{\omega_1 - \omega}{2} t \\
&+ \quad \frac{1}{\omega - \omega_1} \, \cos\frac{\omega_1 + \omega}{2} \, t \sin\frac{\omega_1 - \omega}{2} t
\end{aligned}$$

Owing to the hypothesis

$$\omega_1 \approx \omega$$

we can approximately evaluate the above solution $\zeta(t)$ by neglecting all of its sinusoidal addends except the one of 'large' amplitude $\frac{1}{|\omega-\omega_1|}$, obtaining

$$\zeta(t) \approx \frac{a}{m\omega(\omega-\omega_1)}\ \sin\frac{\omega_1-\omega}{2}t\ \cos\frac{\omega_1+\omega}{2}t$$

As $\frac{\omega_1+\omega}{2} \approx \omega$, we obtain

$$\zeta(t) \approx r(t)\cos\omega t$$

with

$$r(t) := \frac{a}{m\omega(\omega-\omega_1)}\ \sin\frac{\omega_1-\omega}{2}t$$

So the typical DPM

$$\begin{aligned} \mathrm{p}(t) &= \mathrm{o} + z(t)\mathrm{e}_3 \\ &= \mathrm{o} + (z_* + \zeta(t))\,\mathrm{e}_3 \\ &\approx \mathrm{o} + (z_* + r(t)\cos\omega t)\,\mathrm{e}_3 \\ &= \mathrm{o}_* + (r(t)\cos\omega t)\,\mathrm{e}_3 \end{aligned}$$

exhibits the same oscillatory character about $\mathrm{o}_* = \mathrm{o} + z_*\mathrm{e}_3$ as before, but its amplitude $|r(t)|$ is itself a sinusoidal function of time, whose 'large' oscillations correspond to the well known (acoustic) phenomenon of *beats*.

Damped oscillator

Consider the case of a 'small' *viscous force* acting (as well as the weight and the elastic force) on the particle, i.e.

$$\mathrm{F}(\mathrm{p},\mathrm{v}) := m\,\mathrm{g} - k\,(\mathrm{p}-\mathrm{o}) - h\,\mathrm{v}$$

with

$$k > 0\ , \quad \mathrm{o} \in Q\ , \quad 0 < h < \sqrt{4mk}$$

An admissible motion

$$\mathrm{p}(t) = \mathrm{o} + z(t)\,\mathrm{e}_3 \in Q$$

is a DPM of $\mathcal{T}$, iff its coordinate expression

$$z(t) \in \mathbb{R}$$

satisfies Lagrange equation (†), which, in this case, reads

$$m\ddot{z}(t) = -mg\sin\alpha - kz(t) - h\dot{z}(t) \qquad (\dagger)_5$$

or

$$m\ddot{z}(t) + h\dot{z}(t) + kz(t) = kz_*$$

with

$$z^* := -\frac{mg\sin\alpha}{k} \leq 0$$

(i) For $\alpha = 0$ (horizontal axis), the effect of gravity disappears, for equation $(\dagger)_5$ turns into the linear equation

$$m\ddot{z}(t) + h\dot{z}(t) + kz(t) = 0$$

called *damped oscillator*, whose typical maximal solution could be seen – by the same reasoning as the one adopted for the harmonic oscillator – to be given, for all $t \in \mathbb{R}$, by

$$z(t) = e^{-\sigma t} r\cos(\omega t + \varphi)\ , \quad r \geq 0$$

with

$$\sigma := \frac{h}{2m} < \sqrt{\frac{k}{m}}\ , \quad \omega := \sqrt{\frac{k}{m} - \sigma^2} > 0$$

Remark the oscillatory character of such a solution

$$-e^{-\sigma t}\, r \leq z(t) \leq e^{-\sigma t}\, r$$

with an exponentially decreasing amplitude $e^{-\sigma t}\, r$, whence

$$\lim_{t\to+\infty} z(t) = 0$$

As a consequence, the corresponding DPM

$$\mathrm{p}(t) = \mathrm{o} + (e^{-\sigma t}\, r\cos(\omega t + \varphi))\mathrm{e}_3$$

exhibits the character of a *damped oscillatory motion* about the centre o.

(ii) For $\alpha \neq 0$, we can still write equation $(\dagger)_5$ in the linear form

$$m\,\ddot{\zeta}(t) + h\,\dot{\zeta} + k\,\zeta(t) = 0 \tag*{$(\dagger)_5$}$$

by putting

$$\zeta(t) := z(t) - z_*$$

To any maximal solution

$$\zeta(t) = e^{-\sigma t}\, r\cos(\omega t + \varphi)$$

of the above equation, there corresponds the DPM

$$\begin{aligned} \mathrm{p}(t) &= \mathrm{o} + (z_* + e^{-\sigma t}\, r\cos(\omega t + \varphi))\,\mathrm{e}_3 \\ &= \mathrm{o}_* + (e^{-\sigma t}\, r\cos(\omega t + \varphi))\,\mathrm{e}_3 \end{aligned}$$

So the only effect of gravity is that of moving the centre of the damped oscillations down to $\mathrm{o}_* = \mathrm{o} + z_*\,\mathrm{e}_3$.

Resonance

We shall now add a *time-dependent, sinusoidal force* tangent to Q to the weight, the elastic force and the viscous force, i.e.

$$\mathrm{F}(t,\mathrm{p},\mathrm{v}) := m\,\mathrm{g} - k\,(\mathrm{p}-\mathrm{o}) - h\mathrm{v} + (a\sin\omega_1 t)\,\mathrm{e}_3$$

with

$$k>0\,, \quad \mathrm{o}\in Q\,, \quad 0\approx h<\sqrt{4mk}\,, \quad a>0\,, \quad 0<\omega_1\approx\sqrt{\frac{k}{m}}$$

An admissible motion

$$\mathrm{p}(t) = \mathrm{o} + z(t)\,\mathrm{e}_3 \in Q$$

is a DPM of $\mathcal{T}$, iff its coordinate expression

$$z(t)\in\mathbb{R}$$

satisfies Lagrange equation (†), which, in this case, reads

$$m\,\ddot z(t) = -mg\sin\alpha - kz(t) - h\dot z(t) + a\sin\omega_1 t \qquad (\dagger)_6$$

or

$$m\,\ddot\zeta(t) + h\,\dot\zeta + k\,\zeta(t) = a\sin\omega_1 t \qquad (\dagger)_6$$

by putting

$$\zeta(t) := z(t) - z_*\,, \quad z_* := -\frac{mg\sin\alpha}{k}$$

One could easily check that the typical maximal solution of the above equation splits into the sum

$$\zeta(t) = \zeta_o(t) + \zeta_1(t)$$

where $\zeta_o(t)$ is the typical maximal solution of the damped oscillator $(\dagger)_5$ and $\zeta_1(t)$ is a particular maximal solution of equation $(\dagger)_6$.

As $\lim_{t\to+\infty}\zeta_o(t)=0$, we can approximately evaluate the trend of $\zeta(t)$ in the ‘future’ by putting

$$\zeta(t)\approx\zeta_1(t)$$

Now a particular maximal solution $\zeta_1(t)$ of $(\dagger)_6$ is given, for all $t\in\mathbb{R}$, by

$$\zeta_1(t) = A\sin(\omega_1 t - \theta)\,,$$

with suitable values of $A>0$ and $\theta\in(0,\frac{\pi}{2})$.

In fact, the above $\zeta_1(t)$ satisfies $(\dagger)_6$, iff [36]

$$[(k - m\omega_1^2)A - \mathrm{a}\cos\theta]\sin(\omega_1 t - \theta) + [h\omega_1 A - \mathrm{a}\sin\theta]\cos(\omega_1 t - \theta) = 0$$

that is,

$$(k - m\omega_1^2)A = \mathrm{a}\cos\theta\ , \quad h\omega_1 A = \mathrm{a}\sin\theta$$

whence

$$A = \frac{a}{\sqrt{(k - m\omega_1^2)^2 + h^2\omega_1^2}}\ , \qquad \theta = \arccos\frac{k - m\omega_1^2}{\sqrt{(k - m\omega_1^2)^2 + h^2\omega_1^2}}$$

Remark that the hypotheses $\omega_1 \approx \sqrt{\frac{k}{m}}$ and $h \approx 0$ imply $(k - m\omega_1^2)^2 + h^2\omega_1^2 \approx 0$ and hence a high value for the above amplitude A.

Therefore, the harmonic oscillations

$$\begin{aligned} \mathrm{p}(t) &= \mathrm{o} + z(t)\,\mathrm{e}_3 \\ &= \mathrm{o} + (z_* + \zeta(t))\,\mathrm{e}_3 \\ &= \mathrm{o} + (z_* + A\sin(\omega_1 t - \theta))\,\mathrm{e}_3 \\ &= \mathrm{o}_* + (A\sin(\omega_1 t - \theta))\,\mathrm{e}_3 \end{aligned}$$

about $\mathrm{o}_* := \mathrm{o} + z_*\mathrm{e}_3$ –produced by the additional sinusoidal force– exhibit a ‘very large’ amplitude A, which corresponds to the well known (acoustic) phenomenon of *resonance.*

Rotating axis

Let $Q = \mathrm{o} + \mathrm{Span}\,(\mathrm{e}_3)$ be a straight line of a reference space $\mathcal{E}_3$ uniformly rotating, with respect to $\mathcal{E}_\mathcal{T}$, round a vertical axis $\mathcal{S} = \mathrm{o} + \mathrm{Span}\,(\mathrm{g})$. [37]

The force of gravity acting in $\mathcal{E}_3$ on a mass m is [38]

$$\mathrm{F}(\mathrm{p},\mathrm{v}) := m(\mathrm{g} + w^2(\mathrm{p} - \mathrm{p}_*) - 2\mathrm{w}\wedge\mathrm{v})$$

with

$$\mathrm{g} = \text{const.}\ , \quad g := |\mathrm{g}| > 0\ ; \quad \mathrm{w} = -w\,\frac{\mathrm{g}}{g}\ , \quad w := |\mathrm{w}| = \text{const.} > 0\ ; \quad \mathrm{p}_* := \mathrm{o} + \left((\mathrm{p} - \mathrm{o})\cdot\frac{\mathrm{g}}{g}\right)\frac{\mathrm{g}}{g}$$

Note that, if $\alpha \neq \pi/2$, at any $\mathrm{p} = \xi(z) = \mathrm{o} + z\,\mathrm{e}_3 \in Q$ the effective force $F_\mathrm{u}(z)\,\mathrm{e}_3$ exhibits a *repulsive* character with respect to the *centre*

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[36] Recall that $\sin\omega_1 t = \sin(\theta + (\omega_1 t - \theta)) = \sin\theta\,\cos(\omega_1 t - \theta) + \cos\theta\,\sin(\omega_1 t - \theta)$.

[37] Choose an orthogonal Cartesian system $(\mathrm{o}, \mathrm{e}_1, \mathrm{e}_2, \mathrm{e}_3)$ in $\mathcal{E}_3$ in the same way as we did in the section ‘Straight line’.

[38] See 2.1.2 and Appendix 3.1.

$$\mathrm{o}_\star := \mathrm{o} + z_\star \mathrm{e}_3$$

with

$$z_\star := \frac{g \sin\alpha}{\omega^2} \geq 0\,, \quad \omega := w \cos\alpha > 0$$

since

$$F_{\mathrm{u}}(z) = m(\mathrm{g} + w^2(\mathrm{p} - \mathrm{p}_*) - 2\mathrm{w} \wedge \mathrm{v}) \cdot \mathrm{e}_3 = m(-g \sin\alpha + \omega^2 z) \gtreqless 0 \iff z \gtreqless z_\star$$

(whereas, if $\alpha = \pi/2$, the effective force is just the weight mg).

An admissible motion

$$\mathrm{p}(t) = \mathrm{o} + z(t)\,\mathrm{e}_3 \in Q$$

is a DPM of $\mathcal{T}$, iff its coordinate expression

$$z(t) \in \mathbb{R}$$

satisfies Lagrange equation (†), which, in this case, takes the universal form

$$\ddot{z}(t) = -g \sin\alpha + \omega^2 z(t) \tag{$\dagger$)$_7$}$$

or

$$\ddot{z}(t) - \omega^2 z(t) = -\omega^2 z_\star$$

(i) For $\alpha = 0$ (horizontal axis), the effect of gravity disappears, for equation $(\dagger)_7$ turns into the linear equation

$$\ddot{z}(t) - \omega^2 z(t) = 0$$

called *antioscillator* (note that the repulsive centre is now $\mathrm{o}^* = \mathrm{o}$, acting with the maximal *intensity* $\omega^2 = w^2$).

We shall show, by direct calculations, that the typical maximal solution of the above equation is given, for all $t \in \mathbb{R}$, by

$$z(t) = \lambda\, e^{\omega t} + \mu\, e^{-\omega t}\,, \quad (\lambda, \mu) \in \mathbb{R}^2$$

Indeed, by derivation we obtain

$$\dot{z}(t) = \omega(\lambda\, e^{\omega t} - \mu\, e^{-\omega t})\,, \quad \ddot{z}(t) = \omega^2(\lambda\, e^{\omega t} + \mu\, e^{-\omega t}) = \omega^2 z(t)$$

Moreover, for any $(t_o, z_o, v_o) \in \mathbb{R}^3$, there exists one and only one $z(t) = r\cos(\omega t + \varphi)$ satisfying the initial conditions

$$z(t_o) = z_o\,, \quad \dot{z}(t_o) = v_o$$

i.e.

$$\lambda\, e^{\omega t_o} + \mu\, e^{-\omega t_o} = z_o\ , \quad \lambda\, e^{\omega t_o} - \mu\, e^{-\omega t_o} = \frac{v_o}{\omega}$$

namely, the one determined by by the unique solution (λ, μ) of the above couple of non-homogeneous linear equations (with a non-singular matrix of coefficients).

The corresponding DPM is therefore

$$\mathrm{p}(t) = \mathrm{o} + (\lambda\, e^{\omega t} + \mu\, e^{-\omega t})\, \mathrm{e}_3$$

A qualitative analisys of such DPMs can be carried out as follows.

Remark that the antioscillator is the Euler-Lagrange equation associated with the Lagrangian function $L(z, v) := \frac{1}{2}v^2 - V(z)$, with [39]

$$V(z) := -\frac{1}{2}\omega^2 z^2$$

As a consequence, [40] a qualitative portrait of its solutions is exhibited by the time-oriented level lines $H^{-1}(c_o)$ of the Hamiltonian function $H(z, p) := \frac{1}{2}p^2 + V(z)$ (with $p := \frac{\partial L}{\partial v} = v$), described by equation

$$p = \pm\sqrt{2m(c_o - V(z))}$$

Portrait 4 In Fig.4 you will find – as is suggested by the graph of V – the level lines $H^{-1}(c_o)$ in three typical cases.

Let $c_o = 0$.

$H^{-1}(0)$ consists of two straight lines through $(0, 0)$. The singleton $\{(0, 0)\}$ corresponds to the state of rest at the (unique) equilibrium configuration $\mathrm{o} \in Q$, whose instability can be recognized through Chetaiev's criterion or the portrait itself. The open half-lines C_o^1 and C_o^3 correspond on Q to accelerated centrifugal motions, along which the particle asymptotically comes from the repulsive centre o and proceeds towards infinite distance. The open half-lines C_o^2 and C_o^4, on the contrary, correspond on Q to decelerated centripetal motions, along which the particle comes from infinite distance and asymptotically tends to o.

Let $c_o = E > 0$.

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[39] $L(z, v)$ is the coordinate expression of $L(\mathrm{p}, \mathrm{v}) := \frac{1}{2}\mathrm{v}\cdot\mathrm{v} - V(\mathrm{p})$ with $V(\mathrm{p}) := -\frac{1}{2}\omega^2(\mathrm{p} - \mathrm{o})\cdot(\mathrm{p} - \mathrm{o})$, for all $\mathrm{p} = \mathrm{o} + z\mathrm{e}_3 \in Q$ and $\mathrm{v} = v\mathrm{e}_3 \in T_{\mathrm{p}}Q$.

[40] See II. 3.2.2.

$H^{-1}(E)$ is a hyperbola, whose branches C^1_E and C^2_E correspond on Q to motions along which the particle, after a decelerated centripetal phase, arrives at o with non-null velocity and then proceeds in an accelerated centrifugal way.

Let $c_o = E' < 0$.

$H^{-1}(E')$ is a hyperbola, whose branches $C^1_{E'}$ and $C^2_{E'}$ correspond on Q to motions along which the particle, after a decelerated centripetal phase, can only arrive at a point of coordinate $\pm\sqrt{\frac{-2E'}{\omega^2}}$ with null velocity and then is pushed back in an accelerated centrifugal way. □

(ii) For $0 < \alpha < \frac{\pi}{2}$ (inclined axis), we can still write equation $(\dagger)_7$ in the linear form

$$\ddot{\zeta}(t) - \omega^2 \zeta(t) = 0$$

by putting

$$\zeta(t) := z(t) - z^\star$$

(in the present case, $z_\star > 0$).

Then to any maximal solution

$$\zeta(t) = \lambda\, e^{\omega t} + \mu\, e^{-\omega t}$$

there corresponds the DPM

$$\begin{aligned} \mathrm{p}(t) &= \mathrm{o} + z(t)\,\mathrm{e}_3 \\ &= \mathrm{o} + (z_\star + \zeta(t))\,\mathrm{e}_3 \\ &= \mathrm{o} + \left(z_\star + \lambda\, e^{\omega t} + \mu\, e^{-\omega t}\right)\,\mathrm{e}_3 \\ &= \mathrm{o}_\star + \left(\lambda\, e^{\omega t} + \mu\, e^{-\omega t}\right)\,\mathrm{e}_3 \end{aligned}$$

So the combined effect of gravity plus rotation is that of moving the repulsive centre up to $\mathrm{o}_\star = \mathrm{o} + z_\star\,\mathrm{e}_3$ and weakening its repulsive intensity $\omega^2 = w^2 \cos\alpha < w^2$.

(iii) For $\alpha = \frac{\pi}{2}$ (vertical axis), the effects due to the rotation of $\mathcal{E}_3$ vanish at all, since, in such a case, equation $(\dagger)_7$ does not differ from the equation which governs the free fall along a vertical axis in $\mathcal{E}_T$ (i.e. equation $(\dagger)_1$ with $\sin\alpha = 1$).

Circle

Let Q be a circle, of centre $\mathrm{o} \in \mathcal{E}_T$ and radius $l > 0$, of the terrestrial reference space.

We choose an orthogonal Cartesian system $(\mathrm{o}, \mathrm{e}_1, \mathrm{e}_2, \mathrm{e}_3)$ in $\mathcal{E}_{\scriptscriptstyle T}$ s.t.

$$\begin{aligned} Q &= \{\mathrm{p} \in \mathcal{E}_{\scriptscriptstyle T} \mid (\mathrm{p}-\mathrm{o}) \cdot \mathrm{e}_3 = 0\,,\ l^2 - |\mathrm{p}-\mathrm{o}|^2 = 0\} \\ &\subset \mathrm{o} + \mathrm{Span}\,(\mathrm{e}_1, \mathrm{e}_2) \end{aligned}$$

and

$$\mathrm{g} = (g \cos\alpha)\, \mathrm{e}_1 - (g \sin\alpha)\, \mathrm{e}_3$$

with

$$g := |\mathrm{g}|\,, \quad \alpha := \angle(\mathrm{g}, \mathrm{e}_1) \in \left[0, \frac{\pi}{2}\right]$$

$\pi/2 - \alpha$ is said to be the *slope* of Q.

In particular, for $\alpha = 0$, Q lies on a vertical plane (since $\mathrm{g} \cdot \mathrm{e}_3 = 0$, i.e. e_3 is horizontal) and, for $\alpha = \pi/2$, Q lies on a horizontal plane (since $\mathrm{g} = -g\,\mathrm{e}_3$, i.e. e_3 is vertical).
Note that, for any $\alpha > 0$, e_3 points upwards (since $\mathrm{g} \cdot \mathrm{e}_3 < 0$, whence $\angle(\mathrm{g}, \mathrm{e}_3) > \frac{\pi}{2}$).

Then we consider the covering mapping onto Q defined by

$$\xi : \mathbb{R} \to Q : \theta \mapsto \mathrm{p} = \xi(\theta) := \mathrm{o} + (l \cos\theta)\, \mathrm{e}_1 + (l \sin\theta)\, \mathrm{e}_2$$

whence

$$\mathrm{u}(\theta) := \left(\frac{d\mathrm{p}}{d\theta}\right)_\theta = -(l \sin\theta)\, \mathrm{e}_1 + (l \cos\theta)\, \mathrm{e}_2\,, \quad |\mathrm{u}| = l$$

Mathematical pendulum

We start by putting

$$\mathrm{F} := m\,\mathrm{g}$$

whence

$$F_{\mathrm{u}}(\theta) := \frac{1}{l^2}\, m\,\mathrm{g} \cdot \mathrm{u}(\theta) = -\frac{mg\cos\alpha}{l}\, \sin\theta$$

In such a case, a smooth motion

$$\mathrm{p}(t) = \mathrm{o} + (l \cos\theta(t))\, \mathrm{e}_1 + (l \sin\theta(t))\, \mathrm{e}_2 \in Q$$

is the maximal DPM of $\mathcal{T}$, iff its ‘global coordinate expression’

$$\theta(t) \in \mathbb{R}$$

is a maximal solution of Lagrange equation (†), which, in this case, takes the universal form

$$\ddot{\theta}(t) = -\frac{g\cos\alpha}{l}\, \sin\theta(t) \qquad (\dagger)_8$$

and is called *mathematical pendulum.*

(i) For $0 \leq \alpha \neq \pi/2$ (vertical or inclined circle), equation $(\dagger)_8$ is the *non-linear* Euler-Lagrange equation determined by the Lagrangian function $L(\theta, v) := \frac{1}{2}v^2 - V(\theta)$, with [41]

$$V(\theta) := -\frac{g\cos\alpha}{l}\cos\theta$$

As a consequence, [42] the qualitative portrait of its solutions is exhibited by the time-oriented level lines $H^{-1}(c_o)$ of the Hamiltonian function $H(\theta, p) := \frac{1}{2}p^2 + V(\theta)$ (with $p := \partial L/\partial v = v$), described by equation

$$p = \pm\sqrt{2(c_o - V(\theta))}$$

and travelled according to the time-table established by the integral

$$t(\theta_1) - t(\theta_o) = \pm\int_{\theta_o}^{\theta_1}\frac{d\theta}{\sqrt{2(c_o - V(\theta))}}$$

The portrait, in the standard strip $-\pi \leq \theta \leq \pi$, is the following:

Portrait 5 In Fig.5 you will find – as is suggested by the graph of V – the level lines $H^{-1}(c_o)$ in four typical cases:

Let $c_o = E_0 := V(0)$. [43]

$H^{-1}(E_0)$ is the singleton $\{(0,0)\}$, which corresponds to the state of rest at the equilibrium configuration of coordinate $\theta = 0$ (i.e. $\mathrm{p}_0 := \mathrm{o} + l\,\mathrm{e}_1$, the lowest point of Q), whose stability follows from Dirichlet's criterion (or will follow from the portrait itself).

Let $c_o = E_1$ with $E_0 < E_1 < E_\pi := V(\pm\pi)$.

$H^{-1}(E_1)$ is contained in the strip $-\theta_1 \leq \theta \leq \theta_1$ (where $\pm\theta_1$, with $0 < \theta_1 < \pi$, are the points where V takes the value E_1) and corresponds on Q to a 'small' oscillatory motion round the equilibrium configuration p_0 of coordinate $\theta = 0$, along which an oscillation (e.g. from $-\theta_1$ to θ_1) takes the finite time

$$\int_{-\theta_1}^{\theta_1}\frac{d\theta}{\sqrt{2(E_1 - V(\theta))}} = \sqrt{\frac{l}{2g\cos\alpha}}\int_{-\theta_1}^{\theta_1}\frac{d\theta}{\sqrt{\cos\theta - \cos\theta_1}}$$

---

[41] $L(\theta, v)$ (multiplied by l^2) is the coordinate expression of $L(\mathrm{p}, \mathrm{v}) := \frac{1}{2}\mathrm{v}\cdot\mathrm{v} - V(\mathrm{p})$ with $V(\mathrm{p}) := -\mathrm{g}\cdot(\mathrm{p} - \mathrm{o})$, for all $\mathrm{p} = \mathrm{o} + (l\cos\theta)\,\mathrm{e}_1 + (l\sin\theta)\,\mathrm{e}_2 \in Q$ and $\mathrm{v} = v\,\tau(\theta) \in T_{\mathrm{p}}Q = \mathrm{Span}\,\tau(\theta)$.

[42] See II. 3.2.2.

[43] For $c_o < E_0$, $H^{-1}(c_o)$ is empty.

(the above integral converges, since the integrand is infinite of order $1/2$ at $\pm\theta_1$).

Let $c_o = E_\pi$.

$H^{-1}(E_\pi)$ consists of two points $\{(-\pi, 0), (\pi, 0)\}$ and two branches $C^1_{E_\pi}$, $C^2_{E_\pi}$ contained in the open strip $-\pi < \theta < \pi$.
$\{(-\pi, 0), (\pi, 0)\}$ both correspond to the state of rest at the equilibrium configuration of coordinate $\theta = \pm\pi$ (i.e. $\mathrm{p}_\pi := \mathrm{o} - l\,\mathrm{e}_1$, the highest point of Q), whose instability follows from Chetaiev's criterion (or from the portrait itself).
$C^1_{E_\pi}$ or $C^2_{E_\pi}$ correspond on Q to motions which tend, both in the past and in the future, to the above unstable equilibrium configuration in an infinite time

$$\pm\int_{-\pi}^{\pi} \frac{d\theta}{\sqrt{2(E_\pi - V(\theta))}} = \pm\sqrt{\frac{l}{2g\cos\alpha}}\int_{-\pi}^{\pi} \frac{d\theta}{\sqrt{\cos\theta + 1}}$$

(the above integral diverges, since the integrand is infinite of order 1 at $\pm\pi$).

Let $c_o = E_2 > E_\pi$.

$H^{-1}(E_2)$ consists of two branches $C^1_{E_2}$, $C^2_{E_2}$ contained in the strip $-\pi \leq \theta \leq \pi$ (and continued in all of the other similar strips), which corresponds on Q to complete loops performed with never vanishing velocity (and iterated both in the past and in the future). □

Remark If $\theta(t)$ is a 'small' oscillation round the stable equilibrium configuration p_o of coordinate $\theta = 0$, the right hand side of equation $(\dagger)_8$ can be evaluated – along $\theta(t)$ – by neglecting higher-order infinitesimals. Thus we obtain the harmonic oscillator

$$\ddot{\theta}(t) + \omega^2\theta(t) = 0$$

with

$$\omega := \sqrt{\frac{g\cos\alpha}{l}}$$

So the small oscillations of a pendulum round its stable equilibrium configuration p_0 can be regarded as *isochronous* harmonic oscillations on the tangent straight line $\mathrm{p}_0 + T_{\mathrm{p}_0}Q$, all of which exhibiting the period

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{l}{g\cos\alpha}}$$

(ii) For $\alpha = \pi/2$ (horizontal circle), the effects of gravity disappear, for equation $(\dagger)_8$ degenerates into the linear equation

$$\ddot{\theta}(t) = 0$$

whose typical maximal solution

$$t \in \mathbb{R} \mapsto \theta(t) = \theta_o + v_o(t - t_o)$$

(for any $(t_o, \theta_o, v_o) \in \mathbb{R}^3$) just corresponds on Q to a uniform motion (if $v_o \neq 0$) or a state of rest (if $v_o = 0$). Clearly, all of the points of Q are equilibrium configurations, no one of which is stable since, among the DPMs, there do not exist small motions.

Rotating pendulum

Let Q be a circle, of centre $\mathrm{o} \in \mathcal{E}_3$ and radius $l > 0$, in a reference space $\mathcal{E}_3$ uniformly rotating, with respect to $\mathcal{E}\tau$, round an axis passing through the centre o of Q and stationary in a vertical position.

The force of gravity acting in $\mathcal{E}_3$ on a mass m is [44]

$$\mathrm{F}(\mathrm{p}, \mathrm{v}) := m(\mathrm{g} + w^2(\mathrm{p} - \mathrm{p}_*) - 2\mathrm{w} \wedge \mathrm{v})$$

with

$$\mathrm{g} = \text{const.}\,,\ g := |\mathrm{g}| > 0\,;\ \mathrm{w} = -w\,\frac{\mathrm{g}}{g}\,,\ w := |\mathrm{w}| = \text{const.} > 0\,;\ \mathrm{p}_* := \mathrm{o} + \left((\mathrm{p} - \mathrm{o}) \cdot \frac{\mathrm{g}}{g}\right)\frac{\mathrm{g}}{g}$$

Hence, at any $\mathrm{p} = \xi(\theta) = \mathrm{o} + (l\cos\theta)\,\mathrm{e}_1 + (l\sin\theta)\,\mathrm{e}_2 \in Q$ and $\mathrm{v} = v\tau(\theta) \in T_{\mathrm{p}}Q = \operatorname{Span}\tau(\theta)$, [45]

$$F_\tau(\theta) := \frac{1}{l^2}\,m(\mathrm{g} + w^2(\mathrm{p} - \mathrm{p}_*) - 2\mathrm{w} \wedge \mathrm{v}) \cdot \tau(\theta) = -\frac{g\cos\alpha}{l}\,\sin\theta + (w^2\cos^2\alpha)\,\sin\theta\cos\theta$$

In such a case, a smooth motion

$$\mathrm{p}(t) = \mathrm{o} + (l\cos\theta(t))\,\mathrm{e}_1 + (l\sin\theta(t))\,\mathrm{e}_2 \in Q$$

is the maximal DPM of $\mathcal{T}$, iff its ‘global coordinate expression’

$$\theta(t) \in \mathbb{R}$$

is a maximal solution of Lagrange equation (†), which, in this case, takes the universal form

$$\ddot{\theta}(t) = -\frac{g\cos\alpha}{l}\,\sin\theta(t) + (w^2\cos^2\alpha)\,\sin\theta(t)\cos\theta(t) \qquad (\dagger)_9$$

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[44] See 2.1.2 (footnote [18]) and Appendix 3.1.

[45] We choose an orthogonal Cartesian system $(\mathrm{o}, \mathrm{e}_1, \mathrm{e}_2, \mathrm{e}_3)$ in $\mathcal{E}_3$ in the same way as we did in subsection ‘Circle’.

(i) For $0 \leq \alpha \neq \pi/2$ (vertical or inclined circle), equation $(\dagger)_9$ is the *non-linear* Euler-Lagrange equation determined by the Lagrangian function $L(\theta, v) := \frac{1}{2}mv^2 - V(\theta)$, where

$$V(\theta) := -b_\alpha^2 \cos\theta + \frac{1}{2} w_\alpha^2 \cos^2\theta$$

with

$$b_\alpha := \sqrt{\frac{g \cos\alpha}{l}}\,,\ w_\alpha := w \cos\alpha$$

(remark that, on $[-\pi, \pi]$, the first derivative

$$\frac{dV}{d\theta} = (b_\alpha^2 - w_\alpha^2 \cos\theta) \sin\theta$$

vanishes at $\theta = 0, \pm\pi$ and, if $w_\alpha > b_\alpha$, at two more points $\pm\theta_s := \pm \arccos{(b_\alpha/w_\alpha)}^2$ with $0 < \theta_s < \pi/2$).

As a consequence, [46] the qualitative portrait of its solutions is exhibited by the time-oriented level lines $H^{-1}(c_o)$ of the Hamiltonian function $H(\theta, p) := \frac{1}{2}p^2 + V(\theta)$ (with $p := \partial L/\partial v = v$).

The portrait, in the standard strip $-\pi \leq \theta \leq \pi$, is the following:

Portrait 6 If $w_\alpha \leq b_\alpha$, the graph of V is qualitatively similar to the graph shown in Fig.5 and then the portrait does not differ from the one therein drawn. The centrifugal force field due to the rotation of $\mathcal{E}_3$ in $\mathcal{E}_T$ is *not* strong enough to destroy the ordinary effets of gravity on the pendulum.

If, on the contrary, the value of w_α lies beyond the critical value or *bifurcation* point b_α, i.e. $w_\alpha > b_\alpha$, the centrifugal force *is* strong enough to produce deep changes in the graph of V and therefore in the portrait of the pendulum, as is shown in Fig.6 (where we put $E_s := V(\pm\theta_s)$, $E_0 := V(0)$ and $E_\pi := V(\pm\pi)$).

Let $c_o = E_s$. [47]
$H^{-1}(E_s)$ consists of two points $\{(-\theta_s, 0), (\theta_s, 0)\}$, which correspond on Q to the state of rest at the equilibrium configurations of coordinates $-\theta_s$ and θ_s, respectively, whose stability follows from Dirichlet's criterion (or will follow from the portrait itself).

Let $c_o = E'$, with $E_s < E' < E_0$.
$H^{-1}(E')$ splits into disjoint loops, which correspond on Q to 'small' oscillatory motions about the equilibrium configurations of coordinates $-\theta_s$ and θ_s, respectively.

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[46] See II. 3.2.2.

[47] For $c_o < E_s$, $H^{-1}(c_o)$ is empty.

Let $c_o = E_0$.
$H^{-1}(E_0)$ consists of an 'eight', whose branches both pass through the singleton $\{(0,0)\}$. The latter corresponds on Q to the state of rest at the equilibrium configuration of coordinate $\theta = 0$, whose instability follows from Chetaiev's criterion. Any one of the two remaining branches corresponds on Q to a motion which asimptotically tends, both in the past and in the future, to the above unstable equilibrium configuration.

Let $c_o = E_1$ with $E_0 < E_1 < E_\pi$, or $c_o = E_\pi$, or $c_o = E_2$ with $E_\pi < E_2$.
$H^{-1}(E_1)$, $H^{-1}(E_\pi)$ and $H^{-1}(E_2)$ exhibit the same features as in Fig.5.

(ii) For $\alpha = \pi/2$ (horizontal circle), the effects of gravity disappear, for equation $(\dagger)_9$ degenerate into the linear equation $\ddot{\theta}(t) = 0$. [48]

2 degrees of freedom

We shall now consider a test particle with two degrees of freedom in the terrestrial (or a moving) reference space, that is to say, a mechanical system

$$\mathcal{T} := (Q, m, \mathrm{F})$$

whose configuration space Q is a smooth surface –i.e. a 2-dimensional, connected manifold– of the given reference space.

Recall that, if

$$\xi : X \subset \mathbb{R}^2 \to Q : q = (q^1, q^2) \mapsto \mathrm{p} = \xi(q) = \xi(q^1, q^2)$$

is a local chart (or, possibly, a global chart or a covering mapping) of Q, then, at any point $\mathrm{p} = \xi(q)$ of the coordinate domain $\mathcal{U} := \operatorname{Im}\xi$, one has

$$T_{\mathrm{p}}Q = \operatorname{Im} d_q\xi = \operatorname{Span}\left(\frac{\partial \mathrm{p}}{\partial q^1}, \frac{\partial \mathrm{p}}{\partial q^2}\right)_q$$

Moreover, any (admissible) smooth motion $\mathrm{p}(t) \in \mathcal{U}$ admits a C^∞ coordinate expression $q(t) \in X$ through ξ, i.e.

$$\mathrm{p}(t) = \xi(q(t))$$

whence

$$\dot{\mathrm{p}}(t) = \dot{q}^i(t)\left(\frac{\partial \mathrm{p}}{\partial q^i}\right)_{q(t)}$$

---

[48] See 'Mathematical pendulum' (ii).

and

$$\ddot{\mathrm{p}}(t) = \dot{q}^i(t) \left(\frac{\partial \mathrm{p}}{\partial q^i}\right)_{q(t)} + \dot{q}^i(t)\dot{q}^j(t) \left(\frac{\partial^2 \mathrm{p}}{\partial q^i \partial q^j}\right)_{q(t)}$$

Such a motion $\mathrm{p}(t) \in \mathcal{U}$ satisfies d'Alembert's principle

$$\mathrm{F}(t, \mathrm{p}(t), \dot{\mathrm{p}}(t)) - m\,\ddot{\mathrm{p}}(t) \in T^{\perp}_{\mathrm{p}(t)} Q$$

iff

$$(\mathrm{F}(t, \mathrm{p}(t), \dot{\mathrm{p}}(t) - m\,\ddot{\mathrm{p}}(t)) \cdot \left(\frac{\partial \mathrm{p}}{\partial q^h}\right)_{q(t)} = 0\ , \quad h = 1, 2$$

So, $\mathrm{p}(t) \in \mathcal{U}$ is a DPM of $\mathcal{T}$, iff its coordinate expression $q(t) \in W$ is a solution of the couple of (local) Lagrange equations

$$m\,\ddot{\mathrm{p}}(t) \cdot \left(\frac{\partial \mathrm{p}}{\partial q^h}\right)_{q(t)} = \mathrm{F}(t, \mathrm{p}(t), \dot{\mathrm{p}}(t)) \cdot \left(\frac{\partial \mathrm{p}}{\partial q^h}\right)_{q(t)} \tag{‡}$$

where $\mathrm{p}(t)$, $\dot{\mathrm{p}}(t)$ and $\ddot{\mathrm{p}}(t)$ are meant to be given the above coordinate expressions.

If F keeps orthogonal to Q, d'Alembert's principle takes the universal form

$$\ddot{\mathrm{p}}(t) \in T^{\perp}_{\mathrm{p}(t)} Q$$

to which there corresponds the couple of (local) Euler-Lagrange equations

$$\ddot{\mathrm{p}}(t) \cdot \left(\frac{\partial \mathrm{p}}{\partial q^h}\right)_{q(t)} = 0$$

characterizing the geodesic curves of (Q, K) (K being the quadratic form – semi-square of norm – induced on Q by the Euclidean geometry of the environment). [49] The orbits of the above curves – excluding those that degenerate into singletons – are simply said to be the *geodesics* of the surface Q.

The examples in the sequel will take place on simple types of smooth surface (plane, cilinder, halph-cone and sphere).

Plane

Let Q be a plane of the terrestrial reference space $\mathcal{E}_T$.

We choose an orthogonal Cartesian system $(\mathrm{o}, \mathrm{e}_1, \mathrm{e}_2, \mathrm{e}_3)$ in $\mathcal{E}_T$ s.t.

$$\begin{aligned} Q &= \{\mathrm{p} \in \mathcal{E}_T \mid (\mathrm{p} - \mathrm{o}) \cdot \mathrm{e}_1 = 0\} \\ &= \mathrm{o} + \mathrm{Span}\,(\mathrm{e}_2, \mathrm{e}_3) \end{aligned}$$

---

[49] See II. 3.1.2.

and

$$\mathrm{g} = (g\cos\alpha)\,\mathrm{e}_1 - (g\sin\alpha)\,\mathrm{e}_3$$

with

$$g := |\mathrm{g}|\,, \quad \alpha := \angle(\mathrm{g}, \mathrm{e}_1) \in \left[0, \frac{\pi}{2}\right]$$

α is said to be the *slope* of Q.
In particular, for $\alpha = 0$, Q is horizontal (since $\mathrm{g} = g\,\mathrm{e}_1$, i.e. e_1 is vertical) and, for $\alpha = \frac{\pi}{2}$, Q is vertical (since $\mathrm{g}\cdot\mathrm{e}_1 = 0$, i.e. e_1 is horizontal).
Note that, for any $\alpha > 0$, e_3 points upwards (since $\mathrm{g}\cdot\mathrm{e}_3 < 0$, whence $\angle(\mathrm{g}, \mathrm{e}_3) > \frac{\pi}{2}$).

Then we consider the global chart of Q, defining orthogonal Cartesian coordinates, given by

$$\xi : \mathbb{R}^2 \to Q : (y, z) \mapsto \mathrm{p} = \xi(z) := \mathrm{o} + y\,\mathrm{e}_2 + z\,\mathrm{e}_3$$

whence

$$\frac{\partial \mathrm{p}}{\partial y} = \mathrm{e}_2\,, \quad \frac{\partial \mathrm{p}}{\partial z} = \mathrm{e}_3$$

Falls along a sloping plane

We start by putting

$$\mathrm{F} := m\,\mathrm{g}$$

In such a case, a smooth motion

$$\mathrm{p}(t) = \mathrm{o} + y(t)\,\mathrm{e}_2 + z(t)\,\mathrm{e}_3 \in Q$$

is the maximal DPM of $\mathcal{T}$ satisfying, at a time $t_o \in \mathrm{R}$, initial conditions

$$\mathrm{p}(t_o) = \mathrm{p}_o := \mathrm{o} + y_o\,\mathrm{e}_2 + z_o\,\mathrm{e}_3 \in Q\,, \quad \dot{\mathrm{p}}(t_o) = \mathrm{v}_o := v_{oy}\mathrm{e}_2 + v_{oz}\mathrm{e}_3 \in T_{\mathrm{p}_o}Q = \mathrm{Span}\,(\mathrm{e}_2, \mathrm{e}_3)$$

iff its global coordinate expression

$$(y(t), z(t)) \in \mathbb{R}^2$$

is the maximal solution of Lagrange equations (‡), which now take the universal form

$$\ddot{y}(t) = 0\,, \quad \ddot{z}(t) = -g\sin\alpha \tag{‡$_1$}$$

and satisfies the initial conditions

$$(y(t_o), z(t_o)) = (y_o, z_o)\,, \quad (\dot{y}(t_o), \dot{z}(t_o)) = (v_{oy}, v_{oz})$$

Hence, by quadratures, we obtain (for all $t \in \mathbb{R}$)

$$\dot{y}(t) = v_{oy}\,, \quad \dot{z}(t) = v_{oz} - (g \sin\alpha)(t - t_o)$$

and finally

$$y(t) = y_o + v_{oy}(t - t_o)\,, \quad z(t) = z_o + v_{oz}(t - t_o) - \frac{1}{2}(g\sin\alpha)(t - t_o)^2$$

The correspondent DPM $\mathrm{p}(t)$ is then as follows:

(i) For $\alpha \neq 0$ (non-horizontal plane),

$$\begin{aligned} \mathrm{p}(t) &= \mathrm{o} + (y_o + v_{oy}(t - t_o))\,\mathrm{e}_2 + \left(z_o + v_{oz}(t - t_o) - \frac{1}{2}(g\sin\alpha)(t - t_o)^2\right)\mathrm{e}_3 \\ &= \mathrm{p}_o + \mathrm{v}_o(t - t_o) + \frac{1}{2}\,\mathrm{k}\,(t - t_o)^2\,, \quad \mathrm{k} := -(g\sin\alpha)\,\mathrm{e}_3 \end{aligned}$$

exhibits the same qualitative features as a free fall in the vacuum.

(ii) For $\alpha = 0$ (horizontal plane),

$$\begin{aligned} \mathrm{p}(t) &= \mathrm{o} + (y_o + v_{oy}(t - t_o))\,\mathrm{e}_2 + (z_o + v_{oz}(t - t_o))\,\mathrm{e}_3 \\ &= \mathrm{p}_o + \mathrm{v}_o(t - t_o) \end{aligned}$$

is just a uniform rectilinear motion if $\mathrm{v}_o \neq 0$, or degenerates into a state of rest if $\mathrm{v}_o = 0$.

Geodesics of a plane

In the latter case, owing to horizontality, the efffects of gravity disappear and the DPMs coincide with the inertial motions, that is to say, with the geodesic curves of the plane. So the geodesics of the plane, i.e. the orbits of its non-degenerate geodesic curves, are the straight lines.

Double harmonic oscillator

Consider the case of an *elastic force* acting (as well as the weight) on the particle, i.e.

$$\mathrm{F}(\mathrm{p}) := m\,\mathrm{g} - k\,(\mathrm{p} - \mathrm{o})$$

with

$$k > 0\,, \quad \mathrm{o} \in Q$$

An admissible motion

$$\mathrm{p}(t) = \mathrm{o} + y(t)\,\mathrm{e}_2 + z(t)\,\mathrm{e}_3 \in Q$$

is the DPM of $\mathcal{T}$ satisfying, at a time $t_o \in I$, initial conditions

$$\mathrm{p}(t_o) = \mathrm{p}_o := \mathrm{o} + y_o\,\mathrm{e}_2 + z_o\,\mathrm{e}_3 \in Q\,, \quad \dot{\mathrm{p}}(t_o) = \mathrm{v}_o := v_{oy}\mathrm{e}_2 + v_{oz}\mathrm{e}_3 \in T_{\mathrm{p}_o}Q = \mathrm{Span}\,(\mathrm{e}_2, \mathrm{e}_3)$$

iff its global coordinate expression

$$(y(t), z(t)) \in \mathbb{R}^2$$

is the maximal solution of Lagrange equations (‡), which now take the form of a *double harmonic oscillator*

$$m\,\ddot{y}(t) = -k\,y(t)\,, \quad m\,\ddot{z}(t) = -mg\sin\alpha - k\,z(t) \qquad (\ddagger)_2$$

that is,

$$\ddot{y}(t) + \omega^2 y(t) = 0\,, \quad \ddot{\zeta}(t) + \omega^2\zeta(t) = 0$$

with

$$\omega := \sqrt{\frac{k}{m}}\,, \quad \zeta(t) := z(t) - z_*\,, \quad z_* := -\frac{mg\sin\alpha}{k} < 0$$

and satisfies the initial conditions

$$(y(t_o), z(t_o)) = (y_o, z_o)\,, \quad (\dot{y}(t_o), \dot{z}(t_o)) = (v_{oy}, v_{oz})$$

Check that such a maximal solution is given (for all $t \in \mathbb{R}$) by

$$y(t) = y_o\cos\omega(t-t_o) + \frac{v_{oy}}{\omega}\sin\omega(t-t_o)\,, \quad z(t) - z_* = (z_o - z_*)\cos\omega(t-t_o) + \frac{v_{oz}}{\omega}\sin\omega(t-t_o)$$

The correspondent DPM is then given by

$$\begin{aligned}
\mathrm{p}(t) &= \mathrm{o} + (y_o\cos\omega(t-t_o) + \frac{v_{oy}}{\omega}\sin\omega(t-t_o))\,\mathrm{e}_2 + \Big(z_* + (z_o - z_*)\cos\omega(t-t_o) + \frac{v_{oz}}{\omega}\sin\omega(t-t_o)\Big) \\
&= \mathrm{o}_* + (\,\cos\omega(t-t_o))(\mathrm{p}_o - \mathrm{o}_*) + \left(\frac{1}{\omega}\sin\omega(t-t_o)\right)\mathrm{v}_o
\end{aligned}$$

with

$$\mathrm{o}_* := \mathrm{o} + z_*\mathrm{e}_3$$

(i) If

$$\mathrm{p}_o = \mathrm{o}_*\,, \quad \mathrm{v}_o = 0$$

then $\mathrm{p}(t)$ degenerates into a state of rest, namely

$$\mathrm{p}(t) = \mathrm{o}_*$$

Check that o_* is the unique equilibrium configuration, whose stability can be recognized by applying Dirichlet's criterion to the potential energy $V(y,z) := \frac{1}{2}k(y^2+z^2)+(mg\sin\alpha)z$. [50]

(ii) If

$$\mathrm{p}_o \neq \mathrm{o}_* \ , \quad (\mathrm{p}_o - \mathrm{o}_*)\wedge \mathrm{v}_o = 0$$

i.e. $\mathrm{p}_o - \mathrm{o}_* = s_o\mathrm{u}$ and $\mathrm{v}_o = v_o\mathrm{u}$ with $s_o \neq 0$ and $|\mathrm{u}| = 1$, then $\mathrm{p}(t)$ is a harmonic oscillatory motion along the straight line $\mathrm{o}_* + \mathrm{Span}\,(\mathrm{u})$, namely

$$\mathrm{p}(t) = \mathrm{o}_* + s(t)\mathrm{u}$$

where

$$s(t) = \Big(s_o \cos\omega(t-t_o) + \frac{v_o}{\omega}\sin\omega(t-t_o)\Big)\,\mathrm{u}$$

is the maximal solution of the harmonic oscillator

$$\ddot{s}(t) + \omega^2 s(t) = 0$$

determined by the initial conditions

$$s(t_o) = s_o \ , \quad \dot{s}(t_o) = v_o$$

(iii) If

$$\mathrm{p}_o \neq \mathrm{o}_* \ , \quad (\mathrm{p}_o - \mathrm{o}^*)\wedge \mathrm{v}_o \neq 0$$

then the periodic motion $\mathrm{p}(t)$ could be seen (through some boring calculations) to describe an elliptical orbit, centred at o^* and travelled with constant areal velocity.

Loop the loop

We shall now deal with the case of a particle which, owing to a 'non-strict' one-sided constraint [51] (as well as a two-sided constraint), is only allowed to stay within a closed disc lying on a non-horizontal plane of the terrestrial reference space.

We shall show that, among the DPMs of such a particle under tha action of $\mathrm{F} = m\,\mathrm{g}$, there are some describing iterated loops along the the (circular) boundary of the disc on the plane (like in the acrobatic exercise called 'loop the loop').

---

[50] $V(y,z)$ is the coordinate expression of $V(\mathrm{p}) = \frac{1}{2}k(\mathrm{p}-\mathrm{o})\cdot(\mathrm{p}-\mathrm{o}) - m\mathrm{g}\cdot(\mathrm{p}-\mathrm{o})$ for all $\mathrm{p}\in Q$.

[51] See II. 2.2.2, d'Alembert's inequality.

We choose an orthogonal Cartesian system $(\mathrm{o}, \mathrm{e}_1, \mathrm{e}_2, \mathrm{e}_3)$ in $\mathcal{E}_{T}$ in such a way that the above closed disc of centre $\mathrm{o} \in \mathcal{E}_{T}$ and radius $l > 0$ is described by [52]

$$\begin{aligned} Q &= \{\mathrm{p} \in \mathcal{E}_{T} \mid (\mathrm{p}-\mathrm{o}) \cdot \mathrm{e}_3 = 0\,, \quad l^2 - |\mathrm{p}-\mathrm{o}|^2 \geq 0\} \\ &\subset \mathrm{o} + \mathrm{Span}\,(\mathrm{e}_1, \mathrm{e}_2) \end{aligned}$$

with

$$\mathrm{g} = (g \cos\alpha)\,\mathrm{e}_1 - (g \sin\alpha)\,\mathrm{e}_3$$

and

$$g := |\mathrm{g}|\,, \quad \alpha := \angle(\mathrm{g}, \mathrm{e}_1) \in \left[0, \frac{\pi}{2}\right)$$

We shall also make use of the usual covering mapping onto the circle $\partial Q \subset Q$ of centre o and radius l, i.e.

$$\xi : \mathbb{R} \to \partial Q : \theta \mapsto \mathrm{p} = \xi(\theta) := \mathrm{o} + (l \cos\theta)\,\mathrm{e}_1 + (l \sin\theta)\,\mathrm{e}_2$$

whence we obtain, for any $\theta \in \mathbb{R}$, two mutually orthogonal, non-null vectors

$$\tau(\theta) := \left(\frac{d\mathrm{p}}{d\theta}\right)_\theta = -(l \sin\theta)\,\mathrm{e}_1 + (l \cos\theta)\,\mathrm{e}_2\,, \quad |\tau(\theta)| = l$$

and

$$\mathrm{n}(\theta) := \left(\frac{d\tau}{d\theta}\right)_\theta = -(l \cos\theta)\,\mathrm{e}_1 - (l \sin\theta)\,\mathrm{e}_2\,, \quad |\mathrm{n}(\theta)| = l$$

which therefore satisfy

$$\mathrm{Span}\,(\tau(\theta), \mathrm{n}(\theta)) = \mathrm{Span}\,(\mathrm{e}_1, \mathrm{e}_2)$$

At any $\mathrm{p} = \xi(\theta) \in \partial Q$, we have [53]

$$T_{\mathrm{p}}Q \;:=\; \{\delta\mathrm{p} \in E_{T} \mid \delta\mathrm{p} \cdot \mathrm{e}_3 = 0\,,\; -\delta\mathrm{p} \cdot (\mathrm{p}-\mathrm{o}) \geq 0\}$$

---

[52] See subsection ‘Circle’.

[53] Recall that Q is characterized by conditions

$$f(\mathrm{p}) = 0\,, \quad g(\mathrm{p}) \geq 0$$

with

$$f := \psi \cdot \mathrm{e}_3 : \mathrm{p} \in \mathcal{E}_{T} \mapsto \psi(\mathrm{p}) \cdot \mathrm{e}_3 \in \mathbb{R}\,, \quad g := l^2 - \psi \cdot \psi : \mathrm{p} \in \mathcal{E}_{T} \mapsto l^2 - \psi(\mathrm{p}) \cdot \psi(\mathrm{p}) \in \mathbb{R}$$

where $\psi : \mathrm{p} \in \mathcal{E}_{T} \mapsto \psi(\mathrm{p}) := \mathrm{p} - \mathrm{o} \in E_{T}$ is an affinity s.t. $d_{\mathrm{p}}\psi = \mathrm{id}_{\mathrm{E_T}}$. As a consequence, for any $\mathrm{p} = \xi(\theta) \in \partial Q$, $T_{\mathrm{p}}Q$ is characterized by conditions

$$d_{\mathrm{p}}f(\delta\mathrm{p}) = 0\,, \quad d_{\mathrm{p}}g(\delta\mathrm{p}) \geq 0$$

with

$$d_{\mathrm{p}}f = d_{\mathrm{p}}\psi \cdot \mathrm{e}_3 : \delta\mathrm{p} \in E_{T} \mapsto \delta\mathrm{p} \cdot \mathrm{e}_3 \in \mathbb{R}\,, \quad d_{\mathrm{p}}g = -2\, d_{\mathrm{p}}\psi \cdot \psi(\mathrm{p}) : \delta\mathrm{p} \in E_{T} \mapsto -2\,\delta\mathrm{p} \cdot (\mathrm{p}-\mathrm{o}) \in \mathbb{R}$$

We shall also put $\mathbb{R}^+_\circ := \{b \in \mathbb{R} \mid b \geq 0\}$.

$$\begin{aligned} &= \{\delta\mathrm{p} \in \operatorname{Span}(\tau(\theta), \mathrm{n}(\theta)) \mid \delta\mathrm{p}\cdot\mathrm{n}(\theta) \geq 0\} \\ &= \{a\,\tau(\theta) + b\,\mathrm{n}(\theta)\}_{(a,b)\in\mathbb{R}\times\mathbb{R}^+_\circ} \end{aligned}$$

Let

$$\mathrm{p}(t) = \xi(\theta(t)) = (l\cos\theta(t))\mathrm{e}_1 + (l\sin\theta(t))\mathrm{e}_2$$

be a smooth motion along ∂Q, whose derivatives are expressed by

$$\dot{\mathrm{p}}(t) = \dot\theta(t)\tau(\theta(t)) \ , \quad \ddot{\mathrm{p}}(t) = \ddot\theta(t)\tau(\theta(t)) + \dot\theta^2(t)\mathrm{n}(\theta(t))$$

Such a $\mathrm{p}(t) = \xi(\theta(t))$ is a DPM of $\mathcal{T}$, iff it satisfies d'Alembert's inequality

$$(m\,\mathrm{g} - m\,\ddot{\mathrm{p}}(t))\cdot\delta\mathrm{p} \leq 0 \ , \quad \forall\delta\mathrm{p}\in T_{\mathrm{p}(t)}Q$$

that is to say,

$$(\mathrm{g} - \ddot{\mathrm{p}}(t))\cdot(a\,\tau(\theta) + b\,\mathrm{n}(\theta)) \leq 0 \ , \quad \forall(a,b)\in\mathbb{R}\times\mathbb{R}^+_\circ$$

wich is equivalent to

$$\begin{aligned} (\mathrm{g} - \ddot{\mathrm{p}}(t))\cdot\tau((\theta(t)) &= 0 \\ (\mathrm{g} - \ddot{\mathrm{p}}(t))\cdot\mathrm{n}((\theta(t)) &\leq 0 \end{aligned}$$

that is,

$$\ddot\theta(t) = -\frac{g\cos\alpha}{l}\,\sin\theta(t) \qquad (\diamond)_1$$

$$\dot\theta^2(t) + \frac{g\cos\alpha}{l}\,\cos\theta(t) \geq 0 \qquad (\diamond)_2$$

Now consider a solution $\theta(t)$ (for all $t\in\mathbb{R}$) of mathematical pendulum $(\diamond)_1$ with a high energy, say

$$E \geq \frac{3}{2}E_\pi = \frac{3}{2}\frac{g\cos\alpha}{l}$$

Such a function satisfies inequality $(\diamond)_2$ too, since the energy conservation law

$$\frac{1}{2}\dot\theta^2(t) - \frac{g\cos\alpha}{l}\cos\theta(t) = E$$

implies

$$\dot\theta^2(t) \geq \frac{2g\cos\alpha}{l}\cos\theta(t) + \frac{3g\cos\alpha}{l}$$

whence

$$\dot\theta^2(t) + \frac{g\cos\alpha}{l}\,\cos\theta(t) \geq \frac{3g\cos\alpha}{l}(\,\cos\theta(t)+1) \geq 0$$

As a consequence, the correspondent motion $\mathrm{p}(t) = \xi(\theta(t))$, which describes iterated loops along ∂Q, [54] is a DPM of $\mathcal{T}$.

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[54] See Portrait 5 in subsection 'Mathematical pendulum.

Cylinder

Let Q be a cylinder of axis $\mathrm{o}+\operatorname{Span}\mathrm{e}_3 \subset \mathcal{E}_{\scriptscriptstyle T}$ and radius $l > 0$ in the terrestrial reference space $\mathcal{E}_{\scriptscriptstyle T}$ i.e. [55]

$$Q = \{\mathrm{p} \in \mathcal{E}_{\scriptscriptstyle T} \mid |\mathrm{p}-\mathrm{o}|^2 - ((\mathrm{p}-\mathrm{o})\cdot \mathrm{e}_3)^2 - l^2 = 0\}$$

We choose an orthogonal Cartesian system $(\mathrm{o}, \mathrm{e}_1, \mathrm{e}_2, \mathrm{e}_3)$ in $\mathcal{E}_{\scriptscriptstyle T}$ s.t.

$$\mathrm{g} = (g\cos\alpha)\,\mathrm{e}_1 - (g\sin\alpha)\,\mathrm{e}_3$$

with

$$g := |\mathrm{g}|\ , \quad \alpha := \angle(\mathrm{g}, \mathrm{e}_1) \in \left[0, \frac{\pi}{2}\right]$$

As is known, α is the *slope* of the axis $\mathrm{o}+\operatorname{Span}\mathrm{e}_3$ of Q. [56]

We shall also make use of the covering mapping, defining *cylindrical coordinates* onto Q, given by

$$\xi : \mathbb{R}^2 \to Q : (\theta, z) \mapsto \mathrm{p} = \xi(\theta, z) := \mathrm{o} + (l\cos\theta)\,\mathrm{e}_1 + (l\sin\theta)\,\mathrm{e}_2 + z\,\mathrm{e}_3$$

For any $\mathrm{p} = \xi(\theta, z) \in Q$, we have

$$T_{\mathrm{p}}Q = \operatorname{Span}(\tau(\theta), \mathrm{e}_3)$$

since

$$\left(\frac{\partial \mathrm{p}}{\partial \theta}\right)_{(\theta,z)} = \tau(\theta) := -(l\sin\theta)\,\mathrm{e}_1 + (l\cos\theta)\,\mathrm{e}_2$$

and

$$\left(\frac{\partial \mathrm{p}}{\partial z}\right)_{(\theta,z)} = \mathrm{e}_3$$

Falls along a sloping cylinder

Put

$$\mathrm{F} := m\,\mathrm{g}$$

Let

$$\mathrm{p}(t) = \xi(\theta(t), z(t)) = \mathrm{o} + (l\cos\theta(t))\,\mathrm{e}_1 + (l\sin\theta(t))\,\mathrm{e}_2 + z(t)\,\mathrm{e}_3$$

be a smooth motion along Q, whose derivatives are expressed by

$$\dot{\mathrm{p}}(t) = \dot\theta(t)\,\tau(\theta(t)) + \dot z(t)\,\mathrm{e}_3\ , \quad \ddot{\mathrm{p}}(t) = \ddot\theta(t)\,\tau(\theta(t)) + \dot\theta^2(t)\,\left(\frac{d\tau}{d\theta}\right)_{\theta(t)} + \ddot z(t)\,\mathrm{e}_3$$

---

[55] See I. 5.2.2, Example 9.

[56] See subsection ‘Straight line’.

Such a $\mathrm{p}(t)$ is a DPM of $\mathcal{T}$, iff its ‘global coordinate expression’

$$(\theta(t), z(t)) \in \mathbb{R}^2$$

is a solution of Lagrange equations (‡), which now take the universal form

$$\ddot{\mathrm{p}}(t) \cdot \tau(\theta(t)) = \mathrm{g} \cdot \tau(\theta(t))$$

$$\ddot{\mathrm{p}}(t) \cdot \mathrm{e}_3 = \mathrm{g} \cdot \mathrm{e}_3$$

that is,

$$\ddot{\theta}(t) = -\frac{g \cos\alpha}{l} \sin\theta(t) \qquad (\ddagger)_3$$

$$\ddot{z}(t) = -g \sin\alpha \qquad (\ddagger)_4$$

Equations $(\ddagger)_{3,4}$ separate the variables $\theta(t)$ and $z(t)$.

As a consequence initial data

$$t_o \in \mathbb{R}\ , \quad \mathrm{p}_o = \mathrm{o}+(l\cos\theta_o)\,\mathrm{e}_1+(l\sin\theta_o)\,\mathrm{e}_2+z_o\,\mathrm{e}_3 \in Q\ , \quad \mathrm{v}_o = v_{o\theta}\,\tau(\theta_o)+v_{oz}\,\mathrm{e}_3 \in T_{\mathrm{p}_o}Q$$

determine the maximal DPM given, for all $t \in \mathbb{R}$, by

$$\mathrm{p}(t) = \mathrm{p}_o + l(\cos\theta(t) - \cos\theta_o)\,\mathrm{e}_1 + l(\sin\theta(t) - \sin\theta_o)\,\mathrm{e}_2 + (z(t) - z_o)\,\mathrm{e}_3$$

where $\theta(t)$ denotes the maximal solution of equation $(\ddagger)_3$ with initial conditions $(\theta(t_o) = \theta_o,\ \dot{\theta}(t_o) = v_{o\theta})$, and $z(t)$ denotes the maximal solution of equation $(\ddagger)_4$ with initial conditions $(z(t_o) = z_o,\ \dot{z}(t_o) = v_{oz})$.

The above DPM is the composition

$$\mathrm{p}(t) - \mathrm{p}_o = (\mathrm{p}_\theta(t) - \mathrm{p}_o) + (\mathrm{p}_z(t) - \mathrm{p}_o)$$

of a ‘mathematical pendulum motion’

$$\begin{aligned} \mathrm{p}_\theta(t) \ &:= \ \mathrm{p}_o + l(\cos\theta(t) - \cos\theta_o)\,\mathrm{e}_1 + l(\sin\theta(t) - \sin\theta_o)\,\mathrm{e}_2 \\ &= \ \mathrm{p}_{oz} + (l\cos\theta(t))\,\mathrm{e}_1 + (l\sin\theta(t))\,\mathrm{e}_2\ , \qquad \mathrm{p}_{oz} := \mathrm{o} + z_o\,\mathrm{e}_3 \end{aligned}$$

along the generatrix $(\mathrm{p}_{oz} + \mathrm{Span}\,(\mathrm{e}_1, \mathrm{e}_2)) \cap Q$, and a ‘rectilinear motion’

$$\begin{aligned} \mathrm{p}_z(t) \ &:= \ \mathrm{p}_o + (z(t) - z_o)\,\mathrm{e}_3 \\ &= \ \mathrm{p}_{o\theta} + z(t)\,\mathrm{e}_3\ , \qquad \mathrm{p}_{o\theta} := \mathrm{o} + (l\cos\theta_o)\,\mathrm{e}_1 + (l\sin\theta_o)\,\mathrm{e}_2 \end{aligned}$$

along the directrix $\mathrm{p}_{o\theta} + \mathrm{Span}\,(\mathrm{e}_3) \subset Q$.

Check that

(i) owing to $(\ddagger)_4$, the DPM is a mathematical pendulum motion

$$\mathrm{p}(t) = \mathrm{p}_\theta(t)$$

(i.e. $z(t) = z_o$), iff $\alpha = 0$ (horizontal cylinder) and $v_{oz} = 0$;

(ii) owing to $(\ddagger)_3$, the DPM is a rectilinear motion

$$\mathrm{p}(t) = \mathrm{p}_z(t)$$

(i.e. $\theta(t) = \theta_o$), iff $v_{o\theta} = 0$ in the case of $\alpha = \pi/2$ (vertical cylinder), or $\theta_o \equiv 0 \ (\mathrm{mod.}\pi)$ and $v_{o\theta} = 0$ in the case of $\alpha \neq \pi/2$ (non-vertical cylinder);

(iii) in particular, the DPM degenerates into a state of rest

$$\mathrm{p}(t) = \mathrm{p}_o$$

(i.e. $\theta(t) = \theta_o$, $z(t) = z_o$) and then p_o is an (unstable) equilibrium configuration, iff $\alpha = 0$ (horizontal cylinder), $\theta_o \equiv 0 \ (\mathrm{mod.}\pi)$ and $v_{o\theta} = v_{oz} = 0$.

Geodesics of a cylinder

In the above problem, the DPMs of a particle on a cylinder (however sloping) of $\mathcal{E}_T$ are always perturbed, with respect to the inertial motions, by the terrestrial gravity. So, if we want the effect of gravity to disappear, we should consider a cylinder in a freely falling reference space $\mathcal{E}_3$, where the field of gravity (locally) vanishes. [57]

In such a space, owing to $\mathrm{F} = 0$, Lagrange equations $(\ddagger)$ read (in (θ, z)-coordinates)

$$\ddot{\theta}(t) = 0\,, \quad \ddot{z}(t) = 0$$

To their solutions

$$\theta(t) = \theta_o + v_{o\theta}(t - t_o)\,, \quad z(t) = z_o + v_{oz}(t - t_o)$$

there correspond the inertial motions, that is to say, the geodesic curves of the cylinder.

As a consequence, the geodesics of the cylinder, i.e. the orbits of its non-degenerate geodesic curves, are the *helixes* (including generatrixes and directrixes), which correspond to the straight lines of the plane $\mathbb{R}^2$ covering Q.

Half-cone

Let Q be a half-cone in the terrestrial reference space $\mathcal{E}_T$.

We choose an orthogonal Cartesian system $(\mathrm{o}, \mathrm{e}_1, \mathrm{e}_2, \mathrm{e}_3)$ in $\mathcal{E}_T$ s.t. [58]

$$Q = \{\mathrm{p} \in \mathcal{E}_T \mid (\mathrm{p} - \mathrm{o}) \cdot \mathrm{e}_3 > 0\,, \quad |\mathrm{p} - \mathrm{o}|^2 - ((\mathrm{p} - \mathrm{o}) \cdot \mathrm{e}_3)^2 = 0\}$$

---

[57] See 2.2.1 ‘Falls in translating spaces’ (case 3).

[58] See I. 5.2.2, Example 10.

and

$$\mathrm{g} = (g\cos\alpha)\,\mathrm{e}_1 \mp (g\sin\alpha)\,\mathrm{e}_3$$

with

$$g := |\mathrm{g}|\ , \quad \alpha := \angle(\mathrm{g},\mathrm{e}_1) \in \left[0, \frac{\pi}{2}\right]$$

(if the *slope* of the axis $\mathrm{o} + \mathrm{Span}\,(\mathrm{e}_3)$ of Q is $\alpha > 0$, we choose the sign - or + according to whether e_3 points upwards or downwards, that is to say, according to whether Q is the higher or the lower half-cone).

We shall also make use of the covering mapping onto Q given by [59]

$$\xi : \mathbb{R} \times \mathbb{R}^+ \to Q : (\theta, z) \mapsto \mathrm{p} = \xi(\theta, z) := \mathrm{o} + (z\cos\theta)\,\mathrm{e}_1 + (z\sin\theta)\,\mathrm{e}_2 + z\,\mathrm{e}_3$$

For any $\mathrm{p} = \xi(\theta, z)$, we have

$$T_{\mathrm{p}}Q = \mathrm{Span}\,(\tau(\theta), \mathrm{e}_3 - \mathrm{n}(\theta))$$

since

$$\left(\frac{\partial \mathrm{p}}{\partial \theta}\right)_{(z,\theta)} = z\,\tau(\theta)\ , \quad \left(\frac{\partial \mathrm{p}}{\partial z}\right)_{(z,\theta)} = \mathrm{e}_3 - \mathrm{n}(\theta)$$

where $\tau(\theta)$ and $\mathrm{n}(\theta)$ are two mutually orthogonal vectors of unit norm given by

$$\tau(\theta) := -(\sin\theta)\,\mathrm{e}_1 + (\cos\theta)\,\mathrm{e}_2\ , \quad \mathrm{n}(\theta) := -(\cos\theta)\,\mathrm{e}_1 - (\sin\theta)\,\mathrm{e}_2$$

and related to each other by

$$\left(\frac{d\tau}{d\theta}\right)_{\theta} = \mathrm{n}(\theta)\,, \quad \left(\frac{d\mathrm{n}}{d\theta}\right)_{\theta} = -\tau(\theta)$$

Falls along a sloping half-cone

Put

$$\mathrm{F} := m\,\mathrm{g}$$

Let

$$\mathrm{p}(t) = \xi(\theta(t), z(t)) = \mathrm{o} + z(t)\,(\mathrm{e}_3 - \mathrm{n}(\theta(t))) \in Q$$

be a smooth motion along Q, whose derivatives are expressed by

$$\dot{\mathrm{p}}(t) = z(t)\,\dot\theta(t)\,\tau(\theta(t)) + \dot z(t)\,(\mathrm{e}_3 - \mathrm{n}(\theta(t)))$$

and

$$\ddot{\mathrm{p}}(t) = (z(t)\,\ddot\theta(t) + 2\dot z(t)\,\dot\theta(t))\,\tau(\theta(t)) + (z(t)\,\dot\theta^2(t) - \ddot z(t))\,\mathrm{n}(\theta(t)) + \ddot z(t)\,\mathrm{e}_3$$

---

[59] Recall that $\mathbb{R}^+ := \{z \in \mathbb{R} \mid z > 0\}$.

Such a $\mathrm{p}(t) = \xi(\theta(t), z(t))$ is a DPM of $\mathcal{T}$, iff its 'global coordinate expression'

$$(\theta(t), z(t)) \in \mathbb{R} \times \mathbb{R}^+$$

is a solution of Lagrange equations (‡),which now take the univesal form

$$\ddot{\mathrm{p}}(t) \cdot \tau(\theta(t)) = \mathrm{g} \cdot \tau(\theta(t))$$

$$\ddot{\mathrm{p}}(t) \cdot \Big(\mathrm{e}_3 - \mathrm{n}(\theta(t))\Big) = \mathrm{g} \cdot \Big(\mathrm{e}_3 - \mathrm{n}(\theta(t))\Big)$$

that is,

$$z(t)\,\ddot{\theta}(t) + 2\dot{z}(t)\,\dot{\theta}(t) = -g\cos\alpha\sin\theta(t) \qquad (\ddagger)_5$$

$$2\ddot{z}(t) - z(t)\,\dot{\theta}^2(t) = g(\cos\alpha\cos\theta(t) \mp \sin\alpha) \qquad (\ddagger)_6$$

Equations $(\ddagger)_{5,6}$ do *not* separate the variables $\theta(t)$ and $z(t)$.

Nevertheless, it is still easy to recognize the following (partial) results.

(i) $\alpha = 0$ (half-cone with a horizontal axis)

Equations $(\ddagger)_{5,6}$ read

$$z(t)\,\ddot{\theta}(t) + 2\dot{z}(t)\,\dot{\theta}(t) = -g\sin\theta(t)\,, \quad 2\ddot{z}(t) - z(t)\,\dot{\theta}^2(t) = g\cos\theta(t) \qquad (\ddagger)_\circ$$

a) Let

$$\mathrm{p}(t) = \mathrm{p}_o = \mathrm{o} + z_o\Big(\mathrm{e}_3 - \mathrm{n}(\theta_o)\Big)$$

be the state of rest at a point $\mathrm{p}_o \in Q$.

Such a $\mathrm{p}(t)$ would be a DPM, iff $(\theta(t) = \theta_o\,,\ z(t) = z_o)$ satisfied equations $(\ddagger)_\circ$, i.e.

$$\sin\theta_o = 0\,, \quad \cos\theta_o = 0$$

which are manifestly incompatible.

So there do not exist equilibrium configurations.

b) Let

$$\mathrm{p}(t) = \mathrm{o} + z(t)\Big(\mathrm{e}_3 - \mathrm{n}(\theta_o)\Big)\,, \quad 0 < z(t) \neq \text{const.}$$

be a rectilinear motion along a directrix $\mathrm{o} + \mathrm{Span}\,(\mathrm{e}_3 - \mathrm{n}(\theta_o)) \subset Q$.

Such a $\mathrm{p}(t)$ is a maximal DPM, iff $(\theta(t) = \theta_o\,, z(t))$ is a maximal solution of equations $(\ddagger)_\circ$, i.e.

$$\sin\theta_o = 0\ , \quad \ddot{z}(t) = \frac{g}{2}\cos\theta_o$$

whence

$$\theta_o \equiv 0 \pmod{2\pi}\ , \quad \ddot{z}(t) = \frac{g}{2}$$

or

$$\theta_o \equiv \pi \pmod{2\pi}\ , \quad \ddot{z}(t) = -\frac{g}{2}$$

(on an lowerly or upperly bounded interval of time, owing to the constraint $z(t) > 0$).

So, among the maximal DPMs, there exist incomplete rectilinear falls only along the directrixes

$$\mathrm{o} + \mathrm{Span}\,(\mathrm{e}_3 - \mathrm{n}(0)) \,, \quad \mathrm{o} + \mathrm{Span}\,(\mathrm{e}_3 - \mathrm{n}(\pi))$$

c) Let

$$\mathrm{p}(t) = \mathrm{o} + z_o \Big(\mathrm{e}_3 - \mathrm{n}(\theta(t))\Big) \,, \quad \theta(t) \neq \text{const.}$$

be a circular motion along a generatrix $((\mathrm{o} + z_o\mathrm{e}_3) + \mathrm{Span}\,(\mathrm{e}_1, \mathrm{e}_2)) \cap Q$.

Such a $\mathrm{p}(t)$ would be a DPM, iff $(\theta(t)\,, z(t) = z_o)$ satisfied equations $(\ddagger)_\circ$, i.e.

$$\ddot\theta(t) = -\frac{g}{z_o} \sin\theta(t) \,, \quad \dot\theta^2(t) = -\frac{g}{z_o} \cos\theta(t)$$

which, however, do not share any solution $\theta(t) \neq$ const. , since the first equation (mathematical pendulum) implies

$$\frac{1}{2}\dot\theta^2(t) - \frac{g}{z_o} \cos\theta(t) = \text{const.}$$

whence, owing to the second equation,

$$\cos\theta(t) = \text{const.}$$

that is, $\theta(t) =$ const. .

So, among the DPMs, there do not exist mathematical pendulum motions.

(ii) $0 < \alpha < \pi/2$ (half-cone with a sloping axis)

Equations $(\ddagger)_{5,6}$ keep their form with $\cos\alpha > 0$ and $\sin\alpha > 0$.

a) Let

$$\mathrm{p}(t) = \mathrm{p}_o = \mathrm{o} + z_o \Big(\mathrm{e}_3 - \mathrm{n}(\theta_o)\Big)$$

be the state of rest at a point of Q .

Such a $\mathrm{p}(t)$ is a DPM, iff $(\theta(t) = \theta_o\,, z(t) = z_o)$ satisfies equations $(\ddagger)_{5,6}$, i.e.

$$\sin\theta_o = 0 \,, \quad \cos\theta_o = \pm\tan\alpha$$

The above equations admit a solution θ_o , iff $\tan\alpha = |\cos\theta_o| = 1$, i.e. $\alpha = \pi/4$ and, in such a case, θ_o is given by $\theta_o \equiv 0 \pmod{2\pi}$ on the higher half-cone, and by $\theta_o \equiv \pi \pmod{2\pi}$ on the lower half-cone.
So there exist equilibrium configurations only for $\alpha = \pi/4$ and they are all the points of the horizontal directrix

$$\mathrm{o} + \mathrm{Span}\,(\mathrm{e}_3 - \mathrm{n}(\theta_o))$$

b) Let

$$\mathrm{p}(t) = \mathrm{o} + z(t)\left(\mathrm{e}_3 - \mathrm{n}(\theta_o)\right), \quad 0 < z(t) \neq \text{const.}$$

be a rectilinear motion along a directrix $\mathrm{o} + \mathrm{Span}\,(\mathrm{e}_3 - \mathrm{n}(\theta_o)) \subset Q$.

Such a $\mathrm{p}(t)$ is a maximal DPM, iff $(\theta(t) = \theta_o\,, z(t))$ is a maximal solution of equations $(\ddagger)_{5,6}$, i.e.

$$\sin\theta_o = 0\,, \quad \ddot{z}(t) = \frac{g}{2}(\cos\alpha\cos\theta_o \mp \sin\alpha)$$

whence

$$\theta_o \equiv 0 \pmod{2\pi}\,, \quad \ddot{z}(t) = \frac{g}{2}(\cos\alpha \mp \sin\alpha)$$

or

$$\theta_o \equiv \pi \pmod{2\pi}\,, \quad \ddot{z}(t) = \frac{g}{2}(-\cos\alpha \mp \sin\alpha)$$

(on a lowerly or upperly bounded interval of time, owing to the constraint $z(t) > 0$).

So, among the maximal DPMs, there exist incomplete rectilinear motions only along the directrixes

$$\mathrm{o} + \mathrm{Span}\,(\mathrm{e}_3 - \mathrm{n}(0))\,, \quad \mathrm{o} + \mathrm{Span}\,(\mathrm{e}_3 - \mathrm{n}(\pi))$$

Such a rectilinear motion is uniform

$$\ddot{z}(t) = 0$$

iff $\alpha = \pi/4$ and $\theta_o \equiv 0$ (on the higher half-cone) or $\alpha = \pi/4$ and $\theta_o \equiv \pi$ (on the lower half-cone), that is to say, iff it takes place along a horizontal directrix.

c) By the same reasoning as the one adopted in (i), check that, among the DPMs, there do not exist mathematical pendulum motions.

(iii) $\alpha = \pi/2$ (half-cone with a vertical axis)

Equations $(\ddagger)_{5,6}$ read

$$z(t)\,\ddot{\theta}(t) + 2\dot{z}(t)\,\dot{\theta}(t) = 0\,, \quad 2\ddot{z}(t) - z(t)\,\dot{\theta}^2(t) = \mp g$$

Check that

a) There do not exist equilibrium configurations.

b) There exist, among the DPMs, incomplete rectilinear falls

$$\mathrm{p}(t) = \mathrm{o} + z(t)\left(\mathrm{e}_3 - \mathrm{n}(\theta_o)\right), \quad 0 < z(t) \neq \text{const.}$$

along all the directrixes (i.e. for all $\theta_o \in \mathbb{R}$), characterized by

$$\ddot{z}(t) = \mp\frac{g}{2}$$

c) Only on the higher half-cone there exist, among the DPMs, uniform circular motions

$$\mathrm{p}(t) = \mathrm{o} + z_o\Big(\mathrm{e}_3 - \mathrm{n}(\theta(t))\Big)\,, \quad \theta(t) \neq \text{const.}$$

along all the generatrixes (i.e. for all $z_o > 0$), characterized by

$$\dot{\theta}(t) = \sqrt{\frac{g}{z_o}}$$

Geodesics of a half-cone

We shall now consider a halph-cone Q in a freely falling reference space $\mathcal{E}_3$, where the field of gravity locally vanishes. [60]

In such a space, owing to $\mathrm{F} = 0$, Lagrange equations (‡) read (in (θ, z)-coordinates)

$$z(t)\,\ddot{\theta}(t) + 2\dot{z}(t)\,\dot{\theta}(t) = 0\,, \quad 2\ddot{z}(t) - z(t)\,\dot{\theta}^2(t) = 0$$

To their solutions there correspond the inertial motions, that is to say, the geodesic curves of the half-cone.

As a consequence, the geodesics of the half-cone (i.e. the orbits of its non-degenerate geodesic curves) are the curves which unroll onto the straight lines of the holed-plane $\mathcal{A}_\mathrm{o} := (\mathrm{o} + \mathrm{Span}\,(\mathrm{e}_1, \mathrm{e}_2)) - \{\mathrm{o}\}$ through the transformation

$$\mathrm{p} = \mathrm{o} + (z\cos\theta)\,\mathrm{e}_1 + (z\sin\theta)\,\mathrm{e}_2 + z\,\mathrm{e}_3 \in Q \longmapsto \mathrm{q} = \mathrm{o} + x\,\mathrm{e}_1 + y\,\mathrm{e}_2 \in \mathcal{A}_\mathrm{o}$$

with

$$x = z\cos\frac{\theta}{\sqrt{2}}\,, \quad y = z\sin\frac{\theta}{\sqrt{2}}$$

since $(\theta(t), z(t))$ is a solution of the above Lagrange equations, iff $(x(t), y(t)) = (z(t)\cos\frac{\theta(t)}{\sqrt{2}}, z(t)\sin\frac{\theta(t)}{\sqrt{2}})$ is a solution of equations [61]

$$\ddot{x}(t) = 0\,, \quad \ddot{y}(t) = 0$$

---

[60] See 2.2.1 ‘Falls in translating spaces’ (case 3).

[61] Check that

$$\ddot{x}(t) = \big(\ddot{z}(t) - \frac{1}{2}z(t)\,\dot{\theta}^2(t)\big)\cos\frac{\theta(t)}{\sqrt{2}} - \frac{1}{\sqrt{2}}\big(z(t)\,\ddot{\theta}(t) + 2\dot{z}(t)\,\dot{\theta}(t)\big)\sin\frac{\theta(t)}{\sqrt{2}}$$

$$\ddot{y}(t) = \big(\ddot{z}(t) - \frac{1}{2}z(t)\,\dot{\theta}^2(t)\big)\sin\frac{\theta(t)}{\sqrt{2}} + \frac{1}{\sqrt{2}}\big(z(t)\,\ddot{\theta}(t) + 2\dot{z}(t)\,\dot{\theta}(t)\big)\cos\frac{\theta(t)}{\sqrt{2}}$$

Sphere

Let Q be a sphere of centre $\mathrm{o} \in \mathcal{E}_{T}$ and radius $l > 0$ in the terrestrial reference space $\mathcal{E}_{T}$, i.e.

$$Q = \{\mathrm{p} \in \mathcal{E}_{T} \mid |\mathrm{p} - \mathrm{o}| = l\}$$

For any $\mathrm{p} \in Q$, we have [62]

$$T_{\mathrm{p}}Q = \mathrm{Span}^{\perp}(\mathrm{p} - \mathrm{o})$$

We choose an orthogonal Cartesian system $(\mathrm{o}, \mathrm{e}_1, \mathrm{e}_2, \mathrm{e}_3)$ in $\mathcal{E}_{T}$ s.t.

$$\mathrm{g} = -g\, \mathrm{e}_3\,, \quad g := |\mathrm{g}|$$

Remark that two of the Cartesian coordinates can be regarded as local coordinates on Q . For instance, if $\mathrm{p}_o = \mathrm{o} + x_o\mathrm{e}_1 + y_o\mathrm{e}_2 + z_o\mathrm{e}_3$ is a point of Q with $z_o \neq 0$, there exists an open neighbourhood $\mathcal{U}$ of p_o in Q s.t. $\mathcal{U} = \mathrm{Im}\, \xi$, ξ being the local chart defined, on a suitable connected open neighbourhood X of (x_o, y_o) in $\mathbb{R}^2$, by

$$\xi : X \subset \mathbb{R}^2 \to \mathcal{E}_{T} : (x, y) \mapsto \mathrm{p} := \mathrm{o} + x\, \mathrm{e}_1 + y\, \mathrm{e}_2 + z\, \mathrm{e}_3\,, \quad z := \pm\sqrt{l^2 - x^2 - y^2}$$

where we choose the sign + or - according to whether $z_o > 0$ or $z_o < 0$.

Spherical pendulum

Put

$$\mathrm{F} := m\mathrm{g}$$

An admissible smooth motion

$$\mathrm{p}(t) \in Q$$

is a DPM of $\mathcal{T}$, iff it satisfies d'Alembert's principle, which now takes the universal form

$$\mathrm{g} - \ddot{\mathrm{p}}(t) \in T^{\perp}_{\mathrm{p}(t)}Q = \mathrm{Span}\,(\mathrm{p}(t) - \mathrm{o}) \qquad (\ddagger)_7$$

that is,

$$\ddot{\mathrm{p}}(t) \cdot |_{T_{\mathrm{p}(t)}Q} = \mathrm{g} \cdot |_{T_{\mathrm{p}(t)}Q}$$

or

$$\ddot{\mathrm{p}}(t) \cdot |_{T_{\mathrm{p}(t)}Q} = -d_{\mathrm{p}(t)}V\, |_{T_{\mathrm{p}(t)}Q}$$

---

[62] See I. 5.2.2, Example 8. Recall that $^{\perp}$ denotes orthogonal complement in E_{T} .

where

$$V : \mathcal{E}_{T} \to \mathbb{R} : \mathrm{p} \mapsto V(\mathrm{p}) := -\,\mathrm{g} \cdot (\mathrm{p} - \mathrm{o})$$

is the potential energy of g. [63]

Equation $(\ddagger)_7$ is called *spherical pendulum.*

In absence of a global chart or a covering mapping onto Q, the global dynamics of $\mathcal{T}$ cannot be reduced to the study of ordinary differential equations. Nevertheless, it is still easy to recognize the following (partial) results.

(i) First consider Cauchy data $(t_o, \mathrm{p}_o, \mathrm{v}_o) \in \mathbb{R} \times TQ$ with

$$\mathrm{v}_o = 0$$

The maximal solution of $(\ddagger)_7$ determined by such data is given, for all $t \in \mathbb{R}$, by

$$\mathrm{p}(t) = \mathrm{p}_o \in Q$$

iff

$$\mathrm{g} \;\in\; \mathrm{Span}\,(\mathrm{p}_o - \mathrm{o})$$

or, equivalently,

$$\mathrm{p}_o - \mathrm{o} \;\in\; \mathrm{Span}\,(\mathrm{g}) = \mathrm{Span}\,(\mathrm{e}_3)$$

The above condition

$$\mathrm{p}_o = \mathrm{o} + z_o \mathrm{e}_3 \in Q$$

is satisfied by

$$z_o = \pm\, l$$

That shows the existence of two equilibrium configurations, namely

$$\mathrm{p}_1 := \mathrm{o} - l\mathrm{e}_3\,, \quad \mathrm{p}_2 := \mathrm{o} + l\mathrm{e}_3$$

The stability of the equilibrium at p_1 follows from Dirichlet's criterion, since $V|_Q : \mathrm{p} \in Q \mapsto V(\mathrm{p}) = -\,\mathrm{g} \cdot (\mathrm{p} - \mathrm{o}) = gz$ admits a strict minimum at p_1, namely $V(\mathrm{p}_1) = -gl$.

The instability of the equilibrium at p_2 follows from Chetaiev's criterion, since V admits of a (strict) maximum at p_2, namely $V(\mathrm{p}_2) = gl$, which can be recognized –in Cartesian coordinates about p_2– from the negative definiteness of the 'Hessian' quadratic form on $\mathbb{R}^2$ of coefficients [64]

$$\left(\frac{\partial^2 V}{\partial x^2}\right)_{(0,0)} = \left(\frac{\partial^2 V}{\partial y^2}\right)_{(0,0)} = -\frac{g}{l}$$

---

[63] See footnote [9].

[64] $V(x,y) = g\sqrt{l^2 - x^2 - y^2}$ is the (local) coordinate expression of V in a suitable neighbourhod of $\mathrm{p}_2 = \xi(0,0)$.

$$\left(\frac{\partial^2 V}{\partial x \partial y}\right)_{(0,0)} = \left(\frac{\partial^2 V}{\partial y \partial x}\right)_{(0,0)} = 0$$

(ii) Now consider Cauchy data $(t_o, \mathrm{p}_o, \mathrm{v}_o) \in \mathbb{R} \times TQ$ with

$$\mathrm{v}_o \neq 0$$

In such a case,

$$\mathcal{A} := \mathrm{o} + A\,, \quad A := \mathrm{Span}\,(\mathrm{p}_o - \mathrm{o}, \mathrm{v}_o)$$

is a plane through the centre o of Q and then

$$C := \mathcal{A} \cap Q$$

is a *great circle* of Q, such that $\mathrm{p}_o \in C$ and $\mathrm{v}_o \in T_{\mathrm{p}_o} C$. [65]

Let

$$\mathrm{p}(t) \in C$$

(for all $t \in \mathbb{R}$) be the maximal solution –determined by the given Cauchy data– of the mathematical pendulum equation (in implicit form)

$$\mathrm{g} - \ddot{\mathrm{p}}(t) \in T^{\perp}_{\mathrm{p}(t)} C$$

If $\mathcal{A}$ is a vertical plane (i.e. $\mathrm{g} \in A$), the mathematical pendulum motion $\mathrm{p}(t)$ is the maximal solution –determined by the given Cauchy data– of equation $(\ddagger)_7$, since [66]

$$\mathrm{g} - \ddot{\mathrm{p}}(t) \in A \cap T^{\perp}_{\mathrm{p}(t)} C = \mathrm{Span}\,(\mathrm{p}(t) - \mathrm{o})$$

Conversely, if $\mathrm{p}(t) \in C$ satisfies the above equation $(\ddagger)_7$, we obtain

$$\mathrm{g} = (\mathrm{g} - \ddot{\mathrm{p}}(t)) + \ddot{\mathrm{p}}(t) \in A$$

that is, $\mathcal{A}$ is a vertical plane.

That shows the existence, among the DPMs, of mathematical pendulum motions along all and only the great circles lying on vertical planes.

---

[65] Recall that $T_{\mathrm{p}_o} C$ is the orthogonal complement of $\mathrm{Span}\,(\mathrm{p}_o - \mathrm{o}) \subset A$ in A, i.e. $T_{\mathrm{p}_o} C = A \cap \mathrm{Span}^{\perp}(\mathrm{p}_o - \mathrm{o})$.

[66] Recall that $\mathrm{p}(t) \in \mathcal{A}$ implies $\ddot{\mathrm{p}}(t) \in A$ and that the orthogonal complement of $T_{\mathrm{p}(t)} C$ in A, i.e. $A \cap T^{\perp}_{\mathrm{p}(t)} C$, coincides with $\mathrm{Span}\,(\mathrm{p}(t) - \mathrm{o}) \subset A$.

Geodesics of a sphere

Consider now the spherical surface $Q \subset \mathcal{E}_T$ of Earth ($\gamma\epsilon\omega$), where the tangent vector component F_{tang} of the force of gravity $\mathrm{F} = m\,\mathrm{G}_T$ vanishes (the centrifugal supply being negligible). [67]

In such a case, owing to $\mathrm{F}_{tang} = 0$, d'Alembert's principle takes the (universal) form of the geodesic law

$$\ddot{\mathrm{p}}(t) \in T^{\perp}_{\mathrm{p}(t)}Q = \mathrm{Span}\,(\mathrm{p}(t) - \mathrm{o}) \qquad (\ddagger)_8$$

or, equivalently,

$$(\mathrm{p}(t) - \mathrm{o}) \wedge \ddot{\mathrm{p}}(t) = 0$$

The solutions of equation $(\ddagger)_8$ are the gedesic curves of the sphere.

(i) To Cauchy data $(t_o, \mathrm{p}_o, \mathrm{v}_o) \in \mathbb{R} \times TQ$ with

$$\mathrm{v}_o = 0$$

there corresponds the 'degenerate' geodesic curve

$$\mathrm{p}(t) = \mathrm{p}_o \in Q$$

(for all $t \in \mathbb{R}$).

(ii) To Cauchy data $(t_o, \mathrm{p}_o, \mathrm{v}_o) \in \mathbb{R} \times TQ$ with

$$\mathrm{v}_o \neq 0$$

there corresponds a 'non-degenerate' geodesic curve

$$\mathrm{p}(t) \in Q$$

(for all $t \in \mathbb{R}$), whose behaviour is as follows.

As $\dot{\mathrm{p}}(t) \in T_{\mathrm{p}(t)}Q$ and $\ddot{\mathrm{p}}(t) \in T^{\perp}_{\mathrm{p}(t)}Q$, we obtain

$$\frac{d}{dt}\,(\dot{\mathrm{p}} \cdot \dot{\mathrm{p}}) = 2\,\dot{\mathrm{p}} \cdot \ddot{\mathrm{p}} = 0$$

---

[67] Alternatively, you could consider a sphere Q in a freely falling reference space $\mathcal{E}_3$, where the field of gravity locally vanishes.

whence the (scalar) first integral

$$|\dot{\mathrm{p}}(t)| = |\dot{\mathrm{p}}(t_o)| = |\mathrm{v}_o|$$

that is, $\mathrm{p}(t)$ is a uniform motion.

As $(\mathrm{p}(t) - \mathrm{o}) \wedge \ddot{\mathrm{p}}(t) = 0$, we also obtain

$$\frac{d}{dt}\,((\mathrm{p} - \mathrm{o}) \wedge \dot{\mathrm{p}}) = (\mathrm{p} - \mathrm{o}) \wedge \ddot{\mathrm{p}} = 0$$

whence the (vector) first integral

$$(\mathrm{p}(t) - \mathrm{o}) \wedge \dot{\mathrm{p}}(t) = (\mathrm{p}(t_o) - \mathrm{o}) \wedge \dot{\mathrm{p}}(t_o) = (\mathrm{p}_o - \mathrm{o}) \wedge \mathrm{v}_o \neq 0$$

which implies

$$\mathrm{p}(t) - \mathrm{o} \in \mathrm{Span}^{\perp}((\mathrm{p}_o - \mathrm{o}) \wedge \mathrm{v}_o) = \mathrm{Span}\,(\mathrm{p}_o - \mathrm{o}, \mathrm{v}_o) =: A$$

that is, $\mathrm{p}(t)$ is a plane motion

$$\mathrm{p}(t) \in \mathcal{A} := \mathrm{o} + A$$

So

$$\mathrm{p}(t) \in C := Q \cap \mathcal{A}$$

is a uniform circular motion along the great circle C of Q.

As a consequence, the geodesics of the sphere, i.e. the orbits of its non-degenerate geodesic curves, are the great circles.

2.2.3 Rigid body

We shall now adopt the model of a 'small' 3-dimensional rigid body in terrestrial dynamics.

Free falls

Let $\mathcal{R} = (Q, m, \mathrm{F})$ be a mechanical system corresponding to a 'small' 3-dimensional rigid body freely falling in the terrestrial field of gravity $\mathrm{g} = \mathrm{const.} \neq 0$.

At any admissible configuration $\mathrm{p} = (\mathrm{p}_i)_{i=1,\dots,\nu} \in Q$, with centre of mass $\mathrm{c} \in \mathcal{E}_T$, we have

$$\mathrm{F}_i(\mathrm{p}) = m_i\,\mathrm{g}$$

whence

$$\begin{aligned}\mathrm{R}^{\mathrm{F}}(\mathrm{p}) &= \textstyle\sum_{i=1}^{\nu} \mathrm{F}_i(\mathrm{p}) \\ &= \textstyle\sum_{i=1}^{\nu} m_i\, \mathrm{g} \\ &= (\textstyle\sum_{i=1}^{\nu} m_i)\mathrm{g} \\ &= m_o\, \mathrm{g}\,, \quad m_o := \textstyle\sum_{i=1}^{\nu} m_i\end{aligned}$$

and

$$\begin{aligned}\mathrm{T}^{\mathrm{F}}(\mathrm{p}) &= \textstyle\sum_{i=1}^{\nu}(\mathrm{p}_i - \mathrm{c}) \wedge \mathrm{F}_i(\mathrm{p}) \\ &= \textstyle\sum_{i=1}^{\nu}(\mathrm{p}_i - \mathrm{c}) \wedge m_i\, \mathrm{g} \\ &= \textstyle\sum_{i=1}^{\nu} m_i\,(\mathrm{p}_i - \mathrm{c}) \wedge \mathrm{g} \\ &= (\,\textstyle\sum_{i=1}^{\nu} m_i\,(\mathrm{p}_i - \mathrm{c})) \wedge \mathrm{g} \\ &= 0\end{aligned}$$

As a consequence, [68] the DPMs of $\mathcal{R}$ (*free falls*) are the admissible motions characterized by the motions $\mathrm{c} = \mathrm{c}(t)$ of the centre of mass and the motions $A = A(t)$ around the centre of mass satisfying Newton equation

$$\ddot{\mathrm{c}} = \mathrm{g}$$

and Euler geodesic equation

$$\frac{d}{dt}\,(\mathbf{I}_A\omega) = 0$$

respectively, whose discussion has already been treated. [69]

Lagrange's top

Lagrange's top $\mathcal{R}_{\tilde{\mathrm{o}}} = (Q, m, \mathrm{F})$ is a mechanical system corresponding to a 'small' 3-dimensional rigid body with a gyroscopic structure around the axis $\tilde{\mathcal{G}} = \tilde{\mathrm{o}} + \mathrm{Span}(\tilde{\mathrm{e}}_3) \subset \tilde{\mathcal{E}}_3$ through a point $\tilde{\mathrm{o}}$, constrained to stay stationary at a point $\mathrm{o} \in \mathcal{E}_T$, and the centre of mass $\tilde{\mathrm{c}} = \tilde{\mathrm{o}} + z_o\tilde{\mathrm{e}}_3$ ($z_o \neq 0$); at any admissible configuration

$$\mathrm{p} = (\mathrm{p}_i)_{i=1,\ldots,\nu} \;=\; (\mathrm{o} + A(\tilde{\mathrm{p}}_i - \tilde{\mathrm{o}}))_{i=1,\ldots,\nu} \in Q$$

the terrestrial force of gravity is assumed to be described by

$$\mathrm{F}_i(\mathrm{p}) = m_i\, \mathrm{g}$$

---

[68] See HAD, section 2.3.4.

[69] See section 2.2.1, *Falls in the vacuum* and HAD, section 2.3.5, *Euler geodesic equation.*

with $\mathrm{g} = \text{const.} \neq 0$.

As is known, [70] the DPMs of $\mathcal{R}_{\tilde{o}}$ are characterized by the solutions $A = A(t)$ of Euler equation

$$\mathbf{I}_A\dot{\omega} + \omega \wedge \mathbf{I}_A\omega = \mathrm{T}^{(\mathrm{F})}(A)$$

with [71]

$$\begin{aligned}
\mathrm{T}^{\mathrm{F}}(A) &= \mathrm{T}^{\mathrm{F}}(\mathrm{p}) \\
&= \textstyle\sum_{i=1}^{\nu}(\mathrm{p}_i - \mathrm{o}) \wedge \mathrm{F}_i(\mathrm{p}) \\
&= (\textstyle\sum_{i=1}^{\nu} m_i\,(\mathrm{p}_i - \mathrm{o})) \wedge \mathrm{g} \\
&= (\textstyle\sum_{i=1}^{\nu} m_i\,(\mathrm{p}_i - \mathrm{c}) + \textstyle\sum_{i=1}^{\nu} m_i\,(\mathrm{c} - \mathrm{o})) \wedge \mathrm{g} \\
&= (\textstyle\sum_{i=1}^{\nu} m_i)\;(\mathrm{c} - \mathrm{o}) \wedge \mathrm{g} \\
&= m_o z_o \mathrm{e}_3 \wedge \mathrm{g}\,, \quad m_o := \textstyle\sum_{i=1}^{\nu} m_i
\end{aligned}$$

As

$$\mathrm{T}^{\mathrm{F}} \cdot \mathrm{e}_3 = 0$$

(i.e. T^{F} is orthogonal – at every admissible configuration – to the corresponding position of the gyroscopic axis), [72] a solution of Euler equation exhibits an angular velocity $\omega = \omega(t)$ with a constant gyroscopic component

$$\omega \cdot \mathrm{e}_3 = r_o = \text{const.} \tag{$\dagger$}$$

and, if the above angular velocity corresponds to an initial condition

$$\omega(t_o) = r_o\,\mathrm{e}_3(\mathrm{t_o})\,, \quad r_o \gg 0$$

(prescribing a high initial angular velocity around the initial position of the gyroscopic axis), then it approximately satisfies – in a small neighbourhood of t_o – the equation of gyroscopic effects

$$\omega \wedge \mathrm{e}_3 = -\frac{m_o z_o}{I^3 r_o}\,\mathrm{g} \wedge \mathrm{e}_3 \tag{$\ddagger$}$$

that is,

$$\left(\omega + \frac{m_o z_o}{I^3 r_o}\,\mathrm{g}\right) \wedge \mathrm{e}_3 = 0$$

---

[70] See HAD, section 2.3.5.

[71] The position of the centre of mass at $\mathrm{p} \in Q$ is $\mathrm{c} = \mathrm{o} + A(\tilde{\mathrm{c}} - \tilde{\mathrm{o}}) = \mathrm{o} + z_o\mathrm{e}_3$ with $\mathrm{e}_3 := A(\tilde{\mathrm{e}}_3)$.

[72] See HAD, section 2.3.5, *Gyroscopic effects.*

or

$$\omega = \omega_o \, \mathrm{e}_3 - \frac{m_o z_o}{I^3 r_o} \, \mathrm{g} \tag{††}$$

where the time function [73]

$$\omega_o = \omega \cdot \mathrm{e}_3 + \frac{m_o z_o}{I^3 r_o} \, \mathrm{g} \cdot \mathrm{e}_3$$

is constant, since (owing to (†), (‡) and Poisson's formula $\dot{\mathrm{e}}_3 = \omega \wedge \mathrm{e}_3$)

$$\begin{aligned} \dot{\omega}_o &= \frac{m_o z_o}{I^3 r_o} \, \mathrm{g} \cdot \dot{\mathrm{e}}_3 \\ &= \frac{m_o z_o}{I^3 r_o} \, \mathrm{g} \cdot (\omega \wedge \mathrm{e}_3) \\ &= \frac{m_o z_o}{I^3 r_o} \, \mathrm{g} \cdot \left(- \frac{m_o z_o}{I^3 r_o} \, \mathrm{g} \wedge \mathrm{e}_3 \right) \\ &= 0 \end{aligned}$$

So, owing to the equation (††) of gyroscopic effects, the above solution exhibits – in a small time interval – the features of a regular precession, composed of a proper uniform rotation $\omega_1 := \omega_o \, \mathrm{e}_3$ around the moving gyroscopic axis $\mathcal{G} = \mathrm{o} + \mathrm{Span}(\mathrm{e}_3)$ and a precessional uniform rotation $\omega_2 := -\frac{m_o z_o}{I^3 r_o} \, \mathrm{g}$ around the vertical axis $\mathcal{P} = \mathrm{o} + \mathrm{Span}(\mathrm{g})$.

Foucault's gyroscope

Foucault's gyroscope $\mathcal{R}_{\tilde{\mathrm{c}}} = (Q, m, \mathrm{F})$ is a mechanical system corresponding to a 'small' 3-dimensional rigid body with a gyroscopic structure around the axis $\tilde{\mathcal{G}} = \tilde{\mathrm{c}} + \mathrm{Span}(\tilde{\mathrm{e}}_3) \subset \tilde{\mathcal{E}}_3$ through the centre of mass $\tilde{\mathrm{c}}$, constrained to stay stationary at a point $\mathrm{c} \in \mathcal{E}_T$; at any admissible configuration

$$\mathrm{p} = (\mathrm{p}_i)_{i=1,\ldots,\nu} \; = \; \left(\mathrm{c} + A(\tilde{\mathrm{p}}_i - \tilde{\mathrm{o}})\right)_{i=1,\ldots,\nu} \in Q$$

the terrestrial force of gravity is assumed to be described by

$$\mathrm{F}_i(\mathrm{p}, \mathrm{v}) = m_i(\mathrm{g} - 2\omega_T \wedge \mathrm{v}_i)$$

with $\mathrm{g} = \text{const.} \neq 0$, $\omega_T = \text{const.} \neq 0$ and

$$\mathrm{v} = (\mathrm{v}_i)_{i=1,\ldots,\nu} \; = \; \left(\mathrm{w} \wedge (\mathrm{p}_i - \mathrm{c})\right)_{i=1,\ldots,\nu} \in T_{\mathrm{p}} Q$$

---

[73] Recall that $\tilde{\mathrm{e}}_3 \cdot \tilde{\mathrm{e}}_3 = 1$ and then $\mathrm{e}_3 \cdot \mathrm{e}_3 = 1$.

As is known, [74] the DPMs of $\mathcal{R}_{\tilde{c}}$ are characterized by the solutions $A = A(t)$ of Euler equation

$$\mathbf{I}_A\dot{\omega} + \omega \wedge \mathbf{I}_A\omega = \mathrm{T}^{(\mathrm{F})}(A, \omega)$$

with [75]

$$\begin{aligned}
\mathrm{T}^{\mathrm{F}}(A,\omega) &= \mathrm{T}^{\mathrm{F}}(\mathrm{p}, \dot{\mathrm{p}}) \\
&= \textstyle\sum_{i=1}^{\nu} (\mathrm{p}_i - \mathrm{c}) \wedge \mathrm{F}_i(\mathrm{p}, \dot{\mathrm{p}}) \\
&= (\textstyle\sum_{i=1}^{\nu} m_i\,(\mathrm{p}_i - \mathrm{c})) \wedge \mathrm{g} - 2\textstyle\sum_{i=1}^{\nu} m_i\,(\mathrm{p}_i - \mathrm{c}) \wedge (\omega_T \wedge (\omega \wedge (\mathrm{p}_i - \mathrm{c}))) \\
&= 2\textstyle\sum_{i=1}^{\nu} m_i\,(\mathrm{p}_i - \mathrm{c}) \wedge ((\omega \wedge (\mathrm{p}_i - \mathrm{c})) \wedge \omega_T) \\
&= 2\textstyle\sum_{i=1}^{\nu} m_i\,(\mathrm{p}_i - \mathrm{c}) \wedge (\,(\omega_T \cdot \omega)\,(\mathrm{p}_i - \mathrm{c}) - (\omega_T \cdot (\mathrm{p}_i - \mathrm{c}))\,\omega\,) \\
&= -2\textstyle\sum_{i=1}^{\nu} m_i\,(\mathrm{p}_i - \mathrm{c}) \wedge (\omega_T \cdot (\mathrm{p}_i - \mathrm{c}))\,\omega
\end{aligned}$$

where, if the angular velocity $\omega = \omega(t)$ corresponds to an initial condition

$$\omega(t_o) = r_o\,\mathrm{e}_3(\mathrm{t_o})\,, \quad r_o \gg 0$$

(prescribing a high initial angular velocity around the initial position of the gyroscopic axis), it can approximately be substituted by its preponderant gyroscopic component $\omega_3\mathrm{e}_3$, i.e.

$$\begin{aligned}
\mathrm{T}^{\mathrm{F}}(A,\omega) &= (\,-2\textstyle\sum_{i=1}^{\nu} m_i(\omega_T \cdot (\mathrm{p}_i - \mathrm{c}))\,(\mathrm{p}_i - \mathrm{c})\,) \wedge \omega_3\mathrm{e}_3 \\
&= -I_3\omega_3\omega_T \wedge \mathrm{e}_3
\end{aligned}$$

(the last equality being due to the fact that

$$(\,2\textstyle\sum_{i=1}^{\nu} m_i(\omega_T \cdot (\mathrm{p}_i - \mathrm{c}))(\mathrm{p}_i - \mathrm{c})\,) - I_3\omega_T \in \mathrm{Span}(\mathrm{e}_3)$$

as one could easily verify in terms of components in an orthonormal basis $(\mathrm{e}_1, \mathrm{e}_2, \mathrm{e}_3)$).

As

$$\mathrm{T}^{\mathrm{F}} \cdot \mathrm{e}_3 = 0$$

(i.e. T^{F} is orthogonal – at every admissible configuration – to the corresponding position of the gyroscopic axis), [76] the above solution of Euler equation exhibits an angular velocity $\omega = \omega(t)$ with a constant gyroscopic component

$$\omega_3 = \omega \cdot \mathrm{e}_3 = r_o = \text{const.} \qquad (\dagger)$$

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[74] See HAD, section 2.3.5.

[75] Recall that $(\mathrm{u} \wedge \mathrm{v}) \wedge \mathrm{w} = (\mathrm{w} \cdot \mathrm{u})\mathrm{v} - (\mathrm{w} \cdot \mathrm{v})\mathrm{u}$.

[76] See HAD, section 2.3.5, *Gyroscopic effects.*

and then it approximately satisfies – in a small neighbourhood of t_o – the equation of gyroscopic effects

$$\omega \wedge \mathrm{e}_3 = -\omega_T \wedge \mathrm{e}_3 \tag{‡}$$

that is,

$$(\omega + \omega_T) \wedge \mathrm{e}_3 = 0$$

or

$$\omega = \omega_o \, \mathrm{e}_3 - \omega_T \tag{††}$$

where the time function [77]

$$\omega_o = \omega \cdot \mathrm{e}_3 + \omega_T \cdot \mathrm{e}_3$$

is constant, since (owing to (†), (‡) and Poisson's formula $\dot{\mathrm{e}}_3 = \omega \wedge \mathrm{e}_3$)

$$\begin{aligned} \dot{\omega}_o &= \omega_T \cdot \dot{\mathrm{e}}_3 \\ &= -\omega_T \cdot (\omega \wedge \mathrm{e}_3) \\ &= -\omega_T \cdot (\omega_T \wedge \mathrm{e}_3) \\ &= 0 \end{aligned}$$

So, owing to the equation (††) of gyroscopic effects, the above solution exhibits – in a small time interval – the features of a regular precession, composed of a proper uniform rotation $\omega_1 := \omega_o \, \mathrm{e}_3$ around the moving gyroscopic axis $\mathcal{G} = \mathrm{c} + \mathrm{Span}(\mathrm{e}_3)$ and a precessional uniform rotation $\omega_2 := -\omega_T$ around the parallel to the terrestrial axis $\mathcal{P} = \mathrm{c} + \mathrm{Span}(\omega_\mathrm{T})$ (for that reason, Foucault's gyroscope is an instrument which can to give an experimental evidence – and a measure – of the terrestrial 'rotation' ω_T).

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[77] Recall that $\tilde{\mathrm{e}}_3 \cdot \tilde{\mathrm{e}}_3 = 1$ and then $\mathrm{e}_3 \cdot \mathrm{e}_3 = 1$.

Chapter 3

Appendix: reference transformations

Kinematics is a descriptive study of 'motion' ($\kappa\acute{\iota}\nu\eta\mu\alpha$). As the very concept of motion requires the preliminar choice of a reference space, [1] this Appendix will provide the main tools for studying the kinematics of a 'reference transformation' – i.e. the motion of a reference space with respect to another one – and the effects of such a transformation on the kinematics of a particle. [2]

3.1 Reference transformations

A *reference transformation* is a mapping

$$\alpha : \mathbb{R} \times \tilde{\mathcal{E}}_3 \to \mathcal{E}_3 : (t, \tilde{\mathrm{p}}) \mapsto \alpha(t, \tilde{\mathrm{p}})$$

describing the motion of a reference space $\tilde{\mathcal{E}}_3$ in another reference space $\mathcal{E}_3$, that is to say, the motion

$$\alpha_{\tilde{\mathrm{p}}} : \mathbb{R} \to \mathcal{E}_3 : t \mapsto \alpha_{\tilde{\mathrm{p}}}(t) := \alpha(t, \tilde{\mathrm{p}})$$

of each point $\tilde{\mathrm{p}} \in \tilde{\mathcal{E}}_3$ in $\mathcal{E}_3$ or, equivalently, the configuration

$$\alpha_t : \tilde{\mathcal{E}}_3 \to \mathcal{E}_3 : \tilde{\mathrm{p}} \mapsto \alpha_t(\tilde{\mathrm{p}}) := \alpha(t, \tilde{\mathrm{p}})$$

of $\tilde{\mathcal{E}}_3$ in $\mathcal{E}_3$ at each time $t \in \mathbb{R}$.

---

[1] A reference space – 'mathematical extension' of a rigid body carrying an observer – is conceived as a 3-dimensional, oriented, Euclidean affine space.

[2] The consequent effects on particle dynamics are shown in the main text.

Rigid motion

A reference transformation α is assumed to be a *rigid motion*, preserving the Euclidean structure of the spaces, i.e. a C^∞ mapping such that, for all $t \in \mathbb{R}$, α_t is a time-dependent, orientation-preserving, affine isometry. [3]

So, if we choose an 'origin' $\tilde{\mathrm{o}} \in \tilde{\mathcal{E}}_3$, the motion

$$\mathrm{p}_t := \alpha_{\tilde{\mathrm{p}}}(t) = \alpha_t(\tilde{\mathrm{p}})$$

of any $\tilde{\mathrm{p}} \in \tilde{\mathcal{E}}_3$ will be described by

$$\mathrm{p}_t = \mathrm{o}_t + A_t(\tilde{\mathrm{p}} - \tilde{\mathrm{o}})$$

in terms of the value

$$\mathrm{o}_t := \alpha_t(\tilde{\mathrm{o}})$$

of α_t at $\tilde{\mathrm{o}}$ and the linear part (orientation-preserving, linear isometry) [4]

$$A_t : \tilde{E}_3 \to E_3$$

of α_t (one of them, at least, being a non-constant function of time).

Remark that, for any choice of a point $\mathrm{o} \in \mathcal{E}_3$, we can write

$$\mathrm{p}_t = (\mathrm{o} + A_t(\tilde{\mathrm{p}} - \tilde{\mathrm{o}})) + (\mathrm{o}_t - \mathrm{o})$$

where the first term in the right hand side describes how $\tilde{\mathrm{p}}$ may move round the chosen origin $\tilde{\mathrm{o}}$ kept stationary at o and the second term describes how the motion of $\tilde{\mathrm{o}}$ may differ from the state of rest at o.

Velocity field

The *velocity field* $\mathrm{v}(t, \tilde{\mathrm{p}}) \in E_3$ –describing the velocity of any point $\tilde{\mathrm{p}}$ at any time t– is the first derivative [5]

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[3] $\alpha^{-1} : (t, \mathrm{p}) \in \mathbb{R} \times \mathcal{E}_3 \mapsto \alpha^{-1}(t, \mathrm{p}) := \alpha_t^{-1}(\mathrm{p}) \in \tilde{\mathcal{E}}_3$ is then the motion of $\mathcal{E}_3$ in $\tilde{\mathcal{E}}_3$.

[4] $\tilde{E}_3$ and E_3 denote the vector spaces on which $\tilde{\mathcal{E}}_3$ and $\mathcal{E}_3$, respectively, are modelled.

[5] Preliminarily notice that, in any two bases $\tilde{e} : \mathbb{R}^3 \to \tilde{E}_3$ and $e : \mathbb{R}^3 \to E_3$, A_t is represented by the linear isomorphism $A_t^{(\tilde{e},e)} := e^{-1} \circ A_t \circ \tilde{e} : \mathbb{R}^3 \to \mathbb{R}^3$, called 'representative matrix' of A_t. The time derivative $\dot{A}_t^{(\tilde{e},e)}$ is then the 'representative matrix' of linear morphism $\dot{A}_t := e \circ \dot{A}_t^{(\tilde{e},e)} \circ \tilde{e}^{-1} : \tilde{E}_3 \to E_3$ (which turns out to be independent of the choice of the bases). Moreover, there exists one and only one vector $\omega(t) \in E_3$ s.t. $\omega(t) \wedge = \dot{A}_t \circ A_t^{-1}$, whence $\omega(t) = 0 \Longleftrightarrow \dot{A}_t = 0$.

Recall that the *wedge product*

$$\wedge : E_3 \times E_3 \to E_3 : (\mathrm{u}, \mathrm{v}) \mapsto \mathrm{u} \wedge \mathrm{v}$$

$$\begin{aligned} \mathrm{v}(t,\tilde{\mathrm{p}}) &:= \dot{\mathrm{p}}_t \\ &= \dot{\mathrm{o}}_t + \dot{A}_t(\tilde{\mathrm{p}}-\tilde{\mathrm{o}}) \\ &= \dot{\mathrm{o}}_t + (\dot{A}_t \circ A_t^{-1} \circ A_t)(\tilde{\mathrm{p}}-\tilde{\mathrm{o}}) \\ &= \mathrm{v}(t,\tilde{\mathrm{o}}) + \omega(t) \wedge (\mathrm{p}_t - \mathrm{o}_t) \end{aligned}$$

where $\omega(t) \in E_3$, uniquely determined by condition $\omega(t)\wedge\; = \dot{A}_t \circ A_t^{-1}$, is called *angular velocity* of the rigid motion.

The velocity field can also be expressed as a vector field $\tilde{\mathrm{v}}(t,\tilde{\mathrm{p}}) \in \tilde{E}_3$ by putting

$$\tilde{\omega}(t) := A_t^{-1}(\omega(t))$$

and [6]

$$\begin{aligned} \tilde{\mathrm{v}}(t,\tilde{\mathrm{p}}) &:= A_t^{-1}(\mathrm{v}(t,\tilde{\mathrm{p}})) \\ &= \tilde{\mathrm{v}}(t,\tilde{\mathrm{o}}) + \tilde{\omega}(t) \wedge (\tilde{\mathrm{p}}-\tilde{\mathrm{o}}) \end{aligned}$$

Acceleration field

The *acceleration field* $\mathrm{a}(t,\tilde{\mathrm{p}}) \in E_3$ −describing the acceleration of any point $\tilde{\mathrm{p}}$ at any time t− is the second derivative

$$\begin{aligned} \mathrm{a}(t,\tilde{\mathrm{p}}) &:= \ddot{\mathrm{p}}_t \\ &= \ddot{\mathrm{o}}_t + \ddot{A}_t(\tilde{\mathrm{p}}-\tilde{\mathrm{o}}) \\ &= \mathrm{a}(t,\tilde{\mathrm{o}}) + \omega(t) \wedge (\mathrm{v}(t,\tilde{\mathrm{p}}) - \mathrm{v}(t,\tilde{\mathrm{o}})) + \dot{\omega}(t) \wedge (\mathrm{p}_t - \mathrm{o}_t) \end{aligned}$$

---

is the skew-symmetric, bilinear mapping defined as follows:
(i) if (u,v) is a linearly dependent system, then $\mathrm{u}\wedge\mathrm{v} = 0$;
(ii) if (u,v) is a linearly independent system, then

$$|\mathrm{u}\wedge\mathrm{v}| := |\mathrm{u}||\mathrm{v}|\sin\angle(\mathrm{u},\mathrm{v}) \neq 0$$

$$\mathrm{u}\wedge\mathrm{v} \in \mathrm{Span}^{\perp}(\mathrm{u},\mathrm{v})$$

$$(\mathrm{u},\mathrm{v},\mathrm{u}\wedge\mathrm{v}) \in \mathrm{Or}(E_3)$$

where $\mathrm{Or}(E_3)$ denotes the orientation given to E_3 (in an orthonormal basis $(\mathrm{e}_1,\mathrm{e}_2,\mathrm{e}_3)$ belonging to $\mathrm{Or}(E_3)$, the components of $\mathrm{w} = \mathrm{u}\wedge\mathrm{v}$ are given, for all $i = 1,2,3$, by

$$w^i = u^{i+1}v^{i+2} - v^{i+1}u^{i+2}$$

where $(i, i+1, i+2)$ is a circular permutations of $(1,2,3)$).

[6] Any orientation-preserving linear isometry such as A_t^{-1}, satisfies −for all $\mathrm{u},\mathrm{v} \in E_3$− $A_t^{-1}(\mathrm{u}\wedge\mathrm{v}) = A_t^{-1}(\mathrm{u}) \wedge A_t^{-1}(\mathrm{v})$.

The acceleration field can also be expressed as a vector field $\tilde{\mathrm{a}}(t, \tilde{\mathrm{p}}) \in \tilde{E}_3$ by putting [7]

$$\begin{aligned} \tilde{\mathrm{a}}(t, \tilde{\mathrm{p}}) &:= A_t^{-1}(\mathrm{a}(t, \tilde{\mathrm{p}})) \\ &= \tilde{\mathrm{a}}(t, \tilde{\mathrm{o}}) + \tilde{\omega}(t) \wedge (\tilde{\mathrm{v}}(t, \tilde{\mathrm{p}}) - \tilde{\mathrm{v}}(t, \tilde{\mathrm{o}})) + \dot{\tilde{\omega}}(t) \wedge (\tilde{\mathrm{p}} - \tilde{\mathrm{o}}) \end{aligned}$$

Uniform translation

α is said to be a *uniform rectilinear motion of translation*, if $\dot{\mathrm{o}}_t$ and A_t are constant functions of time, say

$$\dot{\mathrm{o}}_t = \mathrm{v} \neq 0 \,, \quad A_t = A \,, \quad \forall t$$

The velocity field is then expressed by

$$\mathrm{v}(t, \tilde{\mathrm{p}}) = \mathrm{v} \quad \text{or} \quad \tilde{\mathrm{v}}(t, \tilde{\mathrm{p}}) = \tilde{\mathrm{v}}$$

with $\tilde{\mathrm{v}} := A^{-1}(\mathrm{v})$. [8]

As a consequence, the acceleration field is expressed by

$$\mathrm{a}(t, \tilde{\mathrm{p}}) = 0 \quad \text{or} \quad \tilde{\mathrm{a}}(t, \tilde{\mathrm{p}}) = \tilde{0}$$

Uniform rotation

α is said to be a *uniform circular motion of rotation*, if o_t and $\omega(t)$ are constant functions of time, say

$$\mathrm{o}_t = \mathrm{o} \,, \quad \omega(t) = \omega \neq 0 \,, \quad \forall t$$

The velocity field is then expressed by

$$\mathrm{v}(t, \tilde{\mathrm{p}}) = \omega \wedge (\mathrm{p}_t - \mathrm{o}) \quad \text{or} \quad \tilde{\mathrm{v}}(t, \tilde{\mathrm{p}}) = \tilde{\omega} \wedge (\tilde{\mathrm{p}} - \tilde{\mathrm{o}})$$

where $\tilde{\omega} := A_t^{-1}(\omega)$ is a non-null, constant function of time. [9]

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[7] As $\dot{A}_t\big(\tilde{\omega}(t)\big) = \dot{A}_t\big(A_t^{-1}(\omega(t))\big) = \omega(t) \wedge \omega(t) = 0$, condition $\omega(t) = A_t\big(\tilde{\omega}(t)\big)$ implies $\dot{\omega}(t) = A_t\big(\dot{\tilde{\omega}}(t)\big)$.

[8] Each $\tilde{\mathrm{p}} \in \tilde{\mathcal{E}}_3$ moves in $\mathcal{E}_3$ uniformly (i.e. with constant scalar velocity $|\mathrm{v}(t, \tilde{\mathrm{p}})| = |\mathrm{v}| \neq 0$) along a rectilinear orbit of direction $\mathrm{Span}(\mathrm{v})$.

[9] As $0 = \dot{\omega}(t) = A_t\big(\dot{\tilde{\omega}}(t)\big)$, we have $\dot{\tilde{\omega}}(t) = 0$.

Check that the *rotation axis* $\tilde{\mathcal{A}} := \tilde{\mathrm{o}} + \mathrm{Span}(\tilde{\omega}) \subset \tilde{\mathcal{E}}_3$ keeps stationary in the position $\mathcal{A} := \mathrm{o} + \mathrm{Span}(\omega) \subset \mathcal{E}_3$ and each $\tilde{\mathrm{p}} \notin \tilde{\mathcal{A}}$ moves in $\mathcal{E}_3$ uniformly (i.e. with constant scalar velocity $|\mathrm{v}(t, \tilde{\mathrm{p}})| = |\tilde{\mathrm{v}}(t, \tilde{\mathrm{p}})| = |\tilde{\omega} \wedge (\tilde{\mathrm{p}} - \tilde{\mathrm{o}})| \neq 0$) along a circular orbit centred at a point of $\mathcal{A}$ and lying on a plane orthogonal to $\mathcal{A}$.

As a consequence, the acceleration field is expressed by

$$\mathrm{a}(t,\tilde{\mathrm{p}}) = \omega \wedge \mathrm{v}(t,\tilde{\mathrm{p}}) \quad \text{or} \quad \tilde{\mathrm{a}}(t,\tilde{\mathrm{p}}) = \tilde{\omega} \wedge \tilde{\mathrm{v}}(t,\tilde{\mathrm{p}})$$

The *centripetal* direction of the acceleration $\tilde{\mathrm{a}}(t,\tilde{\mathrm{p}})$, is exhibited by the following more explicit expression [10]

$$\begin{aligned}
\tilde{\mathrm{a}}(t,\tilde{\mathrm{p}}) &= \tilde{\omega} \wedge \tilde{\mathrm{v}}(t,\tilde{\mathrm{p}}) \\
&= \tilde{\omega} \wedge (\tilde{\omega} \wedge (\tilde{\mathrm{p}} - \tilde{\mathrm{o}})) \\
&= ((\tilde{\mathrm{p}} - \tilde{\mathrm{o}}) \wedge \tilde{\omega}) \wedge \tilde{\omega} \\
&= ((\tilde{\mathrm{p}} - \tilde{\mathrm{o}}) \cdot \tilde{\omega})\tilde{\omega} - \tilde{\omega}^2(\tilde{\mathrm{p}} - \tilde{\mathrm{o}}) \\
&= \tilde{\omega}^2(\tilde{\mathrm{p}}^* - \tilde{\mathrm{o}}) - \tilde{\omega}^2(\tilde{\mathrm{p}} - \tilde{\mathrm{o}}) \\
&= -\tilde{\omega}^2(\tilde{\mathrm{p}} - \tilde{\mathrm{p}}^*)
\end{aligned}$$

where $\tilde{\mathrm{p}}^*$ denotes the orthogonal projection of $\tilde{\mathrm{p}}$ onto the *rotation axis*

$$\tilde{\mathcal{A}} := \tilde{\mathrm{o}} + \mathrm{Span}(\tilde{\omega})$$

3.2 Galileian transformations

Reference transformations produce special effects on the kinematics of a particle. In this context, *Galileian transformations* – i.e. reference transformations determined by uniform rectilinear motions of translation – play a distinguished role

Composition of motions

Let

$$\tilde{\gamma} : I \subset \mathbb{R} \to \tilde{\mathcal{E}}_3 : t \mapsto \tilde{\mathrm{p}}(t)$$

be a smooth motion of a particle in $\tilde{\mathcal{E}}_3$.

---

[10] We shall make use of the following property of wedge product

$$(\tilde{\mathrm{u}} \wedge \tilde{\mathrm{v}}) \wedge \tilde{\mathrm{w}} = (\tilde{\mathrm{u}} \cdot \tilde{\mathrm{w}})\tilde{\mathrm{v}} - (\tilde{\mathrm{v}} \cdot \tilde{\mathrm{w}})\tilde{\mathrm{u}}$$

We also recall that the orthogonal projection of $\tilde{\mathrm{p}}$ onto $\tilde{\mathcal{A}} := \tilde{\mathrm{o}} + \mathrm{Span}(\tilde{\omega})$, is

$$\tilde{\mathrm{p}}^* := \tilde{\mathrm{o}} + \left((\tilde{\mathrm{p}} - \tilde{\mathrm{o}}) \cdot \frac{\tilde{\omega}}{|\tilde{\omega}|}\right) \frac{\tilde{\omega}}{|\tilde{\omega}|}$$

where $|\tilde{\omega}|^2 := \tilde{\omega}^2 := \tilde{\omega} \cdot \tilde{\omega}$.

Such a motion, observed in $\mathcal{E}_3$, will be described by the composition [11]

$$\gamma := \alpha \circ (\tau \times \tilde{\gamma}) : I \subset \mathbb{R} \to \mathcal{E}_3 : t \overset{\tau \times \tilde{\gamma}}{\longmapsto} (t, \tilde{\mathrm{p}}(t)) \overset{\alpha}{\longmapsto} \mathrm{p}(t) = \alpha\,(t, \tilde{\mathrm{p}}(t))$$

since the particle, at any time $t \in I$, occupies the position

$$\tilde{\mathrm{p}}(t) \in \tilde{\mathcal{E}}_3$$

which in turn occupies the position

$$\begin{aligned} \mathrm{p}(t) &= \alpha(t, \tilde{\mathrm{p}}(t)) \\ &= \alpha_t(\tilde{\mathrm{p}}(t)) \\ &= \mathrm{o}_t + A_t(\tilde{\mathrm{p}}(t) - \tilde{\mathrm{o}}) \in \mathcal{E}_3 \end{aligned}$$

Composition of velocities

By derivation

$$\dot{\mathrm{p}}(t) = (\dot{\mathrm{o}}_t + \dot{A}_t(\tilde{\mathrm{p}}(t) - \tilde{\mathrm{o}}))\; +\; A_t(\dot{\tilde{\mathrm{p}}}(t))$$

we obtain the classical *composition law of velocities*

$$\dot{\mathrm{p}}(t) = A_t(\dot{\tilde{\mathrm{p}}}(t))\; +\; \mathrm{v}(t, \tilde{\mathrm{p}}(t))$$

whence

$$\dot{\tilde{\mathrm{p}}}(t) = A_t^{-1}(\dot{\mathrm{p}}(t))\; -\; \tilde{\mathrm{v}}(t, \tilde{\mathrm{p}}(t))$$

Composition of accelerations

By further derivation

$$\ddot{\mathrm{p}}(t) = (\ddot{\mathrm{o}}_t + \ddot{A}_t(\tilde{\mathrm{p}}(t) - \tilde{\mathrm{o}}))\; +\; 2\,\dot{A}_t(\dot{\tilde{\mathrm{p}}}(t))\; +\; A_t(\ddot{\tilde{\mathrm{p}}}(t))$$

we obtain the classical *composition law of accelerations*

$$\ddot{\mathrm{p}}(t) = A_t(\ddot{\tilde{\mathrm{p}}}(t))\; +\; \mathrm{a}(t, \tilde{\mathrm{p}}(t))\; +\; 2\,\omega(t) \wedge A_t(\dot{\tilde{\mathrm{p}}}(t))$$

whence

$$\ddot{\tilde{\mathrm{p}}}(t) = A_t^{-1}(\ddot{\mathrm{p}}(t))\; -\; \tilde{\mathrm{a}}(t, \tilde{\mathrm{p}}(t))\; -\; 2\,\tilde{\omega}(t) \wedge \dot{\tilde{\mathrm{p}}}(t)$$

---

[11] Put $\tau : t \in I \mapsto \tau(t) := t \in \mathbb{R}$ and $\tau \times \tilde{\gamma} : t \in I \mapsto (t, \tilde{\mathrm{p}}(t)) \in \mathbb{R} \times \tilde{\mathcal{E}}_3$.

Acceleration supplies and Galileian transformations

In the above composition law of accelerations, we encounter 'acceleration supplies' given (up to the sign) by the values – along the graph of the tangent lift of $\tilde{\gamma}$ – of the vector field

$$\tilde{\Gamma} : \mathbb{R} \times \tilde{\mathcal{E}}_3 \times \tilde{E}_3 \to \tilde{E}_3 : (t, \tilde{\mathrm{p}}, \tilde{\mathrm{v}}) \mapsto \tilde{\Gamma}(t, \tilde{\mathrm{p}}, \tilde{\mathrm{v}}) := \tilde{\mathrm{a}}(t, \tilde{\mathrm{p}}) + 2\,\tilde{\omega}(t) \wedge \tilde{\mathrm{v}}$$

arising from the reference transformation α.

The distinguished role of Galileian transformations in particle kinematics, is shown by the following result:

Proposition $\tilde{\Gamma}$ *vanishes identically, iff* α *is a Galileian transformation.*

Proof: We have

$$\tilde{\Gamma}(t, \tilde{\mathrm{p}}, \tilde{\mathrm{v}}) = \tilde{\mathrm{a}}(t, \tilde{\mathrm{p}}) + 2\,\tilde{\omega}(t) \wedge \tilde{\mathrm{v}} = 0\ , \quad \forall\, t, \tilde{\mathrm{p}}, \tilde{\mathrm{v}}$$

iff

$$\begin{aligned}
&\tilde{\mathrm{a}}(t, \tilde{\mathrm{p}}) = 0\ ,\ \ \tilde{\omega}(t) \wedge \tilde{\mathrm{v}} = 0\ , \quad \forall\, t, \tilde{\mathrm{p}}, \tilde{\mathrm{v}} \\
&\tilde{\mathrm{a}}(t, \tilde{\mathrm{p}}) = 0\ ,\ \ \tilde{\omega}(t) = 0\ , \quad \forall\, t, \tilde{\mathrm{p}} \\
&\mathrm{a}(t, \tilde{\mathrm{p}}) = 0\ ,\ \ \omega(t) = 0\ , \quad \forall\, t, \tilde{\mathrm{p}} \\
&\mathrm{a}(t, \tilde{\mathrm{o}}) = 0\ ,\ \ \omega(t) = 0\ , \quad \forall\, t \\
&\ddot{\mathrm{o}}_t = 0\ ,\ \ \dot{A}_t = 0\ , \quad \forall\, t \\
&\dot{\mathrm{o}}_t = \mathrm{v} \neq 0\ ,\ \ A_t = A\ ,\ \ \forall t
\end{aligned}$$

i.e. α is a Galileian transformation (uniform, rectilinear motion of translation). □