AMBIGUITY AS A COGNITIVE AND DIDACTIC RESOURCE

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Mathematics is often perceived and described as a sort of domain of perfection, where any polysemy is seen as dangerous and any ambiguity is banned. But if that is true for the "final" dressing of official mathematics, in doing mathematics and, as we claim, in understanding it, things run the opposite way. So, contrasting usual didactic practices, we believe that suitable forms of ambiguity can be seen and used as powerful cognitive resources. In this report, we present reflections on a long lasting didactic activity with students of different grades (from primary to university level), where it is shown how to exploit some ambiguities (in particular linguistic ones), related to the notion of consecutive numbers.

INTRODUCTION

What is the meaning of *ambiguity*? The Webster's 2013 Dictionary states: "*Ambiguity: doubtfulness or uncertainty, particularly as to the signification of language, arising from its admitting of more than one meaning; an equivocal word or expression.*" [1]

In these words we perceive something wrong in any occurrence of an ambiguity. Indeed, according to common sense, ambiguity is a sort of imperfection or a kind of error; and errors should be avoided. This is particularly true within mathematics domain. After all, one of the main features of mathematicians' activity is the ability to expunge any inconsistency and every uncertainty from their arguments. However, although there is no doubt that the final aspiration of mathematics (and, for that, of all science and knowledge), is a total freedom from errors, it is also true that, in the long path toward this goal, errors often play an invaluable role and make a strong contribution towards this goal. Many authors have stressed this view, in different philosophical and theoretical domains, from gnoseology to epistemology. In particular, in the works of Popper, Kuhn, Lakatos, just to mention a few, errors and ambiguities are viewed as powerful stimuli for scientific development.

In the educational domain, the impact of errors as a didactic resource has also been recognized, albeit more recently. For this we essentially refer to the monograph (Borasi, 1996) and to the extensive bibliography therein.

Less widely explored, as far as we know, is a possible positive influence on teaching activities of ambiguity and in particular of *polysemy*, here considered as a form of linguistic ambiguity [2]. Many authors have noticed how and why the confusion that often arises between a naïve and a technical meaning of the same word (*angle*, *continuous*, *square*, *limit*, *increasing,* and so on and so forth) is the source of many students' difficulties (see e. g. Tall & Vinner, 1981, Ferrari, 2004, Bardelle, 2010).

So, the usual approach to ambiguities is to identify them in order to avoid, as far as possible, their disturbing consequences.

Our approach is quite the opposite. According to the general view of (Borasi, 1996), in the sequel we will try to show how ambiguities, in particular linguistic ones, can play a useful role in the development of mathematics learning. To this purpose, we present an example of an educational activity proposed over several years to students of different grades (from primary to university students). Then we will discuss some real and potential developments of the activity and will make comments on its effectiveness. Finally some reflections will be done on the role of the teacher within a general vision of the educational process.

THEORETICAL FRAMEWORK

One can not assign a positive value to errors and inaccuracies of students if one has a vision of teaching as transmission. Indeed, our claims on ambiguity and its important role in mathematics learning processes require an active role of learners as conceived in the inquiry approach (Borasi, 1992), to which we feel very close:

> *Whereas in traditional mathematics classes ambiguity, anomalies, and contradictions are carefully eliminated so as to avoid a potential source of confusion, in an inquiry classroom these elements would be highlighted and capitalized on as a motivating force* (Borasi, 1996, p. 25)

Starting from this main premise, the research work we are going to present finds its roots in the Vygotskian sociocultural vision of learning. In particular in our work, the language plays a central role: if errors are of several types, ambiguity, as clearly indicated in the quotation from Webster, is a typical linguistic affair. It is therefore necessary to frame our proposal in the context of studies on the importance of language for mathematics and mathematics education. For this, we refer to general studies of functional linguistics (e. g. Halliday, 1985) and to the elaboration of these ideas in the domain of Math Education, especially (Ferrari, 2004). From there we draw some essential constructs, described below.

The pivotal notion connecting texts to contexts is that of *register*, that is, a variety of language depending on usage. Whenever an individual uses the language for a particular purpose, he selects a register which is suitable for that purpose. Of course, the choice is also bounded by the resources that are available for the individual. For further details on this meaning of register, see (Ferrari, 2004), where the sense is also contrasted with the same term used differently by (Duval, 2006). In this framework the key distinction is between *literate* and *colloquial* registers. The difference is functional, in the sense that the same people can use a literate register in a certain context and a colloquial register in a different one. The literate registers are typically used in communications belonging to scientific, legal, political, literary, etc. domains, in most of the narrative and often in speeches among educated people. The colloquial register is typically used in all more or less informal speeches. Of course, a fully satisfying definition is impossible. For math educators, what is particularly important is to compare or to contrast the literate registers used within formal mathematics with the colloquial ones used in life and also in the everyday discourses in teaching-learning environments. This comparison represents a tool by means of which it is possible to interpret some students' difficulties. For example in (Bardelle, 2010) it is used to analyse the behaviour of a sample of university students involved in the study of the monotonicity of a function and of the properties of its graph.

In many articles, the ambiguity coming from two different uses of the same word or locution in the colloquial and in the literate (mathematical) register, is shown as an obstacle for the correct acquisition of some mathematical notion. [3] This is certainly true, but at the same time we believe that, in many mathematical contexts, ambiguity, far from being an obstacle for learning, can be viewed and exploited as a resource, acting as a stimulus for a deeper and more critical advancement of knowledge. More generally, we are convinced that it is quite illusory to try to expunge any ambiguity from the mathematical discourse; or, may be, from mathematics itself, according with Sfard's (2008) radical identification of mathematics as a discourse.

In the history of mathematics, we know that such intricate paths are very frequent (see, e. g., the exemplary analysis of Lakatos (1976) about the Euler's theorem for polyhedra). We do not think that the individual cognitive development reproduces faithfully the cultural evolution, but we just want to observe that whenever an epistemologically difficult notion is encountered [4], both in the history and in a learning environment, many attempts have to be made before the notion can be assimilated. Many of these attempts include more or less conscious shifts from a linguistic register to another and changes in the meaning of some key word.

Therefore, according to a sociocultural vision of learning-teaching processes, our way of working with students is based on a continuous interplay between the linguistic components of knowledge and an epistemological analysis of the disciplinary concepts involved. For this kind of analysis, we refer mainly to the theoretical construct CAC (*Cultural Analysis of the Content*) of Boero and Guala (2008). According with it, mathematics is seen as an evolving discipline "*with different levels of rigor both at a specific moment in history (according to the cultural environment and specific needs), and across history, and as a domain of culture as a set of interrelated cultural tools and social practices, which can be inherited over generations*" (*ibid*., p. 223). In turn, this vision of mathematics forces a conception of mathematics education activities that leads teachers and educators "*to radically question their beliefs concerning mathematics in general and specific subject matter in particular"* (ibid., p. 223).

A CULTURAL-LINGUISTIC ANALYSIS OF *CONSECUTIVE*

In this study we take into account the notion of *consecutiveness*, in particular its occurrence in the locution *consecutive numbers*. We deeply analyse this notion within the solution and discussion of the following arithmetic word problem, submitted, with some variants, to students of different grades.

Take four consecutive numbers. Multiply the two middle numbers by each other, then multiply the first one by the last one and calculate the difference between the two results obtained. Repeat the exercise several times, using different numbers. Do you observe any regularity?

Of course the consecutiveness notion is not encountered here for the first time as an argument of discussion within math education (see, for just one remarkable example, Boero, Chiappini, Garuti, & Sibilla, 1995). Indeed, this notion is suitable to trigger various arithmetic explorations, or to develop argumentation and proof activities in arithmetics, guided by careful teaching mediations. Here, we focus on some subtle ambiguity inherent to the notion of "consecutiveness", trying to show how, if handled with care, it can represent a powerful resource for education and knowledge purposes. With this aim, in this section we analyse some cultural and epistemological aspects of that notion, starting from the above problem, and according with the CAC perspective (Boero & Guala, 2008).

First of all the text refers to consecutive numbers without specifying the number domain. That is not a problem. In fact, from one side the use of the word "consecutive" in a literate register, like the text of a word problem, should suggest that the correct domain is the set of natural numbers or perhaps of integers; from the other side the common use in a colloquial register of the word 'number' without specifications, always refers to natural numbers. But what is the meaning of "consecutive natural numbers"? Even in the scientific register we can recognize (at least) two possible meanings assigned to these words: i) the first one is framed within the order relation; ii) and the second one comes from the additive structure (to be picky, one might even subtly distinguish this meaning from the one embodied in Peano's successor operator) [5]. Let us give a closer look at these two meanings, and at their formal renderings:

i) Probably the most natural meaning of the term "consecutive" corresponds to the idea "to be immediately subsequent to", like, for example, in "Monday and Tuesday are consecutive days". In mathematical words this requires the presence of an order relation (better, a total and discrete one). So to say in a (partially) formalized language that two elements a and b are consecutive, we have to say that "$a < b$ and there is no element c such that $a < c < b$". From the logical point of view, we have the conjunction of two statements, an atomic one and the negation of the existential of a conjunction (there <u>does not exist</u> any c such that $a < c$ <u>and</u> $c < b$). So, we see that the formal translation of a (relatively) simple notion, has a logical structure far from simple. Worse, we could even argue that the notion of consecutive is 'symmetric' (that is, if 2 and 3 are consecutive, also 3 and 2 are consecutive), and so its algebraic translation becomes even more complicated.

ii) The other meaning of consecutive can be expressed as: “Two numbers are consecutive if the second one is obtained adding one unity to the first one”, which in algebraic language is written as “a and b are consecutive if $b = a + 1$”. Unfortunately, this definition is good for natural numbers (and for integers too), but not for other numerical domains, for example the sets of even or of odd numbers. But one can generalize the above notion of consecutiveness by taking into account the more general ‘additive’ relationship existing between two closest terms of any arithmetical progression, i.e., with obvious meaning of the variables, $a_{n+1} = a_n + d$.

What happens is that the two above meanings of the word consecutive coincide for natural numbers, but they do not for other kind of numbers: either in the sense that the first meaning disappears (for rational numbers a and b it is meaningless to say that they are consecutive while it makes sense to say that $b = a + 1$); or in the sense that the second possibility vanishes (two even numbers a and b can be consecutive but it can never happen that $b = a + 1$); or in the sense that both conditions are acceptable but bear different meanings (e. g. in the case of rational numbers truncated to, say, the second decimal place). In any case the two properties keep their own autonomous importance.

We will try to show how this muddle of issues concerning the notion of consecutiveness can represent a precious opportunity for mathematics education. To this purpose, in the next section we will show and analyze some students behaviours and the development of their mathematical knowledge coming out in the attempts to manage the ‘fuzzy’ situation produced by the ambiguity of the notion of consecutiveness in the problem presented above.

SOME EXPERIMENTAL EVIDENCE

We present some critical points of the experiences that we have been living during these years, working with the problem presented in the previous section. We have submitted the above problem or some variants of it for several years to students of different grades (from elementary school up to prospective elementary and secondary school teachers) with slightly different goals, according to the context and to the age of students (see Iannece & Romano, 2009, Mellone, 2011a and 2011b).

According with our teaching philosophy, based on the inquiry approach, as outlined in (Borasi, 1992), in the management of mathematical activities, rather than packing ready-made solutions and imposing them to students, we prefer to spend most of the time in mathematical discussions, believing that this increases students’ opportunities to build a flexible and critical mathematical knowledge. For this reason we often use open tasks, without posing limits to their developments and, most of all, taking seriously all students’ reasoning and feedback. This way of working, besides the outcomes concerning students, allows us to grow in our awareness as educators.

During these years, we have collected a lot of data regarding students' behaviour when dealing with problems of the type above described. Despite their different ages and experiences, we have observed many common points in their answers and reasoning. However, to be more definite, here we report some notable behaviours of students of a mathematics class of prospective elementary school teachers. In particular, we comment on students' typical reactions to a specific articulation of the problem that we have proposed several times.

First of all, we have noticed that the majority of students, after the first arithmetic explorations and the discover/recognition of different kinds of regularities, are naturally led to express the notion of consecutiveness in a general form. Among the first attempts to translate the relationship among the four numbers into the algebraic language, the students very often propose to use four letters, typically four "consecutive" letters in the alphabetic order, like a, b, c, d, for denoting the four consecutive numbers. Of course, such a choice can be easily labelled as naïve and unproductive, since the letters of the alphabet do not support an algebraic structure, and therefore the circumstance that b follows immediately a does not express adequately the analogous relationship between the two numbers. In front of this situation, one could be tempted to push directly toward the more effective algebraic translation a, $a + 1$, $a + 2$, $a + 3$, which is, of course, completely faithful and, above all, suitable for the usual algebraic manipulations, including those required in the problem. We too have done this many times, before becoming aware of the fact that by doing so we were losing a great opportunity. Now, we prefer to recognize the value of this choice, where a sort of isomorphism between two order structures (numbers and alphabet) is clearly glimpsed and exploited. After all, the fact that this attempt is not effective is akin to what normally happens to mathematicians when they reach their results by trial and error.

Giving time and confidence to students who choose the four letters a, b, c, d, for representing the four numbers, allows most of them to implement an interesting bridge behaviour between the initial idea of the four consecutive letters and the final four expressions a, $a + 1$, $a + 2$, $a + 3$: in fact, they prefer to maintain the four different letters, but accompanying them with the three conditions $b = a + 1$, $c = b + 1$, $d = c + 1$. In our opinion, this behaviour testifies that very often a system of equations is easier to be conceived in comparison to the more concise and effective solution a, $a + 1$, $a + 2$, $a + 3$, that, in a sense, requires the quite sophisticated ability to conceive four numbers and simultaneously their relationship. In other words a, $a + 1$, $a + 2$, $a + 3$ appear as the mental result of a transformation applied to the three above equations [6]. Anyway, the habit to discuss (and a sufficient dose of patience from the teacher) guarantees that the best representation (a, $a + 1$, $a + 2$, $a + 3$) emerges in any case. But the possibility for the students to discuss about the benefits or the disadvantages of using different algebraic translations represents an invaluable way to converge with full awareness toward the best one.

The typical step that follows in our activity is to ask students to consider the problem for four consecutive even numbers. We have experienced that many students propose without hesitation an algebraic representation like a, $a + 2$, $a + 4$, $a + 6$, and this is true especially for those students who had represented the four consecutive natural numbers as a, $a + 1$, $a + 2$, $a + 3$. Of course the use of the additive representation for natural numbers fosters a similar form also in the case of even numbers. The fact that the students do not manifest any hesitation in shifting from the operator "+1" to the operator "+2", shows, in our opinion, that the meaning of consecutiveness associated with the order relation prevails over its additive interpretation in the students' perception of this notion. Indeed they have no problem in abandoning the transcription of "consecutive of n" as "$n + 1$". However, their favourite representation is a, $a + 2$, $a + 4$, $a + 6$, in contrast with $2n$, $2n + 2$, $2n + 4$, $2n + 6$, selected by relatively few students, although the second representation is of course the only one that correctly expresses the evenness of the four numbers. This inaccuracy turns out to be useful and fruitful. In our opinion, it reveals that students, somehow unconsciously, prefer to focus on the arithmetic progression as the real nature of the problem, rather than on the kind of numbers involved (in this case, even). Indeed, when asked, in the next step, to represent four consecutive odd numbers, the students who had represented the even numbers as $a,$ $a + 2$, $a + 3$, $a + 4$ easily realize that their representation also works with odd numbers, while among the students who proposed $2n$, $2n + 2$, $2n + 4$, $2n + 6$ only a few of them succeed in finding the correct representation $2n + 1$, $2n + 3$, $2n + 5$, $2n + 7$ [7]. Finally the discovery that in both cases (even and odd numbers) the searched difference is always 8, moves the mathematical discourse toward the arithmetic progressions with common difference 2.

The last step of our typical way of managing the problem (with students of suitable age), has the crucial goal of addressing the concept of density of the rational numbers. Our request sounds as a true provocation, namely to consider four consecutive decimal numbers. The persistence of the *concept image*, in the sense of (Tall & Vinner, 1981), of the "discrete" order relation as unique prototype of all the possible order relationships, is the trigger element at the base of our proposal. We have experienced that this condition of cognitive break represents a very fertile tool to face the epistemological knot of the density of rational numbers. Among the students' attempts, we have observed that very often they propose to consider decimal numbers truncated at the first decimal digit (for example 2.1; 2.2; 2.3; 2.4), or, less frequently, at the second digit.

No doubt that this answer is a mistake, but, this error offers to students the opportunity to start new explorations on arithmetic progressions. In fact they easily discover that for decimal numbers truncated as above the difference in the problem is 0.02, and after that, moving to consider decimal numbers cut at the second digit, they realize that the difference is 0.0002. These are good premises to explore and discover the more general rule according to which, given four consecutive numbers

a, $a + d$, $a + 2d$, $a + 3d$ of an arithmetic progression, the difference $(a + d)(a + 2d) - a(a + 3d)$ is $2d^2$.

The path shown is an example of how successive generalizations can be exploited to make evident to students the irreplaceable usefulness of the algebraic language for doing manipulations and seeing relationships. But we claim that here we have more, namely the possibility of exploiting the ambiguity of a term like "consecutive" for didactic purposes. Indeed, the two meanings associated to the term coincide in the case of natural numbers, but they split when passing to other domains of numbers. So, thanks to the provocation of moving the problem toward the field of rational numbers, the students have the opportunity to become aware of this double meaning, realizing that, according with the first meaning, there cannot be consecutive rational numbers, while according with the second meaning, but enlarging it to sequences of numbers separated to each other by a constant difference, new fascinating arithmetic structures can be glimpsed and explored. The interesting linguistic-epistemological phenomenon of the splitting of a notion into two different ones, when switching from a source domain to new enlarged environments, already studied from several points of view (Lakatos, 1976, and many others) appears here in a new guise as a learning resource.

EDUCATIONAL OUTCOMES AND CONCLUSIVE REMARKS

Several studies in Mathematics Education have shown that many students' difficulties in mathematics come from their inability to juggle between the daily life use of a word and the formal use of the same word in mathematics (see for example Bardelle, 2010). On the other hand to have a word with different meanings and different uses depending on the needs of the communication is undoubtedly a definite advantage for the people who are in possession of this variety (see for example the recent Morris, 2014). In this direction, we are convinced that the ambiguity of words used in mathematics rather than to be hidden, as most education practices usually do, should be exploited as resources for mathematics education. At the same time we are convinced that in order to do this in an effective way, it is necessary to develop deep reflections about the epistemological and linguistic features concerning the use of words in mathematics (and not only). A particularly useful framework for this analysis is the CAC construct of (Boero & Guala, 2008).

In our opinion, the case of the word "consecutive" examined above offers a particularly rich context, but it is just an example [8]. In our teaching experiences we have understood how the polysemy of such a simple word like "consecutive", rather than being an avoidable flaw in a mathematical text, is something to explore with our students. This also amounts to do interdisciplinary work between mathematics and language and to include mathematics fully in Human Sciences.

During these years, our way of organizing the mathematical activities has allowed the teachers involved in them to grow in their awareness as educators, but also to gain deeper competence about the mathematical topics explored with students.

Indeed, the inquiry approach (Borasi, 1996), by which our method is partly inspired, first of all requires from teachers a strong ability to give space and attention to every proposal and idea that students might have and to carefully guide them. It also entails that the teachers themselves be involved in the mathematical work in an atmosphere genuinely oriented to discovery, where even the possible lack of a prompt response to students' questions does not appear as a diminution of their authority. So, we believe in a total and deep engagement in which students and educators/teachers live together a real and mutual learning experience. To say it in Radford's words, "*Teachers and students are in the same boat, producing knowledge and learning together. In their joint labour, they sweat, suffer, and find gratification and fulfillment with each other*" (Radford, 2014, p. 19).

NOTES

1. http://www.webster-dictionary.org/definition/ambiguity, retrieved 25th August, 2014
2. "The ambiguity of an individual word or phrase that can be used (in different contexts) to express two or more different meanings" (The Webster's 2013 Dictionary: http://www.webster-dictionary.org/definition/polysemy, retrieved 25th August, 2014)
3. However, a recent study (Morris, 2014) reports the case of a population where, due to the poorness of the language, the mathematical notions are introduced using words which are completely new. It is interesting to see how in this case the total absence of ambiguity does not facilitate in any way the understanding of mathematics by pupils.
4. Here the notion of epistemological obstacle (Brousseau, 1997) is clearly involved.
5. Strictly speaking, none of these meanings (and their algebraic transcriptions) can be taken as the faithful translation of the meaning in the daily register. Indeed, the word "consecutive", and the locution "consecutive numbers" too, like any word, is inherently ambiguous and points, to be understood and used to communicate something, on the previous knowledge of all the people involved in the communication, in other words to the pragmatic features of the communication. Therefore its meaning cannot be considered as uniquely determined.
6. This cognitive behaviour suggests that the usual curricular hierarchy, according to which equations are treated rigorously before systems should perhaps be partly revised. But this is a totally different question.
7. We suggest here a tentative interpretation of this difficulty. Namely, while in $2n$, $2n + 2$, $2n + 4$, $2n + 6$, the term $2n$ is easily used as a standard way of representing an even number, and +2, +4 and +6 are standard ways of adding two units at a time, in $2n + 1$, $2n + 3$, $2n + 5$, $2n + 7$, the term $2n + 1$ is still a common representation of a generic odd number but $2n + 3$ (and $2n + 5$ and $2n + 7$) has to be obtained from $(2n + 1) + 2$, where the two +'s play different syntactic-semantic roles.
8. For an analogous deep investigation done on the word "triangle", see (Castagnola & Tortora, 2009).

REFERENCES

Bardelle, C. (2010). Interpreting monotonicity of functions: a semiotic perspective, in Pinto M.F. & Kawasaki T.F., *Proceedings of PME 34*, Vol. 2, 177-184. Belo Horizonte, Brazil: PME.

Boero, P., Chiappini, G., Garuti, R., & Sibilla, A. (1995). Towards Statements and Proofs in Elementary Arithmetic: An Exploratory Study About the Role of Teachers and the Behaviour of Students, In *Proceedings of PME 19,* Vol. 3, pp. 129-136. Recife, Brazil: PME.

Boero, P., & Guala, E. (2008). Development of Mathematical Knowledge and Beliefs of Teachers. *The Role of Cultural Analysis of the Content to Be Taught*, in Sullivan P. &

Wood T. (eds), *Knowledge and Beliefs in Mathematics Teaching and Teaching Development*, 223-244.

Borasi, R. (1992). *Learning mathematics through inquiry*, Portsmouth, NH : Heinemann.

Borasi, R. (1996). *Reconceiving Mathematics Instruction: a Focus on Errors*, Norwood, NJ: Ablex Publishing Co.

Brousseau, G. (1997). *Theory of didactical situations in mathematics*. Dordrecht, the Netherlands: Kluwer Academic Publishers.

Castagnola, E., & Tortora, R. (2009). Can any definition of triangle be considered the best one? In Kourkoulos M. & Tzanakis C. (eds), *Proceedings of the 5th International Colloquium on the Didactics of Mathematics.* Rethymnon, Greece, Vol. 2, p. 219-232.

Duval, R. (2006). A cognitive analysis of problems of comprehension in a learning of mathematics. *Educational Studies in Mathematics*, 61, 103-131.

Ferrari, P.L. (2004). Mathematical language and advanced mathematics learning. In Høines M.J. & Fuglestad A.B. (eds), *Proceedings of PME 28*, Vol. 2, pp. 383-390. Bergen, Norway: PME.

Iannece, D., & Romano, P. (2009). What does it mean to have a 'scientific approach' for a teacher? A reflection. In Kourkoulos M. & Tzanakis C. (eds), *Proceedings of the 5th International Colloquium on the Didactics of Mathematics*, Rethymnon, Greece, Vol. 2, p. 409-419.

Lakatos, I. (1976). *Proofs and Refutations*. Cambridge, GB: Cambridge University Press.

Mellone, M. (2011a). The influence of theoretical tools on teachers' orientation to notice and classroom practice: a case study. *Journal of Mathematics Teacher Education*, 14, 269-284.

Mellone, M. (2011b). "Looking for tricks": a natural strategy, early forerunner of algebraic thinking. In Pytlak M. & Swoboda E. (eds) *Proceedings of CERME 7,* University of Rzeszow, Poland, pp. 1882-1890.

Morris, N. (2014). Probability, Uncertainty and the Tonga way. In Liljedahl P., Nicol C., Oesterle S. & Allan D. (eds), *Proceedings of the Joint Meeting of PME 38 and PME-NA 36*, Vol. 4, pp. 241-248. Vancouver, Canada: PME.

Radford, L. (2014). On teachers *and* students: an ethical cultural-historical perspective. In Liljedahl P., Nicol C., Oesterle S. & Allan D. (eds), *Proceedings of the Joint Meeting of PME 38 and PME-NA 36*, Vol. 1, pp. 1-20. Vancouver, Canada: PME.

Sfard, A. (2008). *Thinking as communicating: Human development, development of discourses, and mathematizing*. Cambridge: Cambridge University Press.

Tall, D. & Vinner, S. (1981). Concept image and concept definition in mathematics, with special reference to limits and continuity, *Educational Studies in Mathematics*, 12, 151-169.