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Reconceiving Mathematics Instruction



Reconceiving Mathematics Instruction:

A Focus on Errors

by Raffaella Borasi



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ISBN: 1-56750-168-0

*Reconceiving Mathematics Instruction:
A Focus on Errors*

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AMERICAN PSYCHOLOGICAL ASSOCIATION

Contents

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Printed in the United States of America

Borasi, Raffaella.

Reconceiving mathematics instruction : a focus on errors /
Raffaella Borasi.

p. cm. — (Issues in curriculum theory, policy, and research)
Includes bibliographical references and index.

ISBN 1-56750-167-2. ISBN 1-56750-168-0 (pbk.)

1. Mathematics—Study and teaching. 2. Errors. I. Title.

II. Series.

QA11.B6385 1996

510'.71—dc20

96-3766

CIP

Ablex Publishing Corporation
355 Chestnut Street
Norwood, New Jersey 07648

P

In order to keep this title in print and available to the academic community, this edition was produced using digital reprint technology in a relatively short print run. This would not have been attainable using traditional methods. Although the cover has been changed from its original appearance, the text remains the same and all materials and methods used still conform to the highest book-making standards.

Acknowledgments

vii

1 Introduction

Motivation and scope of the book

A new metaphor for error making

A first illustration of using errors as springboards
for inquiry—Error case study A

Content and organization of the book

1
1
4
7
10

2 Reconceiving mathematics education within an inquiry framework

Current mathematics teaching practices:
An exemplification of the transmission paradigm

Major critiques of a transmission pedagogy

Key elements of an inquiry framework

16
17
23

3 Alternative views on errors

Contributions to a view of errors as springboards for Inquiry

A critical view of existing uses of errors in
Mathematics education

27
27

4 Errors and the history of mathematics

Some historical error case studies

A first analysis of using errors as springboards
for inquiry in mathematics

36
45
45

5 Unlocking the potential of errors to stimulate inquiry within the mathematics curriculum

Error case studies generated by my own exploration of specific errors

First thoughts about using errors as springboards for inquiry
as an instructional strategy

69
69

6 Capitalizing on errors in mathematics instruction: A first analysis

Error case studies reporting on error activities experienced
by secondary students within a teaching experiment

Important variations within the instructional strategy
of capitalizing on errors

108
131

Potential benefits of capitalizing on errors
in mathematics instruction

119
120

131
139

v

7 Capitalizing on errors in mathematics instruction: A teaching experiment	149
Overview of the teaching experiment	150
Brief description and analysis of all the error activities developed in the teaching experiment	151
Evaluation of what the students gained from the experience and from the uses of errors made in it	164
8 Capitalizing on errors in mathematics instruction: Examples from the classroom	169
Error case studies reporting on error activities developed in secondary and college mathematics classes	169
Further considerations on capitalizing on errors in regular classrooms	202
9 Errors as springboards for inquiry and teacher education	209
Error case studies experienced by teachers	209
Potential benefits of engaging mathematics teachers in a use of errors as springboards for inquiry	253
10 Creating a learning environment supportive of inquiry	259
The assumptions informing a use of errors as springboards for inquiry revisited	260
Major implications of adopting an inquiry approach to school mathematics	267
Supporting teachers in the implementation of an inquiry approach in their classrooms	272
11 Conclusions	277
Using errors as springboards for inquiry in mathematics instruction: A summary	277
Looking ahead	282
Appendix A: Summary of categories, codes, and abbreviations employed in the book	285
Appendix B: Title and abstract of error case studies	291
References	301
Author Index	309
Subject Index	313

Acknowledgments

This book is the culmination of more than 10 years of work on developing the implications of an inquiry approach to mathematics instruction, and on the idea of "using errors as springboards for inquiry" more specifically. Over this period of time, my ideas and research benefited greatly from my interactions with several people, in a variety of contexts and roles. Here I would like to acknowledge at least those contributions that were most influential with respect to the study informing this book, while also providing some glimpses into the "history" behind the results reported in the following chapters (since the process that led to them was not as "linear" and smooth as it may be suggested by the logical organization finally achieved in the book).

It is quite difficult for me to identify when and how the idea of "using errors as springboards of inquiry" first occurred to me. Some important seeds, however, were planted even more than 10 years ago as I attended, as a graduate student in mathematics, some very interesting courses on the history of calculus and of infinity offered by Pascal Dupont and Ferdinando Arzarello, respectively, at the University of Torino (Italy). Further inspiration came from my interactions with Stephen I. Brown, when I was a doctoral student in mathematics education at SUNY/Buffalo, as he introduced me to the works of philosophers and historians of science such as Kuhn, Lakatos, and Kline, to more humanistic and constructivist views of mathematics, and to exciting instructional techniques such as his "What-if-not" strategy to encourage students' problem posing in mathematics.

As I began to develop the idea that a study of errors could provide a vehicle for student inquiry and problem posing in mathematics, I was very fortunate to encounter unusual encouragement and support from all the members of my doctoral dissertation committee—Stephen I. Brown (who chaired the committee), Gerald R. Rising, and Hugh G. Petrie. I look with sincere gratitude not only at the intellectual and emotional support they provided me during my dissertation work, but especially at their willingness to sponsor a dissertation study that did not fit the traditional canons and expectations—as it involved the first and "risky" stages of developing a very new strategy, mainly through conceptual analysis and the development of a few in-depth illustrations (a selection of

which is now reported in the "error case-studies" included in Chapters 4 and 5), and did not have an "empirical research" component except for the ethnographic report of a mathematics education course, designed and implemented for mathematics teachers, where my beginning ideas about this strategy were first put into practice in the context of teacher education (the only instructional context easily accessible for me at the time).

This course taught at SUNY/Buffalo, and a revised version of the same which I taught a year later at the University of Rochester, were very important for me, as they not only enabled me to begin to explore what my proposed strategy would "look like in practice," but also provided me with an opportunity to share my initial ideas with experienced teachers and benefit from their feedback. I would like to give special thanks to all the people who participated in these courses and encouraged me to pursue my initial ideas through their enthusiasm and commitment to the course. I would also like to express special recognition to Richard Fasse, John Sheedy, and Barbara Rose, as they agreed to include in this book some of the work they produced in the context of these courses (see error case studies S, T, and U, in Chapter 9).

Both my dissertation study and the course taught at the University of Rochester were necessary preliminary stages that enabled me, later on, to design a sound school-based research project where the idea of "using errors as a springboard of inquiry" could be put into practice in secondary mathematics classes and further elaborated as a result of these experiences. This study was made possible by a grant from the National Science Foundation (award #MDR-8651582) and by the collaboration of several schools and mathematics teachers. Among them, I would like to especially recognize the support of the School Without Walls, an alternative urban high school within the Rochester City School District, where I was able to teach some experimental courses myself, and of the following teachers who volunteered their classes for teaching experiments involving the proposed strategy: Dave Baker, Judi Fonzi, Tracy Martham, Maria Gajary, and Barbara Rose. In this empirical project, at different points in time, I also benefited from the competent assistance of Richard Fasse, Doug Noble, Marilu Raman, Barbara Rose, Donna Rose, and Constance Smith, as research assistants. A special thanks goes also to all the students who participated in these experiments, and especially to Katya McElfresh and Mary Israel, the two students who participated in the teaching experiment described in Chapter 7.

Although the "data" that constituted the basis for this study (and for all the error case studies reported in Chapters 4 through 9) were all essentially collected by 1989, it still took me a long time (almost 4 years!) before I was ready to process and organize the information they had provided me in the form of this book. In the meantime, I got involved in other projects that, although not addressing explicitly the topic of errors, were quite influential in my growing understanding of the potential of the notion of inquiry as a theoretical framework for reconceiving mathematics instruction more generally, as well as for

the specific approach to errors I had been developing. My collaboration with Marjorie Siegel in the context of another project funded by the National Science Foundation (entitled "Reading to Learn Mathematics for Critical Thinking," award #MDR-8850548) was especially important to this respect. Further insights on what it takes to implement radical instructional innovation in mathematics classrooms, and its implications for teacher education, were further provided by my experience in the teacher enhancement project "Supporting Middle School Learning Disabled Students in the Mainstream Mathematics Classroom" (NSF award #TPE-9153812), and benefited from the many fruitful discussions I had with Judi Fonzi, Barbara Rose, and Constance Smith in this context.

Finally, I would like to thank Beatriz D'Ambrosio, Judi Fonzi, Arthur Woodward, and Ian Westbury for their thoughtful reading of earlier drafts of this manuscript and for the valuable feedback they provided me, so that I could make the book more readable and effective.

This book, however, is dedicated to my son Madhu—it is an interesting coincidence that my second book happened to be written a little after the birth of my second son, just as my first book followed the birth of my first son!

Chapter 1

*Introduction**

MOTIVATION AND SCOPE OF THE BOOK

Dissatisfaction with the current status of school mathematics is growing across the world. Evidence for this can be found not only in the results of mathematics education research reported in professional journals and books, but also in the increasing attention given by the media to this issue. Most recently, some professional organizations have also produced a number of influential reports making recommendations for school mathematics reform, both in the United States (NCTM, 1989, 1991; NRC, 1989, 1990, 1991b) and in the United Kingdom (HMSO, 1982).

Calls for reform in mathematics instruction, however, are not just a recent phenomenon. Even if we just look at the U.S. educational scene for the past 40 years, several movements recommending radical changes in mathematics instruction can be identified, starting with the "New Math" projects of the post-Sputnik era. Although all these movements have been unanimous in criticizing the dire state of mathematics instruction, and more specifically in pointing out its inadequacy to prepare future citizens for the demands of our increasingly technological world, they often differed in their proposals for, as well as their approach to, school mathematics reform.

For example, most New Math projects in the 1960s focused essentially on the curriculum—what mathematical content should be taught, in what sequence, and with what kind of supporting instructional materials. In the 1970s, many researchers and reformers shifted their attention to the way mathematics is learned and taught, as reflected for example in the call for a "focus on problem solving" characterizing the 1980 *Agenda for Action* of the National Council of Teachers of Mathematics (NCTM, 1980). In the 1980s, while new experimental curricula continued to be developed and further research on how students

* The teaching experiences reported in this book were made possible by a grant from the National Science Foundation (Award No. MDR-8651582). The opinions and conclusions reported here, however, are solely the author's.

learn and solve problems in mathematics was pursued, mathematics educators also became increasingly aware of the need to address other neglected issues essential for the success of school mathematics reform—such as the impact of teachers' as well as students' beliefs and expectations regarding various aspects of school mathematics, the implicit messages conveyed by the everyday practices and discourse taking place in the mathematics classroom, the need to rethink teacher education both at the preservice and in-service levels, and the crucial role played by a number of institutional and political forces in making educational reform possible (as reflected, e.g., in the compendium of mathematics education research reported in Grouws, 1992).

The recent reports mentioned earlier (HMSO, 1982; NCTM, 1989, 1991; NRC, 1989, 1990, 1991b) all show awareness of both the complexity involved in school mathematics reform and the contributions provided by educational research in the last few decades for approaching this task in an informed way. These reports are also remarkably consistent with respect to both the goals and the practices they recommend for future school mathematics. They all suggest abandoning the current emphasis on the acquisition of specific mathematical facts and techniques and, instead, trying to enable students to develop more important mathematical skills such as the ability to pose and solve a variety of math-related problems, to reason and communicate mathematically, and to appreciate the value and potential applications of mathematics. To achieve these new goals, mathematics teachers are asked to abandon a view of mathematics teaching as the direct transmission of established knowledge and to try instead to develop learning environments in their classes that would:

- Encourage students to explore.
- Help students to verbalize their mathematical ideas.
- Show students that many mathematical questions have more than one right answer.
- Teach students, through experience, the importance of careful reasoning and disciplined understanding.
- Provide evidence that mathematics is alive and exciting.
- Build confidence in all students that they can learn mathematics. (NRC, 1991a, p. 7)

There are several reasons that make mathematics educators hope that the recommendations offered in these reports will be more successful than past movements in achieving some real change in the way mathematics is taught in schools. First of all, these reports reflect the combined position of important professional organizations that have a real investment in what happens in school mathematics, such as national organizations of mathematics teachers, educators, and researchers both in mathematics and mathematics education. Furthermore,

their recommendations are informed by the results of educational research and try to address several variables that can affect instructional change, such as curriculum content, classroom discourse and activities, evaluation practices, the role of mathematics teachers as professionals, and teacher education.

Yet, significant instructional changes will be achieved only if each educator involved with school mathematics engages in a personal rethinking of various aspects of mathematics instruction and identifies some concrete ways in which his or her current practice could be modified to reflect the results of such rethinking. Therefore, before such a vision can be meaningfully implemented in today's mathematics classrooms, it is important that mathematics teachers, and educators more generally, examine critically both the assumptions and implications of the vision for school mathematics these reports propose.

This book is intended to support educators in such a challenging enterprise by focusing attention on "errors" and their use in mathematics instruction. Throughout the book, an approach to errors as opportunities for learning and inquiry will be developed and employed both as a means to create the kinds of instructional experiences advocated for school mathematics reform and as a heuristic to invite reflections about both school mathematics and mathematics as a discipline.

This approach to mathematical errors, however, will require a considerable departure from how errors have traditionally been viewed, especially within an educational context. Most people, it would seem, have negative feelings toward errors, as making mistakes often generates feelings of frustration or disapproval that we would like to avoid, sometimes even at the cost of not trying at all. Negative attitudes toward errors, although more or less explicitly expressed, are often present on the educational scene as well and have even found some theoretical justification.

On the contrary, some schools of thought have been aware that errors are not only inevitable, but also a healthy part of one's education—as suggested by the popular motto "You learn from your mistakes." Despite the positive connotations of this message, however, the concept of error making in education has not yet received adequate analysis and consequently the education community has so far failed to find imaginative ways of using errors constructively in formal instruction. More specifically, although a positive role for errors has certainly been recognized in recent years in some areas of mathematics education research, student errors have primarily been employed by researchers and teachers as a tool to identify learning difficulties and to plan curriculum and teaching material accordingly or, more generally, as a means to understand students' conceptions and learning processes. However valuable, these approaches have not invited the students themselves to capitalize on their errors as learning opportunities, nor they have enabled educators to take advantage of the educational potential of errors in ways that go beyond diagnosis and remediation.

In contrast, the approach to errors informing this book recognizes the potential of errors to provide the source of valuable opportunities for mathematical exploration, problem solving, and reflection, for students as well as teachers. I have tried to summarize this approach with the expression "capitalizing on errors as springboards for inquiry" or, more briefly, "capitalizing on errors." Before discussing in more detail the content and organization of the book in the concluding section of this chapter, I would like to provide an introduction to this view of errors in two complementary ways. First, in the next section I develop the implications of using the familiar situation of getting lost in a city as a metaphor to help us think differently about errors. To give a flavor of what using mathematical errors as springboards for inquiry may look like, I then briefly sketch some questions and explorations that could be generated from looking at an error commonly made by students when adding fractions.

A NEW METAPHOR FOR ERROR MAKING

It has been recognized that the metaphors we carry with us may considerably affect the way we perceive a phenomenon (Lakoff & Johnson, 1980; Ortony, 1979; Schol, 1963, 1979). This has certainly been the case with respect to error making in the context of school mathematics, where two metaphors have been commonly employed, especially in the research literature. The first and most popular metaphor, revealed by the use of terminology such as *diagnosis* and *remediation*, or *clinical* interview, is clearly borrowed from the medical field. Errors are seen here as the *symptoms of a disease* (the latter being the student's misconception or learning difficulty that caused the error). More recently, though, errors have also been referred to as *bugs* (especially in research informed by a cognitive science framework—e.g., Brown & Burton, 1978; Brown & Van Lehn, 1982; Maurer, 1987). This time the metaphor has been borrowed from the field of computer science. Errors are here equated to *inappropriate instructions in a computer program* that do not allow the learner to reach the desired outcome.

Both metaphors have contributed to a more positive and constructive approach to errors in mathematics education research, as mentioned in the previous section. For instance, the views of errors implied by these metaphors have suggested the inefficiency of attempting to eliminate student errors simply by re-explaining a topic or by assigning more practice and, instead, pointed out the value of errors as sources of information about students' real problems in learning. At the same time, there are some negative connotations connected with both diseases and bugs in a program, which in turn support the view that errors are something to be eliminated and avoided whenever possible. In addition, the medical metaphor brings along the dangerous premise that you need an ex-

pert—the teacher or the researcher—in order to be able to make use of errors, leaving the student quite passive in the process.¹

Some of these negative assumptions and implications can be best identified and challenged once we consider an alternative metaphor to think about error making—getting lost in a city. In what follows, I identify and discuss specific aspects of the familiar situation of "getting lost" in the effort to shed new light on the phenomenon of error making in an educational context.² Readers are encouraged to think about their own experiences when getting lost and to draw their own conclusions as they read on.

A first crucial difference between the medical metaphor and "getting lost" immediately stands out. Whenever you get lost, it is essentially up to *you* to do something to resolve the situation. This suggests that the students themselves should be given the opportunity to analyze their own errors, rather than expecting the teacher to do it for them all the time.

Memories of occasions when we got lost are also likely to bring with them a lot of emotions, thus reminding us of the importance of dealing with the affective issues connected with making errors in school. At the same time, both what you feel when getting lost and what you do about it may differ considerably in different circumstances. To better appreciate the significance and implications of this realization, it may be helpful to consider the following possible scenarios:

- *Scenario 1:* If you need to reach your destination as quickly as possible—as when you are trying to make an important appointment or to reach the nearest hospital in an emergency—getting lost will obviously be perceived as a nuisance and a problem, and your efforts are likely to focus on finding a way to reach the original destination without delay (probably by asking someone for directions, or quickly checking your map to try to figure out where you are and how you could get to your destination from there). It is quite conceivable that in this scenario you will be quite upset, frustrated, and unwilling to take any risk that could cause you to lose even more time.

- *Scenario 2:* Your reactions might be quite different instead, if you got lost when going back home after work having just moved to a new neighborhood or city. Because this is a path you expect to follow many times in the future, perhaps with variations as required by picking up a friend or running an errand along the way, your goal this time may not be limited to getting home as

¹ It is important to note that this as well as other limitations of the medical metaphor have already been pointed out in the mathematics education literature (e.g., Kilpatrick, 1987a) and led to alternative definitions of diagnostic/prescriptive mathematics teaching that attempt to explicitly eliminate these unwanted implications (see especially the definitions offered by Engelhardt [1988, p. 48] as quoted in Gräber & Johnson [1991, p. III-11]).

² A full discussion of the implications of developing the metaphor of "error-making as getting lost in a city" generatively à la Schon can be found in Borsi (1988b).

quickly as possible. Rather, you may be motivated to study your map to find some alternative routes to go from home to work, look around and try to spot landmarks that will help you find your way better in the future, and, more generally, use this occasion as an opportunity to get to know the neighborhood better. In other words, even if you still want to reach your original destination eventually, there may be other long-term agendas operating that will invite you to take advantage of what you can learn from getting lost.

• *Scenario 3:* Suppose instead that you are on vacation and you are visiting a city for the first time. Although in this case you may still be moving around with a specific destination in mind, you may be more willing to relinquish your original goal if something more interesting comes your way—because, after all, you are interested mainly in getting to know the city and enjoying yourself in the process. Paradoxically, when you are a tourist you may even welcome getting lost, as it could provide you with unexpected opportunities for “adventures”—such as visiting a part of the city not mentioned in your guide book or conversing with some of the local people—and thus result in a better understanding of the city’s peculiar layout, architecture, and inhabitants. In fact, one may even argue that in this scenario getting lost almost loses its meaning, because the original destination was not so important to start with.

The last two scenarios suggest the possibility that making an error in school should not always be perceived as something negative to be avoided, because it could occasionally provide learning opportunities that can be capitalized on. As suggested by the situation depicted in the second scenario, analyzing one’s errors could in fact contribute not only to the completion of the task that had been originally set, but also to a better understanding of the topic under study. It is worth noting that a realization of this potential value of errors has already been recognized in the context of computer programming, where “debugging” one’s programs is considered an integral part of learning to write efficient programs. The tourist scenario goes even further in uncovering new roles for errors in mathematics instruction, as it suggests that errors could occasionally provide students with the opportunity and the stimulus to engage in mathematical inquiries that, in turn, may yield some valuable and unexpected results—regardless of whether they contribute to the solution of the original task.

At the same time, the different reactions to getting lost described in the three scenarios should make educators reflect on the key role played by the *context* in which the error is made. As long as students perceive making errors as evidence of academic failure and the cause of lower grades, and continue to operate under time pressures to “cover” a rigidly prescribed curriculum sequence, it is not surprising that most of them react to errors with frustration and show only the desire to be told how to get the right answer. It is also unlikely that students will be willing and interested in spending the considerable time and effort necessary to pursue a study of their errors, unless they expect such activity to be worthwhile and rewarded.

Therefore, in order to enable students to appreciate and take advantage of errors along the lines suggested by the second and third scenarios, it is necessary that a compatible learning climate be established in the classroom. Among other things, this may require a change of emphasis from *product* to *process* in the evaluation of students’ learning, the creation of incentives for conceptual understanding and risk taking, and some flexibility with respect to learning objectives so as to allow for the degree of leisure necessary to pursue the “digressions” that could be motivated by errors.

In sum, thinking of error-making as getting lost in a city suggests the value of developing an instructional strategy that would encourage and support the students themselves to capitalize on errors in school mathematics. At the same time, such a strategy should not be conceived or implemented in isolation, because its success would largely depend on a more comprehensive rethinking of the goals and practices of current mathematics instruction. The challenge, however, seems worth meeting, once one realizes the learning opportunities that students could derive from the use of such a strategy, as illustrated by the example reported in the next section.

A FIRST ILLUSTRATION OF USING ERRORS AS SPRINGBOARDS FOR INQUIRY:

ERROR CASE STUDY A: $\frac{a}{b} + \frac{c}{d} = \frac{a+c}{b+d}$ (“Ratios” case study [A/1]³)

By looking at a specific example, in what follows I try to give a flavor of the kinds of questions and explorations that mathematical errors could invite when approached as springboards for inquiry rather than in a diagnosis and remediation spirit. When students first learn to add fractions, they often make the mistake of adding numerators and denominators separately, as illustrated by the following examples:

$$\frac{3}{4} + \frac{6}{7} = \frac{9}{11} \qquad \frac{2}{3} + \frac{5}{7} = \frac{7}{10}$$

This specific error has received considerable attention in the mathematics education literature (e.g., Ashlock, 1986; Lankford, 1974; Vinner, Herskowitz, & Bruckheimer, 1981), because of its high frequency and resilience and because of the many reasons that may lead students to such a misconception. However, once the concern for diagnosing the cause of this error and elimi-

³ The abbreviated title given in parentheses will be used in the rest of the text to refer to this specific error case study; the code reported in square brackets is intended to facilitate the location of the case study in the book, as it indicates first the consecutive letter associated to that case study and then the chapter in which the case study is reported.

nating it is temporarily left aside, one can also realize that this error may invite many mathematical questions worth investigating. In what follows, I briefly sketch some possible avenues for exploration that can be stimulated by this erroneous method of adding fractions (see Borsari, 1987, and Borsari & Michaelson, 1985, for a more developed analysis of some of these questions).

- Leaving aside for a while the concern for remediating the original error, we could challenge the status quo and question whether there are cases in which operating in this way would make sense. Depending on how we interpret this general question, we may decide to pursue the following lines of inquiry:
 - Are there other operations with fractions where numerators and denominators are separately combined?
 - Are there some fractions for which the result of adding with the standard algorithm and this alternative one are the same, or at least "close enough?"
 - Are there real-life situations that are described by this way of adding?
- The answer to the first question may motivate a review and comparison of all the operations dealing with fractions the students know. This, in turn, may result in a better understanding of these algorithms and their relationships (and, indirectly, contribute to clarifying the confusion between the algorithms for multiplication and addition of fractions that has been identified as a main cause of this error—e.g., Ashlock, 1986). The second question, instead, could be addressed by trying to solve an "unusual" equation in several variables ($\frac{ad+bc}{bd} = \frac{a+c}{b+d}$), and/or by searching for patterns in the results obtained by applying the standard and the "alternative" addition algorithms to several fractions. In both cases, this exploration is likely to involve some creative mathematical problem solving and could benefit from the aid provided by calculators or computers for executing calculations and generating data.

The last question, somewhat surprisingly, leads to the realization that there are indeed many situations where it makes sense to "add" in this way. Consider for instance the following examples:

- Baseball batting average: If a player gets 3 hits out of 4 times at bat in one game and 6 hits out of 7 times at bat in another, his average is 9 out of 11 (and not $\frac{45}{28}$, which is the result of adding the fractions $\frac{3}{4}$ and $\frac{6}{7}$).

- Keeping a record of "game results": If I won 2 out of 3 games yesterday, and 5 out of 7 games today, altogether I have won 7 out of 10 games (and not $\frac{29}{21} = \frac{2}{3} + \frac{5}{7}$).

A closer analysis of these examples, however, also reveals that when computing batting averages or the rate of success at certain games the "numerical objects" one is dealing with are not really *fractions* but rather *ra-*

tios The very fact that these two kinds of "number" have a different addition algorithm shows that they are not the same. This, in turn, may invite further exploration, along the lines sketched in the next subsection.

- Once we realize that *fractions* and *ratios* are distinct mathematical entities, we may be curious to identify more precisely what the differences are between the two and, even more generally, what kind of "number system" ratios constitute. By looking at more familiar number systems (such as the natural numbers, integers, fractions, etc.) as a reference and inspiration, the following kinds of questions could be raised and pursued:
 - When are two ratios considered equal?
 - What other operations (besides addition) could be introduced in this number system? How would these operations "work?"
 - Could we further extend this system? (e.g., would it make sense to talk of "negative" or "improper" ratios?)

Addressing these questions will engage us in the creation of a new number system—much as mathematicians in the past must have done when they conceived the possibility of considering "new" numbers such as negative numbers, irrational numbers, or imaginary numbers. This creative enterprise is likely to bring along some surprises (such as the realization that ratios cannot be "ordered" easily; the fact that, although 0/0 as a fraction is undefined, 0:0 as a ratio makes sense and plays an important role in the system; the difficulty of creating a meaningful algorithm for multiplying ratios or making sense of "negative ratios"). These surprising results, in turn, may invite a re-examination of properties of the standard number systems that one might have previously taken for granted.

- In the process of trying to introduce certain operations among ratios or to make sense of some puzzling results about these new "numbers," several questions may be raised about the number systems usually studied in the school mathematics curriculum, such as:
 - What is the meaning of "multiplication" in various number systems?
 - How is "order" determined in a number system? Are there some standard number systems that cannot be ordered?
 - How were existing number systems successively extended by mathematicians, and why?

The reflections and explorations motivated by these questions have the potential to generate a better understanding of how standard number systems "work" and provide a greater appreciation of their complexity and of the creativity that mathematicians showed in creating them.

- Realizing that what could have initially been considered an obvious mistake may instead be acceptable under different conditions may also challenge more generally many people's expectations about mathematical rules and results, especially if the following questions are raised and pursued:

- Are there other cases when in mathematics something can be right and wrong at the same time? How could that be?
- How can we decide whether a given rule is right or wrong in mathematics? Will it always be possible to do so?
- Can the same symbol be used in mathematics to mean different things? If so, why?

Notice how these kinds of questions will involve not so much technical explorations but rather reflections and discussions about the nature of mathematics as a discipline. Mathematical inquiries of this kind may be especially important to challenge some common misconceptions about the nature of mathematics that may have contributed to many students' dislike and disaffection toward this subject matter in school.

Indeed, many valuable mathematical inquiries could be developed by addressing any of the questions articulated in this section. The direction and depth of these inquiries will obviously depend on the mathematical background of the person engaging in them. Yet, most of the questions I have listed would be accessible, at least to a certain extent, to most secondary school mathematics students.

Although I am not arguing that *any* mathematical error could turn out to be as rich in generating questions worth exploring as the one chosen for this illustration, I suggest that mathematics teachers and educators have so far overlooked the potential of many errors to stimulate reflection and inquiry about both specific mathematical content and mathematics as a discipline. One of my main goals in writing this book has been to uncover this potential and propose ways that would enable mathematics teachers to capitalize on errors in their classes, so as to offer their students additional opportunities to engage in meaningful mathematical problem solving and problem posing as well as to challenge some of their dysfunctional views about school mathematics.

CONTENT AND ORGANIZATION OF THE BOOK

Despite their obvious limitations, I hope that the previous example and the metaphor of getting lost succeeded in giving a sense of the proposed approach to errors as springboards for inquiry in mathematics instruction. In the remaining chapters, this approach to errors will be progressively articulated, on the basis of theoretical considerations as well as empirical data, and it will also be used as a heuristic to examine some important issues about mathematics and its teaching.

In the previous discussion of the metaphor of error making as getting lost I already suggested that the pedagogical implications of a different approach to errors could not be addressed in isolation, because they greatly depend on the

overall approach to mathematics education that has been assumed. Thus, the next two chapters are devoted to a discussion of the theoretical framework informing and supporting the proposed strategy. More specifically, in Chapter 2 I review some fundamental critiques of the transmission paradigm characterizing much of current mathematics instruction and use these critiques to articulate the basic assumptions in terms of mathematics, learning, and teaching characterizing the *inquiry approach* to mathematics instruction informing my work. In Chapter 3, theoretical support for a view of errors as springboards for inquiry will then be derived by developing specific implications of the framework developed in Chapter 2, and by discussing other relevant contributions provided by researchers and practitioners in various fields. A brief review of the multiple ways errors have been viewed and employed in mathematics education so far will also help further characterize by contrast the nature of the strategy proposed in this book. This analysis also reveals that such a strategy requires a radical departure from traditional school mathematics expectations and practices regarding errors.

In Chapters 4 through 9, an approach to errors as springboards for inquiry is then progressively articulated as an instructional strategy for mathematics instruction, by looking at its application in various contexts. (Note: Because in my analysis of the proposed strategy I found it useful to define and use a number of concepts and their abbreviations, I have reported a glossary of these terms in Appendix A for the readers' convenience.) More specifically, by looking first at the roles played by errors in the development of mathematical knowledge in chapter 4, I argue that the analysis of perceived errors has often led mathematicians to unexpected discoveries and new insights, and sometimes has even opened entirely new areas of research and caused radical changes in the way mathematics itself has been conceived. To dispel the doubt that this use of errors might be a prerogative of professional mathematicians and to illustrate its full potential for mathematics instruction, in the following Chapter 5 I then show how errors can lead to valuable explorations and reflections even when dealing with mathematical content addressed by the K–12 curriculum. Based on a number of experiences conducted with mathematics students in various instructional settings (mostly at the secondary school level), I then consider some implications of capitalizing on errors in mathematics instruction—in terms of possible variations within the strategy, potential benefits and drawbacks of each variation, and obstacles that could be encountered when implementing the strategy (Chapters 6, 7, & 8). Finally, in Chapter 9 I argue for the value of engaging mathematics teachers, too, in experiences involving a use of errors as springboards for inquiry in the context of teacher education initiatives. In this chapter I also discuss how mathematics teachers interested in implementing the proposed strategy in their classroom could be supported in this challenging enterprise.

Throughout Chapters 4 to 9, a number of well-developed illustrations of how specific errors have been capitalized on—by mathematicians in the history of

mathematics, by myself and other mathematics teachers, and by students in experimental implementations of the strategy—play a key role. The illustration reported earlier in this chapter is a first example of these error case studies. Overall, the following 21 error case studies are reported in the book:

- A. $\frac{a}{b} + \frac{c}{d} = \frac{a+c}{b+d}$.
- B. Lack of Rigor in the Early Development of Calculus and its Positive Outcomes.
- C. The Surprising Consequences of Failing to Prove the Parallel Postulate.
- D. Dealing With Unavoidable Contradictions Within the Concept of Infinity.
- E. Progressive Refinements of Euler's Theorem on the "Characteristic" of Polyhedra.
- F. $\frac{1}{6} = \frac{1}{4}$. How can such a Crazy Simplification Work?
- G. Incorrect Definitions of Circle—A Gold Mine of Opportunities for Inquiry.
- H. The Unexpected Value of an Unrigorous Proof.
- I. Students' Analysis of Incorrect Definitions of circle.
- J. Debugging an Unsuccessful Homework Assignment.
- K. Students Using Errors Constructively When Developing a Theorem About Polygons.
- L. Students Dealing With an Unresolvable Contradiction: The case of 0° .
- M. Building on Errors to Construct the Formula for the Probability of "A or B" in a Middle School Class.
- N. Students Dealing Creatively with Errors When Doing Geometric Constructions.
- O. Problems Encountered when Discussing the "Crazy" Simplification $\frac{1}{6} = \frac{1}{4}$ in a Secondary Classroom.
- P. College Students Dealing with "Undefined" Results in Mathematics.
- Q. Teachers' Analysis of Incorrect Definitions of circle.
- R. Teachers' Reflections and Problem-Solving Activities Around an Unrigorous Proof.
- S. "Numbering Systems without Zero": A Teacher-Generated Exploration.
- T. "Beyond Straight Lines": A Teacher's Reflections and Explorations into the History of Mathematics.
- U. Building on Probability Misconceptions: A Student Activity Created by a College Teacher.

Abstracts of all these case studies can be found in Appendix B, along with the abbreviated titles I have used to refer to specific case studies in the text of the

book. Some summary information about the mathematical topics and contexts involved in these case studies is also provided in Table 1.1.

Taken as a whole, the error case studies developed in the book go beyond merely illustrating the variety of ways in which mathematical errors can lead to valuable mathematical activities and results. Rather, they provide concrete contexts in which the strategy of capitalizing on errors can be studied "in action." Their analysis thus empirically grounds the theoretical arguments and working hypotheses developed in the book, and generates valuable insights about the learning environment necessary to support an implementation of the suggested strategy. Furthermore, I hope that the experiences reported in the error case studies developed within mathematical classrooms will provide a concrete image of the kinds of mathematical experiences, learning environments, and discourse recommended by the most recent calls for reform, thus implicitly contributing to a better understanding of the radical changes that have been suggested for mathematics instruction as well as their potential benefits.

The more general issue of what is involved in reconceiving mathematics in construction in a spirit of inquiry is returned to even more explicitly in chapter 10, where, in light of the examples and arguments developed in the previous chapters, I explicitly discuss how the different approach to errors that I am propos-

TABLE 1.1. Topics and Contexts for the Error Case Studies Developed in the Book.

Math topic/ error content:	People using the error as "springboard":			
	The Mathematicians author	Math teachers (in math ed. courses)	Math students: college	Math students: secondary school
NUMBERS:				
problems with "zero"				
fractions vs. ratio				
ALGEBRA:				
$16/64 = 1/4$		A/1	S/9	P/8
				L/6
GEOMETRY:				
definition of circle				
a theorem on circle				
polygons/ polyhedra	E/4	G/5	Q/9	O/8
geometric constructions				
non-Euclidean geometry				
PROBABILITY:				
P(A or B)				
common misconceptions				
CALCULUS:				
infinite expressions	B/4	H/5	R/9	U/9
infinity	D/4	D/4		M/8

Note. In this table and hereafter, each error case study has been identified by a consecutive letter followed by a number indicating the chapter where such a case study is reported.

ing challenges the common perception of mathematics as the discipline of certainty and absolute truth and leads instead to the appreciation that mathematics is the product of the human mind, at the same time considerably challenging curriculum choices and teaching approaches that shape current mathematics instruction.

In sum, I hope that this book will contribute to the current rethinking of the school mathematics curriculum and of mathematics teaching practices in two complementary ways: (a) by suggesting a new instructional strategy that can help achieve the new goals set forth by recent calls for school mathematics reform, and (b) by inviting a rethinking of the nature of mathematics as a discipline as well as of mathematics learning and teaching more generally. Although the content of this book will obviously be most relevant for mathematics educators interested in improving mathematics instruction, I expect that it will interest researchers and teachers in other fields of education as well. If indeed errors can be used as springboards for inquiry in mathematics, the discipline perceived as the most dualistic in schools, such a use of errors should be even more possible in other academic disciplines. It is my hope that this book will invite other educators to translate the implications of my work on mathematical errors for the learning and teaching of their own subject area—an issue that is raised and partially addressed in the concluding chapter.

Chapter 2

Reconceiving Mathematics Education Within an Inquiry Framework

The previous discussion of the metaphor of getting lost suggested that both teachers' and students' approaches to errors will be greatly influenced by their overall views of school mathematics. In this chapter, I make explicit the fundamental assumptions about mathematics, learning and teaching that inform the strategy of using errors as springboards for inquiry and contrast this theoretical framework with the one that underlies the teaching practices that are most common in school mathematics today.

More specifically, I start by highlighting the most typical elements of today's mathematics classroom and then identifying the characteristics of the *transmission model* of mathematics instruction informing them. The key goals and assumptions of such a model are then critically examined, building on the work of several philosophers, psychologists, mathematicians, and mathematics educators. The major assumptions and goals of an *inquiry approach* to mathematics instruction emerging from several of these critiques are then explicitly articulated.¹ Besides enabling the reader to better appreciate the rationale and scope of the proposed use of errors in mathematics instruction, the discussion developed in this chapter is also intended to contribute to positioning this work with respect to the various movements that have recently called for school mathematics reform.

¹ The analysis and arguments developed in this chapter have been based on the most recent articulation of an inquiry framework for instruction that I have developed together with Marjorie Siegel (see Borasi & Siegel [1992, 1994] and Siegel & Borasi [1994]), as well as the earlier discussions of a *humanistic inquiry approach* to mathematics education reported in Borasi (1991a, 1992).

CURRENT MATHEMATICS TEACHING PRACTICES: AN EXEMPLIFICATION OF THE TRANSMISSION PARADIGM

One of the most remarkable features of U.S. mathematics classes (especially at the secondary school level) is their predictable routine. Whether the topic addressed is fractions, geometry, graphing, probability, or even calculus, the lesson is likely to develop as a sequence of review of homework, presentation of new material by the teacher, practice exercises done individually by the students, and assignment of similar exercises for homework. This pattern has not changed much in the last several years. Welch's (1978) description of mathematics classes in the 1970s, reflecting the results of a large study of mathematics instruction in the United States supported by the National Science Foundation, could fit most of today's mathematics classes as well:

In all math classes that I visited, the sequence of activities was the same. First, answers were given for the previous day's assignment. The more difficult problems were worked on by the teacher or the students at the chalkboard. A brief explanation, sometimes none at all, was given of the new material, and the problems assigned for the next day. The remainder of the class was devoted to working on homework while the teacher moved around the room answering questions. The most noticeable thing about math classes was the repetition of this routine. (p. 6, as cited in NCTM, 1991, p. 1).

The pervasiveness and persistence of the teaching practices described in this quote should not be surprising once one realizes that they are the natural consequence of the following set of assumptions informing most traditional instruction:

- A view of *mathematical knowledge* as a body of established facts and techniques that are hierarchically organized, context-free, value-free, and thus able to be broken down and passed along by experts to novices.
- A view of *learning* as the successive accumulation of isolated bits of information and skills that are achieved mainly by listening/observing, memorizing and practicing.
- A view of *teaching* as the direct transmission of knowledge that can be achieved effectively as long as the teacher provides clear explanations and the students pay attention to them and follow them with memorization and practice.

The transmission model of mathematics instruction described by these assumptions has also been instrumental in defining the goals of school mathematics reflected in most current K-12 mathematics curricula and standardized

tests. These goals focus essentially on enabling students to perform correctly and efficiently a predetermined set of mathematical techniques (consisting mainly in "computations" dealing with more and more complex types of numbers as well as algebraic symbols; e.g., Bishop, 1988; NCTM, 1989; NRC, 1989). Clear teacher explanation and/or demonstration of these techniques, followed by sustained practice by the students, may in turn seem a reasonable approach to ensure the attainment of such goals.

It is obvious, therefore, that attempts at changing the way mathematics classes are currently taught and at introducing innovative strategies are not likely to succeed unless the pedagogical assumptions of a transmission model of mathematics instruction are challenged at the same time. Although each of these assumptions may at first seem dictated by common sense, they have all been criticized by scholars working within different disciplines such as philosophy, psychology, sociology, anthropology, and mathematics education. In the following section, I briefly summarize the most influential of these critiques.

MAJOR CRITIQUES OF A TRANSMISSION PEDAGOGY

A first critique of the transmission model that has become especially popular in the United States builds on economic reasons. The previously mentioned reports published by professional associations in the United States (NCTM, 1989, 1991; NRC, 1989, 1990, 1991b) pointed out that the kind of mathematical knowledge and skills that have traditionally been the goals of direct teaching (i.e., some basic factual knowledge and computational skills) are no longer what our society requires, given the rapid changes that continuously occur and the availability of more and more sophisticated technology. Rather, to fully function in today's world, students should become good mathematical problem solvers and critical thinkers, confident in their mathematical ability and able to apply what they know in novel situations and learn new content on their own. This analysis is revealed quite explicitly in *Everybody Counts* (NRC, 1989):

Jobs that contribute to this world economy require workers who . . . are prepared to absorb new ideas, to adapt to change, to cope with ambiguity, to perceive patterns, and to solve unconventional problems. It is *these* needs, not just the need for calculation (which is now done mostly by machines), that makes mathematics a prerequisite to so many jobs. (p. 1)

A definition of the goals of school mathematics purely in terms of acquiring some specific techniques has also been criticized, although from a different perspective, by a group of mathematicians and mathematics educators supporting a more "humanistic" view of mathematics (e.g., Brown, 1982; Lerman, 1990a, 1990b; White, 1993). These scholars have argued that, in order to portray the

true nature of mathematics as a discipline, learning mathematics in schools should not be reduced to technical content, but rather should also explicitly address elements such as the history and philosophy of mathematics, applications of mathematics that reveal the social, political, and ethical dimensions of this discipline, and even affective issues connected with the learning of mathematics. An awareness of these elements (ignored in traditional curricula informed by a transmission approach) is crucial if students are to challenge the common belief that mathematics is a deterministic, black-and-white and cut-and-dried domain where there is no place for reasoning, creativity, or personal judgment (Borasi, 1990; Schoenfeld, 1989). This dualistic view of mathematics has proven dysfunctional for many students, potentially causing math avoidance and anxiety, as well as expectations and behaviors that are likely to reduce a student's chances of success in the discipline. Explicitly addressing the elements identified by the supporters of a humanistic view of mathematics in the mathematics curriculum would thus help counteract these negative effects, also enabling students to become aware of some important aspects of the mathematical culture and, in turn, making them feel more a part of the mathematical community of practice.

A second kind of critique of the transmission model is instead based on more philosophical grounds. Peirce's perspective on knowledge presented a first challenge to the transmission assumption that absolute knowledge can be achieved. First of all, Peirce argued that all knowledge is indirect, because we know the world through signs, and signs must be interpreted by other signs. Furthermore, Peirce warned against the hope of achieving "absolute truth":

Peirce does not set up "truth" as the goal. Unlike practitioners of conventional logic, Peirce understands that we have to abandon any hope of knowing that something is true once and for all and be satisfied with the idea that we can only be certain about something for the time being. (Siegel & Carey, 1989, pp. 21-22)

At the same time, Peirce suggested that the uncertainty permeating human knowledge has some positive implications, because it can cause the kind of "doubt" that promotes continuous inquiry in the effort to achieve more and more refined explanations of the world around us. In sum, Peirce proposed a dynamic view of knowledge as a "process of inquiry motivated by uncertainty," as illustrated by his metaphor of "walking on a bog":

[W]e never have firm rock beneath our feet; we are walking on a bog, and we can be certain only that the bog is sufficiently firm to carry us *for the time being* [emphasis added]. Not only is this all the certainty that we can achieve, it is also all the certainty we can rationally wish for, since it is precisely the tenuousness of the ground that propels us forward. . . .

Only doubt and uncertainty can provide a motive for seeking new knowledge. (Skagstad, 1981, p. 18)

A similar view of knowledge as inquiry also informed the work of Dewey on logic, as reflected by his definition of "reflective thought" as involving "(1) a state of doubt, hesitation, perplexity, mental difficulty, in which thinking originates, and (2) an act of searching, hunting, inquiring, to find material that will resolve the doubt, settle and dispose of the perplexity" (Dewey, 1933, p. 12).

This dynamic view of knowledge finds support in the works of Kuhn (1970), Lakatos (1976) and Kline (1980) on the history of science and mathematics. These philosophers of science have provided several historical examples that show how some scientific theories and mathematical concepts were challenged and changed over time. By showing the fallibility of results held "true" by great scientists for long periods of time, these historical analyses also warn us that the body of knowledge we currently rely on in any subject matter (even mathematics) may not be as "secure" as we would like to believe. In fact, it is always possible that new events and discoveries may challenge even what today's are the most taken for granted "truths."

A similar position about the nature of mathematical knowledge characterizes mathematics educators who identify themselves as "radical constructivists" (e.g., von Glasersfeld, 1990, 1991). These mathematics educators argue that even in mathematics (perceived by many as the "discipline of certainty" par excellence) knowledge is socially constructed and therefore neither predetermined nor absolute. Confrey (1990b) has clearly articulated this position, as reported in the following quote:

[Radical] constructivism can be described as essentially a theory about the limits of human knowledge, a belief that all knowledge is necessarily a product of our own cognitive acts. We can have no direct or unmediated knowledge of any external or objective reality. We construct our understanding through our experiences, and the character of our experience is influenced profoundly by our cognitive lenses. . . . Mathematicians act as if a mathematical idea possesses an external, independent existence; however the constructivist interprets this to mean that the mathematician and his/her community have chosen, for the time being, not to call the construct into question, but to use it as if it were real, while assessing its worthiness over time. (Confrey, 1990b, pp. 108-109)

Notice that, even if the notion of absolute knowledge is rejected, this does not mean that "anything goes" in mathematics. Rather, radical constructivists suggest that the whole notion of "mathematical truth" needs to be reconceived as the result of social negotiations within the mathematical community of the time. The important role played by such a community and by the set of shared

values, beliefs and practices that characterizes it, has recently received increasing attention in mathematics education (e.g., Ernest, 1991; Resnick, 1988; Schoenfeld, 1992; von Glasersfeld, 1991). This theme is further explicated in the following description of mathematics proposed by Schoenfeld (in press-a):

Mathematics is an inherently social activity, in which a community of trained practitioners (mathematical scientists) engages in the science of patterns—systematic attempts, based on observation, study, and experimentation, to determine the nature or principles of regularities in systems defined axiomatically or theoretically ("pure mathematics") or models of systems abstracted from real world objects ("applied mathematics"). . . .

Truth in mathematics is that for which the majority of the community believes it has compelling arguments. In mathematics truth is socially negotiated, as it is in science.

Several psychological studies on how individuals learn and attain knowledge have provided a third major critique of the transmission paradigm, as they radically challenge some of the behavioristic assumptions about learning on which such a paradigm relies. The 1980s and 1990s, especially, have seen a wealth of research on children's mathematical learning and problem solving based on the premise, based largely on Piaget's (1970) model of cognitive development, that children construct concepts and cognitive structures through interactions with their environment. These studies (e.g., Baroody & Ginsburg, 1990; Ginsburg, 1983, 1989; Steffe, von Glasersfeld, Richards, & Cobb, 1983) suggest that in order to learn mathematics effectively, students need to make sense and construct a personal understanding of specific concepts, rules or algorithms—they cannot simply "absorb" these from teacher's explanations or even demonstrations. The role played by individuals' background knowledge, cognitive structures, interests, and purposes, in creating such personal understanding has also been emphasized:

Many decades of research on human learning of complex subjects suggests that people do not always learn things bit by bit from the ground up. . . . They often jump into any situation with some knowledge, however rudimentary or inaccurate, and, even before they have mastered specific techniques, they begin fitting their knowledge into a larger picture. Students bring their own interpretation of tasks and concepts to the instructional process. (Sliver, Kilpatrick, & Schlesinger, 1990, p. 6)

The view of learning just described, although also referred to as "constructivist" in the mathematics education literature, does not always share the assumptions about knowledge discussed earlier as characteristic of radical constructivism.

Rather, as suggested by Goldin (1990), it could be compatible with instructional approaches that assume the mathematical knowledge to be learned as pre-established, even though it needs to be personally reconstructed by each individual in order to learn it:

It is important to recognize that one does not need to accept radical constructivist epistemology in order to adopt a model of learning as a constructive process, or to advocate increased classroom emphasis on guided discovery in mathematics. . . . [A] scientific, moderately empiricist epistemology is equally compatible with such views. (Goldin, 1990, p. 40)

However, there are some fundamental differences between instructional approaches informed by a radical constructivist orientation and those adopting the more "moderate" constructivist viewpoint about learning characterized by the assumption that "knowledge is radically constructed by the cognizing subject, not passively received from the environment" (Kilpatrick, 1987b, p. 7). This distinction is well captured by the following analysis of "discovery learning":

Discovery learning certainly rested on some assumptions that constructivists share. It stressed the importance of: 1) involving the student actively in the learning process; 2) emphasizing the process of "coming to know" over the rapid production of correct answers; and 3) extracting and making increasingly visible the structure of a concept. . . . However, the statement in the first claim, "genuinely engaged in solving a problem," entails, for the constructivist, more than reasoning inductively to conclude with a predesignated generalization carefully manufactured by using a set of examples. . . . Discovery learning was a model for promoting more effective learning—the epistemological content (the claims about the mathematical knowledge to be learned) remained relatively untouched. . . . For the constructivist, mathematical insights are always constructed by individuals and their meaning lies within the framework of that individual's experience. Students' explanations, their inventions, have legitimate epistemological content and are the primary source for investigation (other potential sources include the beliefs of teachers and mathematicians). For the constructivist, mathematical ideas are created and their status negotiated within a culture of mathematicians, of engineers, of applied mathematicians, statisticians or scientists, and, more widely, society as a whole, as it conducts its activities of commerce, construction and regulation. (Confrey, 1991, pp. 112–113)

Research on learning informed by the work of Vygotsky (1962, 1978), although sharing the fundamental constructivist assumption that learners have to construct their own knowledge, has also added another crucial dimension to the

study of students' learning by pointing out the importance of social interactions on learning:

[A Piagetian] constructivist approach to cognitive modelling, while offering an account of the psychological processes involved in children's mathematical development, has tended to downplay the importance of social interaction in the learning process. . . . Like Piaget, Vygotsky views learners as active organizers of their experience but, in contrast, he emphasizes the social and cultural dimensions of development. (Cobb, Wood, & Yackel, 1990, p. 126).

In contrast with Vygotsky's original theory and based on more recent research informed by a "social constructivist" approach, Cobb and his colleagues have also argued that:

Social interaction is not a source of processes to be internalized. Instead it is the process by which individuals create interpretations of situations that fit with those of others for the purpose at hand. In doing so, they negotiate and institutionalize meaning, resolve conflicts, mutually take others' perspectives and, more generally, construct consensual domains for coordinated activity (Bauersfeld, 1988; Bishop, 1985; Blumer, 1969; Maurana, 1980; Perret-Clermont, 1980). These compatible meanings are continually modified by means of active interpretative processes as individuals attempt to make sense of situations while interacting with others. Social interactions therefore constitutes a crucial source of opportunities to learn mathematics in that the process of constructing mathematical knowledge involves cognitive conflict, reflection, and active cognitive re-organization. (Cobb, Wood, & Yackel, 1990, p. 127)

Social constructivist theories of learning and development are receiving increasing attention from the educational research community and have provided a framework for valuing social norms and classroom dynamics. This emphasis on the social nature of learning well complements the critiques of the transmission model by philosophers and historians of science discussed earlier in this section, as well as recent studies of learning and thinking undertaken within an anthropological framework (e.g., Lave, 1988; Lave & Wenger, 1989; Rogoff & Lave, 1984). The latter studies have also made mathematics educators more aware of the crucial role played by the *community of practice* (Lave & Wenger, 1989) within which the learning act occurs, as this community shapes the goals and expectations of the learner and, even more importantly, implicitly provides a set of viewpoints, perspectives, and values that inform the learner's activity as well as his or her perceptions and interpretations. This awareness has led several mathematics educators (e.g., Bishop, 1988; Lave, Smith, & Butler, 1988;

Resnick, 1988; Schoenfeld, 1992) to posit that mathematics education should be reconceived not so much in terms of *instruction*, but rather as a process of *enculturation* or *socialization* where acquiring a "mathematical viewpoint" becomes central. This new emphasis is well articulated by Resnick (1988):

[T]he reconceptualization of thinking and learning emerging from the body of recent work on the nature of cognition suggests that becoming good mathematical problem solver—becoming a good thinker in any domain—may be as much a matter of acquiring habits and dispositions of interpretation and sense-making as of acquiring any particular set of skills, strategies or knowledge. If this is so, we may do well to conceive of mathematics education less as an instructional process (in the traditional sense of teaching specific, well-defined skills or items of knowledge), than as a socialization process. . . . If we want students to treat mathematics as an ill-structured discipline—making sense of it, arguing about it, and creating it, rather than merely doing it according to prescribed rules—we will have to socialize them as much as to instruct them. This means that we cannot expect any brief or encapsulated program on problem solving to do the job. Instead, we must seek the kind of long-term engagement in mathematical thinking that the concept of socialization implies. (p. 58)

The critiques of the transmission paradigm articulated in this section, although coming from different disciplinary perspectives and emphasizing complementary aspects, seem to share a view of inquiry as a social, constructed and contingent process of knowing. In the next section I try to identify more explicitly the basic assumptions and implications of an approach to mathematics instruction based on such a view of inquiry.

KEY ELEMENTS OF AN INQUIRY FRAMEWORK

The inquiry model that emerges from the critiques to the transmission paradigm summarized in the previous section is grounded in the following set of assumptions about mathematics, knowledge, learning, and teaching:

- A view of *mathematics* as a humanistic discipline; that is, the belief that mathematical knowledge is socially constructed and fallible as well as shaped by cultural and personal values.
- A view of *knowledge* more generally as constructed through a process of inquiry where uncertainty, conflict and doubt provide the motivation for the continuous search for a more and more refined understanding of the world.

- A view of *learning* as a generative process of meaning making, requiring both social interaction and personal construction, and informed by context and purposes.
- A view of *teaching* as stimulating and supporting the students' own inquiry and establishing a learning environment conducive to such inquiry.

A mathematics classroom informed by these assumptions would look quite different from traditional mathematics classrooms, as illustrated by the instructional experiences reported later in Chapters 6, 7, and 8. In what follows, I identify some of the most important implications of these assumptions for mathematics instruction (which are further revisited in Chapter 10).² In the course of this discussion, key similarities and differences between the inquiry approach I have assumed in the study reported in this book and other important movements within mathematics education, especially the most recent calls for school mathematics reform in the United States, are also highlighted.

An inquiry approach is compatible with and would support many of the changes in mathematics curricula and teaching practices proposed by the already mentioned reports recently produced in the United States by influential associations such as the National Council of Teachers of Mathematics (NCTM, 1989, 1991) and the National Research Council (NRC, 1989, 1990, 1991b). Most notably, the views of learning and knowledge articulated earlier support a shifted emphasis from *product* to *process*, and from *teacher's explanations* to *students' constructions*, as well as the new importance given to classroom discourse and mathematical communication, underlying the NCTM and NRC recommendations. At the same time, as a result of some of the explicit assumptions articulated here, an inquiry approach also emphasizes a number of dimensions that have been somewhat neglected in these documents.

First of all, although appreciating the more extensive and meaningful content proposed by the *NCTM Standards* for the K-12 mathematics, the assumption of a humanistic view of mathematics also leads to critiquing such a curriculum as too restrictive insofar as it is still defined mainly in terms of *technical* content. Within an inquiry approach, instead, it would be considered important also to include, as legitimate curriculum content of each mathematics course, experiences enabling students to appreciate the historical development of specific mathematical concepts and/or areas, the power and limitations of the techniques studied to solve problems in various domains, and the cognitive and emotional components of learning the course content. Furthermore, any curriculum should be designed and conceived of with sufficient flexibility so as to allow for the unexpected explorations that genuine student inquiries could invite.

²See also the instructional experience reported and analyzed in depth in my book *Learning Mathematics Through Inquiry* (Borasi, 1992) for a more contextualized and well-developed illustration of an inquiry approach to mathematics instruction "in action."

Another consequence of both a humanistic view of mathematics and a view of knowing as inquiry is a call for uncovering uncertainty in the mathematical content studied so as to generate genuine doubt and, consequently, invite student inquiry. Whereas in traditional mathematics classes ambiguity, anomalies, and contradictions are carefully eliminated so as to avoid a potential source of confusion, in an inquiry classroom these elements would be highlighted and capitalized on as a motivating force leading to the formation of questions, hunches, and further exploration. In other words, mathematics teachers would be expected to help their students use the confusion that these elements can originate as a starting point for problem posing and data analysis, rather than trying to clear up confusion for them.

The radical constructivist epistemology assumed in an inquiry approach and characteristic of the current work of several mathematics education researchers (e.g., Balacheff, 1988, 1991; Borasi, 1992; Brown & Walter, 1990, 1993; Cobb, Wood, & Yackel, 1990; Confrey, 1990b; Lampert, 1986, 1990; Schoenfeld, 1988, 1992, in press-a, in press-b), also calls for a greater emphasis on students' problem posing and initiative in school mathematics. In most interpretations of a problem-solving approach, problems are set by the teacher, who knows their solution even if the students do not. Also in discovery learning, as discussed in the previous section, what the students discover is predetermined and unquestioned. In an inquiry model, on the contrary, the process the students engage in is expected to be more open ended and generative, insofar as neither the teacher nor the students may know what the outcome of the inquiry may be as they engage in it and, furthermore, the students are actively involved in making decisions about the scope and directions of an inquiry at all stages. This does not mean, however, that within an inquiry approach mathematics teachers should totally relinquish their role in planning and orchestrating classroom activities. Far from diminishing their responsibilities, an inquiry approach imposes new demands on mathematics teachers as they are now expected to provide both stimuli and structure to support the students' own inquiries:

Teachers create an inquiry environment by design and careful planning. A consensual domain that fosters inquiry is not generated simply by removing structure, but rather by creating a different structure that provides support for the students to participate. . . . Students will not become active learners by accident, but *by design*, through the use of the plans that we structure to guide exploration and inquiry. (Richard, 1991, p. 38)

Furthermore, within an inquiry classroom, the teacher has the added responsibility of creating a learning environment that is conducive and supportive of student inquiry. This will involve establishing a radically different set of social norms and values in the classroom, which may take a long time and explicit efforts from the part of the teacher. Among other things, the teacher may need to

model how to approach various aspects of the inquiry process and provide both explicit and implicit messages to help students value all that is involved in being inquirers (Harsic & Short with Burke, 1988). As students take center stage in inquiry classrooms their roles also change, requiring a willingness to listen and negotiate with others as well as a much greater degree of risk taking on the part of the student. As Lampert (1990) wrote:

From the standpoint of the person doing mathematics, making a conjecture . . . is taking a risk; it requires the admission that one's assumptions are open to revision, that one's insights may have been limited, that one's conclusions may have been inappropriate. Although possibly gaining recognition for inventiveness, letting other interested persons in on one's conjectures increases personal vulnerability. (p. 31)

In sum, an inquiry approach would endorse all the fundamental goals for school mathematics articulated by the *NCTM Standards*—that is, enabling students to value mathematics, become confident in their ability to do mathematics, become competent problem solvers, learn to communicate mathematically, and learn to reason mathematically (NCTM, 1989, p. 5). In addition, it would expect students to gain some initiative and ownership in their mathematical activity, by learning to pose problems and questions for exploration on their own as well as to monitor their own mathematical work. Mathematics students would also be expected to gain a better understanding of mathematics as a discipline by coming to appreciate the roots and the forces behind its historical development, the tentative and socially constructed nature of its results, the power and limitations of its application to solve problems in other domains, and the positive role played by ambiguity, uncertainty, and controversy in the creation of mathematical knowledge. As a result, one would hope that they would develop more realistic expectations and functional behaviors as mathematics students. As one proceeds in reading this book, it is important to keep in mind that the strategy of using errors as springboards for inquiry was informed by these alternative views on teaching and learning mathematics and was conceived as a means to support mathematics teachers in creating a learning environment and activities that will facilitate the attainment of these goals.

Chapter 3

Alternative Views on Errors

The inquiry framework developed in the previous chapter proposes views of knowledge, learning, and teaching that support an educational approach to errors as springboards for inquiry. In this chapter, I first of all discuss how assuming a constructivist epistemology requires us to re-examine both the nature and the role of errors in an educational context. Other contributions to the rationale of the proposed approach to errors are also identified. Finally, I briefly review the most salient uses of errors made so far in mathematics education research as well as school mathematics and contrast them with the instructional approach to errors advocated in this book.

CONTRIBUTIONS TO A VIEW OF ERRORS AS SPRINGBOARDS FOR INQUIRY

Explicating the Role of Errors Within a Constructivist Epistemology

The constructivist view of how knowledge is attained, discussed earlier in Chapter 2, has important implications for an educational approach to errors. Let us consider, first of all, the key role assigned to uncertainty, and even more specifically to anomalies, in Peirce's view of knowledge as "a process of inquiry motivated by doubt." Anomalies are here defined as "something that does not make sense" (Siegel & Carey, 1989, p. 23) and as such they are considered likely to motivate the kind of doubt that can set the inquiry process in motion. Furthermore, an anomaly can also inform the direction taken by such an inquiry and its results by raising specific questions that can in turn suggest new exploratory hypotheses. This is especially evident in the role Peirce attributed to anomalies within *abduction*, one of the three key components of the process of inquiry described here:

Peirce defined reasoning as a continuous cycle of abduction, deduction and induction. These three ways to make inferences can be distinguished

in the following manner: first, hypotheses are generated through abduction; the possible consequences of those hypotheses are then developed through deduction; finally, these consequences are tested out against experience so that the hypotheses can be either accepted or modified. . . . Anomalies, those perceptual judgements that seem unexpected, play a crucial role in abductive reasoning: they motivate the formation of new connections among prior judgements in such a way as to generate an hypothesis that explains the unexpected (Siegel & Carey, 1989, pp. 23–24)

Because errors, by definition, are results that do not meet expectations, they can be considered a prototypical example of an anomaly. Thus they, too, can be viewed as a natural stimulus for reflection and exploration and as a means to support inquiry.

Kuhn's (1970) interpretation of the development of science further supports this view of errors, or anomalies more generally, as a key element in triggering "scientific revolutions":

Confronted with anomaly or crisis, scientists take a different attitude towards existing paradigms, and the nature of their research changes accordingly. . . . Scientific revolutions are inaugurated by a growing sense . . . that an existing paradigm has ceased to function adequately in the exploration of an aspect of nature to which that paradigm itself had previously led the way. (pp. 90–91)

Indeed, some of the historical examples Kuhn (1970) identified as catalysts for "scientific revolutions" consisted of unacceptable results or unsolvable problems that could have been interpreted by scientists of the time, or now in retrospect, as some sort of "error." Kuhn's work thus suggests that the analysis of errors by scientists may lead to results other than the elimination of the error as originally perceived. Indeed, there are significant examples to this regard in the history of mathematics, as I illustrate and discuss later in Chapter 4.

Finally, Lakatos' thesis that the growth of mathematical knowledge often occurs through a dialectic process where initial (and often partially incorrect) hypotheses are progressively refined through the analysis of both supporting examples and counterexamples (Lakatos, 1976, 1978) gives an even more explicit recognition to the crucial role played by errors in mathematical inquiries. Lakatos characterized this process of "proofs and refutations" as consisting essentially of the following three rules:

Rule 1. If you have a conjecture, set out to prove it and to refute it. Inspect the proof carefully to prepare a list of non-trivial lemmas (proof-analysis); find counterexamples both to the conjecture (global counterexamples) and to the suspect lemmas (local counterexamples).

Rule 2. If you have a global counterexample discard your conjecture, add to your proof-analysis a suitable lemma that would be refuted by the counterexample, and replace the discarded conjecture by an improved one that incorporates that lemma as a condition. Do not allow a refutation to be dismissed as a monster.

Rule 3. If you have a local counterexample, check to see whether it is not also a global counterexample. If it is, you can easily apply Rule 2. (Lakatos, 1976, p. 50)

(See "Euler theorem" historical case study [E/4] for a better articulation of this method in the context of the development of a specific conjecture.) Lakatos not only posited this activity of identifying the limitations of an initial conjecture and finding ways to overcome them as an effective approach to improve already hopeful conjectures. He went further, suggesting the value of proposing and studying false as well as hopefully correct conjectures. He suggested that mathematicians' work can be most productive if indeed they approach all of their initial conjectures as something to be proved and refuted at the same time, where the goal would not be reduced to making the original conjecture more rigorous, but rather conceived as expanding on such an initial conjecture by means of the insights provided by the analysis of potential counterexamples. Thus, one can say that in Lakatos' "proofs and refutations" approach to the construction of mathematical knowledge errors are not approached as something to be possibly avoided, but rather as a necessary and constructive step in the creation of mathematical knowledge.

In sum, the philosophical contributions summarized here suggest that errors have the potential to raise constructive doubt and questions that can, in turn, lead to worthwhile mathematical inquiries. Furthermore, such inquiries do not always need to be reduced to a search for the causes of the error, with the ultimate goal of eliminating it. Rather, if we are open to pursuing more challenging questions—such as "what would happen if we accepted this result?" or "in what circumstances could this result be considered correct?"—then the analysis of an error might lead to a reformulation of the problem under study, a deeper understanding of the context in which the problem was generated, and even to some unexpected and novel results.

Re-examining the Notion of "Error" From a Radical Constructivist Perspective

A constructive epistemology has radical implications not only for the roles and uses of errors, but for the very notion of *error* as well. When all knowledge is viewed as fallible, and results are accepted as true only "for the time being," what constitutes an error may not always be clear cut. I suggest that Balacheff's

observations about contradictions, as reported in the following quote, can be easily generalized to other kinds of errors as well:

[A] contradiction does not exist by itself, but only with reference to a cognitive system. For instance, a contradiction could be recognized by the teacher in the mathematics classroom, but ignored by the students. . . . On the other hand, a contradiction exists only with reference to a disappointed expectation, or with reference to a refuted conjecture. (Balacheff, 1991, pp. 90-91)

Indeed, most often the decision of whether something constitutes an error may depend both on the context and the person making that decision. Especially when one operates in an inquiry mode, whether something is correct or not may no longer be obvious (at least for the student, if not for the teacher). Rather, whether a tentative or puzzling result should be accepted or not will often be an open question, to be resolved by deriving and evaluating its potential consequences as well as its logical justification and by considering a number of other factors as well—such as the context in which one is operating and the specific problem studied. Most importantly for our discussion, the benefit of engaging in such an analysis with respect to learning mathematics may be essentially the same regardless of whether students conclude by deciding that the result examined was correct or not.

Thus, although the ambiguity uncovered in the previous paragraphs makes the definition of mathematical error a difficult and controversial philosophical task, such ambiguity may be turned to advantage in a pedagogical context. Given that my ultimate goal is to uncover new ways to promote student inquiry in school mathematics, hereafter I have chosen to interpret the term mathematical error in the most comprehensive way possible, so as to maximize the occasions of using the proposed strategy in mathematics instruction. Thus, within this book, even borderline cases such as contradictions, tentative hypotheses and definitions, contrasting results, or results that do not make sense, are considered legitimate starting points for *error activities*—that is, instructional activities designed so as to capitalize on the potential of “errors” to initiate and support inquiry.

Implications of a Constructivist View of Learning

Theoretical support for a view of errors as springboards for inquiry is also provided, from a slightly different perspective, by the work of several psychologists informed by a constructivist approach. First of all, an appreciation of the motivational power of anomalies is embedded in the Piagetian conception of learning as “re-establishing equilibrium” after encountering a puzzling situation that caused “disequilibrium” because no explanation or solution was immedi-

ately available from what the learner already knew. A view of errors as catalysts for learning, however, is even more explicitly embedded in the concept of *accommodation*, a process that Piaget posited, together with *assimilation*, as one of the fundamental mechanisms of child development. Consider the following description of such a process: “In a sense, accommodation occurs because of the *failure* [emphasis added] of the current structures to interpret a particular object or event satisfactorily. The resulting reorganization of thought leads to a different and more satisfactory assimilation of the experience” (Miller, 1983, p. 72).

Notice how such failure can either cause some error or be interpreted by the subject as an error in itself. It is also important to note that the main result of accommodation, as described in the quote, is not so much the remediation of such an error (although this may also follow as a secondary result), but rather a more general revision of a theory or even a way of seeing the world—much akin to the “scientific revolutions” discussed by Kuhn (1970) in his reconstruction of the development of scientific knowledge.

“Inducing” cognitive disequilibrium or cognitive conflict—that is, creating learning environments or activities that are likely to cause errors or conflict needing to be resolved—can thus be seen as a fundamental component of instruction informed by a Piagetian constructivist viewpoint (as shown, for example, in some of the instructional situations designed and studied in Inhelder, Sinclair, Bovy, 1974). Flavell (1977) articulated this principle even more explicitly in his identification of the following phases in a child’s cognitive progress:

- (1) noticing both of the apparently conflicting elements in a situation, (2) interpreting and appreciating the two as conflicting, (3) attempting an explanation of the differences rather than “clinging defensively to” the “initial belief or refusing to have anything more to do with the problem,” and (4) constructing a new conceptualization that accommodates both of the elements, thereby reaching equilibrium. (Graeber & Johnson, 1991, p. V-3)

More generally, constructivist psychologists support the idea that *conflict* or *cognitive dissonance* are catalysts for learning and development. Errors are likely to naturally create such conflictual situations and, thus, can make students aware of the need to critically review their procedures, get more information, or even “adjust their theories” (Confrey, 1990a, 1990b; Graeber & Johnson, 1991).

Further support for the positive role that error can play in student mathematical activities is indirectly provided by several research studies on mathematical learning and problem solving, informed by constructivist and cognitive science perspectives, that have highlighted the important role played by monitoring one’s activity and by “metacognition” more generally (e.g., Brown, 1987; Jones, Palincsar, Ogle, & Carr, 1987; Lester, Garofalo, & Kroll, 1989; Schoen-

feld, 1985, 1987, 1992). As Schoenfeld (1992) pointed out, the term *metacognition* has been used quite loosely in the literature to include various categories of thinking, including "(1) individuals' declarative knowledge about their cognitive processes, (2) self-regulatory procedures, including monitoring and 'on-line' decision-making, and (3) beliefs and affects and their effects on performance" (p. 347). Research in this area has revealed that:

[M]odel learners are aware of and control their efforts to use particular skills and strategies. . . . Awareness refers not only to knowledge of specific cognitive strategies but also to knowledge of how to use them and when they should be used. Control refers, in part, to the capability to monitor and direct the success of the task at hand, such as recognizing that comprehension has failed, using fix-up strategies, and checking an obtained answer against an estimation. Additionally, a large part of controlling strategy use relates to learners' perseverance in motivating themselves, in making decisions about the importance of the task, in managing their time, and in their attribution of success or failure. (Jones, Palincsar, Ogle, & Carr, 1987, p. 15)

These results suggest that an explicit focus on errors could be beneficial to mathematics students as it could contribute to developing some of the metacognitive skills identified as necessary to become independent and efficient problem solvers. First of all, such a focus could enable students to become familiar with specific strategies to critically review and check their mathematical work, at the same time developing the expectation that identifying and correcting errors is mainly the learners' responsibility rather than the teacher's (contrary to what many students tend to believe as a result of traditional mathematics instruction [Gajary, 1991]). Furthermore, because they create a conflict or produce a result that is not acceptable, errors may force a mathematics learner or problem solver to explicitly look at and discuss their work (i.e., their goals, the strategies they used, their feelings about it, etc.) from a metacognitive viewpoint, and also provide a concrete starting point for doing so.

Constructive Uses of Errors Made in Instructional Contexts Other Than the Mathematics Classroom

The value of engaging the students themselves in a constructive use of their errors has already been explicitly recognized in some academic subject areas other than mathematics. In what follows, I briefly review a few examples that can help us rethink the potential role of errors in mathematics instruction.

An academic area where in the last few decades the approach to errors has been radically reconceived is that of writing instruction. On one hand, attention

has shifted away from more "trivial" errors such as spelling or grammar mistakes, so as to minimize their potential interference with the real crux of the act of writing—that is, elaborating and communicating meaning on paper. At the same time, both students and teachers have been asked to accept a different kind of "error"—that is, their early tentative drafts, requiring considerable revisions before being considered "acceptable"—as an integral part of the writing process. Indeed, many recent writing programs, especially those influenced by a "writing to learn approach" (e.g., Connolly, 1989; Emig, 1977; Gere, 1985; Mayher, Lester, & Pradl, 1983; Young & Fuwiler, 1986), encourage students to put on paper their thoughts, however tentative and incomplete, as a means to both articulate such beginning thoughts and provide a concrete starting point to critically examining, refining, and elaborating on them.

These shifts are quite suggestive for mathematics instruction as well. Contrary to most current practices, especially in early grades, a "writing to learn" approach suggests that mathematics teachers, too, should decrease their attention to and concern for computational errors, so as to free their students to focus on mathematical thinking and problem solving. At the same time, students should be led to accept errors such as tentative definitions and hypotheses, partial results, or only partially successful procedures as an integral part of their mathematical activity—an approach that is quite foreign to most mathematics students, who tend to believe that either you remember the right way to solve a mathematical problem or you might as well give up (Borasi, 1990; Schoenfeld, 1989, 1992). In other words, a "successive draft approach" to mathematics learning would encourage students to pursue the solution to novel mathematical problems by attempting alternative approaches and critically examining and building on their tentative results, whether correct or not. (Good illustrations of these principles in action can be found in the "Students' polygon theorem" [K/6] and "Students' geometric constructions" [N/8] case studies).

Computer programming is another instructional area in which errors have received some explicit attention. In computer programming courses, in fact, it is now an established practice to expect students to debug their own incorrect programs. That is, when a program does not work as its creator wished, or it does not run at all, the student is expected to identify and eliminate the error with minimal help from the outside (in the form of computer error messages, or the appeal to a computer consultant). Although often a difficult task, debugging constitutes a challenging problem-solving activity and may yield not only a better program, but also a deeper understanding of the problem in question and a better knowledge of the potential and limitations of the computer language used. Papert (1980), for example, reported the case of a child who began to appreciate the power of using subprocedures and a "structured programming" approach to writing LOGO programs as a result of realizing the difficulty of debugging his own unstructured programs. It is interesting to note that although the use of words such as *bug* and *debug* can be found occasionally in the cur-

rent mathematics education literature, most educators seem to have missed an important element in the practice of debugging in computer programming. That is, the role of analyzing and resolving errors should pass from the teacher to the students themselves if mathematics educators want to exploit the potential of errors to initiate problem solving and improve mathematical understanding.

It is also important to point out that the educational value of errors when working with computers can go even beyond debugging one's program so as to "make it work." This can be especially seen in Papert's (1980) suggestion of LOGO as a learning environment that could stimulate a new kind of approach to learning in school. In Papert's vision, as students engage in genuine inquiry within the LOGO environment, programming errors that cause unexpected results may not only invite reflections about the "Turtle" language, but also generate new questions worth exploring. Consider, for example, the following hypothetical dialogue between two students trying to draw a flower using LOGO, intended by Papert to provide an image of the kind of learning environment that could be encouraged through LOGO (Note: In the following quote, the text in *italic* reports my own explanations of figures or explanations in the original text that have been omitted here for the sake of brevity):

- Let's make a petal by putting two QCIRCLES (*i.e.*, a program previously written to draw a quarter of a circle) together.
- OK. What size?
- How about 50? (*the students write the sequence of instructions "QCIRCLE 50; QCIRCLE 50" and obtain a semicircle*)
- It didn't work.
- Of course! Two QCIRCLES make a semicircle.
- We have to turn the Turtle between QCIRCLES.
- Try 120°.
- OK, that worked for triangles.
- And let's hide the Turtle by typing HIDE TURTLE. (... The sequence of commands "QCIRCLE 50; LEFT 120; QCIRCLE 50" produces Figure 3.1.)
- What's going on?
- Try a right turn.
- Why don't we just stick with the bird? We could make a flock. (Papert, 1980, pp. 79-81)

Note how in this case the "buggy" program produced an unexpected result (the "bird" illustrated in Figure 3.1) that, regardless of its inadequacy with respect to accomplishing the original task of drawing a petal, had merits of its own and suggested a worthwhile digression such as writing a program that could draw flocks of similar birds of different sizes and in different positions (see Papert, 1980, pp. 90-92). Analogous to this example, I suggest that a sim-

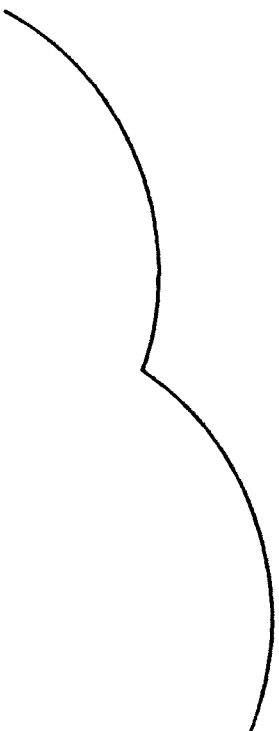


FIGURE 3.1. Figure produced by the set of commands "QCIRCLE 50; LEFT 120; QCIRCLE 50" in LOGO.

ilar attitude toward mathematical errors could invite student problem posing and exploration within school mathematics.

It is also important to note, however, that although learning to debug one's programs is considered an important goal even for beginning programmers, some authors have shown that students may find considerable difficulties with this task. First of all, if left on their own, students may not be able to develop efficient debugging strategies—as shown, for example, by Carver (1988), who suggested that debugging strategies should explicitly be taught to beginning programmers. Even more importantly, the students' rather dualistic expectations about school learning and knowledge may provide a considerable obstacle to engage in debugging:

Children often develop a "resistance" to debugging. . . . I have seen this in many children's first sessions in a LOGO environment. The child plans to make the Turtle draw a certain figure, such as a house or a stick man. A program is quickly written and tried. It doesn't work. Instead of being debugged, it is erased. Sometimes the whole project is abandoned. Sometimes the child tries again and again and again with admirable persistence but always starting from scratch in an apparent attempt to do the thing "correctly" in one shot. . . . It is easy to empathize. The ethic of school has rubbed off too well. What we see as a good program with a small bug, the child sees as "wrong," "bad," a "mistake." School teaches that errors are bad; the last thing one wants is to pore over them, dwell on them, or think about them. . . . The debugging philosophy suggests an opposite attitude. Errors benefit us because they lead us to study what happens, to understand what went wrong, and, through understanding, to fix it. (Papert, 1980, pp. 113-114)

The difficulties with debugging identified here should be taken into consideration by any attempt to capitalize on errors in mathematics instruction as well.

More specifically, they suggest that specific strategies to use errors constructively should be explicitly introduced to students and that students' beliefs that could run contrary to such a use of errors be addressed.

Finally, another interesting example of an instructional use of errors that involves the students themselves in the analysis of their errors is provided in the context of science instruction informed by a "conceptual change" approach (see Confrey, 1990a, for a review of research in this area). Such an approach pays considerable attention to students' *misconceptions* or *preconceptions* of specific scientific concepts, which are conceived of as "mini-theories: configurations of beliefs that [can be] likened to the broad theoretical commitment held by communities of scientists" (Confrey, 1990a, p. 20). Although this line of educational research has focused more on identifying and understanding students' conceptions rather than on instructional implications, some authors (e.g., Hewson, 1981; Nussbaum & Novick, 1982; Rowell & Dawson, 1979) have suggested that students' specific misconceptions could be used as a way to generate conflicts which, in turn, may expose and challenge the students' limited theories about how the world operates and invite the consideration of new ones. These initiatives are derived from the constructivist positions about the development of scientific knowledge and the process of learning summarized in earlier sections, as explicitly revealed by Confrey's (1990a) analysis:

Working in the area of Newtonian and Einsteinian physics, Posner, Strike, Hewson, and Gertzog (1982) borrow and refine the Piagetian terms *assimilation* and *accommodation*. They use accommodation to describe the times when a student may need to replace or reorganize his or her existing conceptions and argue for the conditions under which this is likely to occur. They require that a student be dissatisfied with an existing conception and find a new conception intelligible, plausible and fruitful. They further indicate that accommodation is facilitated when anomalies exist within their current belief system; when analogies and metaphors assist the student in accepting a new conception and make it more intelligible; and when their epistemological, metaphysical, and other beliefs support such a change. Hewson (1981) elaborates on this position by discussing how conceptions can be in competition with each other and how, in such cases of conflict, a student will raise or lower the status of one conception relative to another. (p. 23).

A CRITICAL REVIEW OF EXISTING USES OF ERRORS IN MATHEMATICS EDUCATION

To fully appreciate the radical nature of approaching errors as springboards for inquiry in mathematics instruction, it is also important to realize that such an ap-

proach is quite at odds with most teachers' and students' views of errors and also differs considerably from the uses of errors made by most mathematics education researchers to date. In what follows, I briefly identify and describe some of these positions and contrast them with the approach to errors advocated in this book. However, I would like to clarify up front that this is *not* intended to be a comprehensive review of the rich research literature on error analysis and students' misconceptions (which can be found, instead, in the works of Confrey, 1990a, Graeber & Johnson, 1991, and Radatz, 1979, 1980). Such a review would be quite beyond the scope of this section, given the focus of the book on developing and evaluating an instructional strategy that would engage the students themselves in activities that capitalize on the potential of errors to stimulate and support mathematical inquiry. Rather, my main goal in this section is to better articulate an instructional use of errors as springboards for inquiry by contrasting it with existing approaches to errors in mathematics education.

The Negative Perception of Errors in Traditional Mathematics Instruction

Most mathematics students, as well as teachers, have negative feelings about errors and approach them as unfortunate events that need to be eliminated and possibly avoided at all times. Implicit evidence of these beliefs can be found in many common practices, such as the fact that making errors automatically lowers a student's test grade, that most often incorrect answers to a question posed by the teacher in class are rejected or ignored until the correct one is produced, and that teachers try to assign tasks that "good" students should be able to complete without making errors.

These beliefs and attitudes should come as no surprise because they have been given theoretical justification within the behaviorist view of learning informing a transmission paradigm (as discussed earlier in Chapter 2). Behaviorist research, in fact, suggests that learning is enhanced when correct responses are rewarded (positive reinforcement) and incorrect ones are either punished (negative reinforcement) or extinguished through lack of attention (withholding of positive reinforcement; Miller, 1983). Within this framework, students and teachers are obviously not invited to see errors in a positive light and, furthermore, paying explicit attention to errors in class may even be considered "dangerous," because it could interfere with "fixing" the correct result in the student's mind.

Contributions and Limitations of Research on "Error Analysis"

The vast body of research in error analysis is evidence of the fact that mathematics education researchers have long recognized the value of looking care-

fully at students' errors. Since the beginning of the century there has been great interest in and research on students' systematic errors in mathematics education. The reviews of this body of research compiled by Radatz (1979, 1980), Confrey (1990a), and Graeber and Johnson (1991) have pointed out some common assumptions that characterize the approach to errors assumed by these studies and can help us see the limitations of such an approach.

First of all, these studies challenge the explanation of errors as due to uncertainty and carelessness, and set themselves the task of proving that student errors:

- are causally determined, and very often systematic;
- are persistent and will last for several school years, unless the teacher intervenes pedagogically;
- can be analyzed and described as error techniques;
- can be derived, as to their causes, from certain difficulties experienced by students while receiving and processing information in the mathematical learning process, or from effects of the interaction of variables acting on mathematics education (teacher, curriculum, student, academic environment, etc.). (Radatz, 1980, p. 16)

The overall concern for remediation characterizing these studies of mathematical errors is also well illustrated by the following quote from Radatz' (1980) review:

Student errors "illustrate" individual difficulties; they show that the student has failed to understand or grasp certain concepts, techniques, problems, etc., in a "scientific" or "adult" manner. Analyzing student errors may reveal the faulty problem-solving process and provide information on the understanding of and the attitudes towards mathematical problems. (p. 16)

Given these overall beliefs about errors, it is not surprising that error analysis research in mathematics education mainly relied on the use of diagnostic tests and statistical data, and focused on:

- Attempting to determine and classify potential causes of errors.
- Identifying potential error techniques (especially dealing with procedural knowledge and areas of the curriculum such as arithmetic and elementary algebra).
- Determining the frequency distribution of these error techniques.
- Attempting to classify and group errors.
- Documenting the persistency of individual error techniques.
- (More recently) devising computer programs that could duplicate specific student errors.

- (To a lesser degree) developing didactic aids for treating particular learning difficulties and errors.

These studies have undoubtedly provided interesting information about common errors experienced by mathematics students within the traditional mathematics curriculum. A good summary of research on the difficulties most students encounter when operating with fractions and decimals, when solving simple equations, and with the notions of ratio, proportion, probability, and limit, can be found in Graeber & Johnson (1991). These insights can inform instruction by highlighting common pitfalls mathematics teachers should try to avoid and/or suggesting alternative instructional approaches that might diminish, or even eliminate, the occurrence of common errors in the first place.

For instance, research has shown that often students who add fractions by adding numerators and denominators separately (i.e., $5/6 + 2/3 = 7/9$) do so because they (a) confuse the algorithm of multiplication of fractions with that of addition, and/or (b) do not have a sound concept of what a fraction is and what adding fractions means (e.g., Ashlock, 1986; Lankford, 1974). This information suggests that more instructional time should be spent developing a sound conceptual understanding of fractions, by using manipulatives and various forms of representation, before introducing any formal algorithm to operate with fractions. Furthermore, students should be encouraged to explicitly articulate the differences between various algorithms for operations with fractions, as well as to recognize the fundamental differences between operating with fractions instead of whole numbers (Graeber & Johnson, 1991).

Although these examples illustrate the potential contributions of error analysis research to mathematics instruction, it is important to note that most researchers in this area have left the task of deriving instructional implications to teachers and curriculum developers. Another important limitation in this area of research is that "errors are seen from the perspective of the expert" (Confrey, 1990a, p. 37) and the ultimate scope of their analysis is that of remediating or avoiding them in mathematics instruction; thus, the potential of errors to generate new questions and explorations is not likely to be exploited or even recognized. Most importantly, although mathematics students may eventually benefit from the diagnosis of their errors conducted by researchers and/or their teacher, they are not involved in the analysis itself, thus missing a valuable opportunity to come to appreciate the value of errors.

Contributions and Limitations of Research on "Students' Misconceptions"

More recently, the study of student mathematical errors has developed in directions that are more consistent with a constructivist view of learning (e.g.,

Novak, 1987; Novak & Helm, 1983). Here student errors are seen as an inevitable and integral part of learning and a valuable source of information about the learning process, a clue that researchers and teachers should take advantage of for uncovering what a student really knows and how he or she has constructed such knowledge. This more positive approach to errors is revealed by the use of terms such as *misconceptions*, *alternative conceptions*, or *implicit theories* instead of errors. Characteristic of these studies has also been a methodological approach based on clinical interviews and teaching experiments rather than statistical studies, intended to use the students themselves as the major source of information as the researchers try to reconstruct the thinking, problem-solving, or learning processes that led to the misconception.

Some important distinctions between these studies on student misconceptions and the error analysis studies discussed earlier are well captured in the following quote by Confrey (1991):

[R]esearch on student misconceptions . . . documented that students hold mini-theories about scientific and mathematical ideas and that the theories and their forms of argument must be understood and directly addressed if students are to come to a more acceptable understanding of the concept. Unlike error patterns, these mini-theories relate formal scientific/mathematical meanings for terms with their everyday usage, examine how the theories relate to historical development and discuss how the theories reflect the child/student's view of science or mathematics as a whole. (p. 121)

Confrey (1990a) identified two main categories of research within this area: (a) studies that have developed in the tradition of genetic epistemology founded by Piaget, and (b) studies that have been informed by philosophies of science such as those developed by Kuhn, Popper, Lakatos, and Toulmin. In both cases, the study of students' misconceptions can be characterized by an emphasis on:

- The role of students' theories of learning.
- "Understanding the roots" of students' misconceptions, rather than "attempting to eliminate" them.
- Assuming the learner's rather than the expert's viewpoint (hence the common use of terminology such as *student conceptions* or *alternative conceptions* instead of *misconceptions*).
- Appreciating the reasonableness of the misconception to the learner and the need for the learner to experience its limitations as a necessary prerequisite to modifying it.

This last point, in particular, suggests the potential value of errors as catalysts of conceptual change in instruction, as implicit in Confrey's (1990a) discussion of the role played by anomalies more generally:

Von Glasersfeld . . . emphasizes that it is through discrepancy, perturbation, or encounters with the unexpected that we can envision the qualities of our constructs; these key moments in our activities of reflection are opportunities to glimpse our own constructs. However, he also warns us that our "problems," that is, our perceptions of deviance, may not coincide (and probably will not coincide) with those of children. Thus, if we want to investigate their conceptions, we need to seek out their problems, not impose ours. (pp. 14-15)

It is interesting, however, to note that there has been so far little effort (or even interest) to translate the interesting results of this line of research in terms of implications for classroom instruction (Confrey, 1990a). Yet, these studies have provided many valuable contributions to mathematics education by showing the importance for teachers to listen to their students and respect their thinking and, furthermore, by providing even more valuable information than error analysis research with respect to understanding students' learning difficulties in mathematics. At the same time, it is important to identify the limitations of this approach to errors with respect to inviting the students themselves to take advantage of errors as learning opportunities in their mathematical activities, because within this approach it is still only researchers and teachers—not the students—who engage in the creative activity of analyzing errors.

Students' Constructive Uses of Errors Proposed for Mathematics Instruction

As already mentioned in the previous subsections, research on error analysis and student misconceptions, although providing valuable information and suggestions for classroom instruction, has rarely resulted in the development of teaching strategies that make constructive uses of errors. There are, however, a few important exceptions, based on the idea that students' specific errors could be used as a means to generate conflicts that, in turn, may expose and challenge the students' misconceptions. Referring to Graeber and Johnson (1991) for a more in-depth review of these instructional ideas, I would like to mention here *conflict teaching*—a strategy developed specifically for mathematics instruction by Bell and his colleagues (Bell, 1983, 1986, 1987; Bell, Brekke, & Swan, 1987; Bell & Purdy, 1986; Swan, 1983).

This instructional approach aims at eliminating students' misconceptions through a series of steps that involves: (a) an *intuitive phase*, consisting of activities designed to elicit and expose specific misconceptions; (b) a *conflict phase*, in which alternative conceptions are generated to present a contrast with the previous ones and thus generate cognitive conflict; (c) a *resolution phase*, involving class discussions where the students face the inconsistencies thus re-

vealed and try to resolve them by identifying the "better" conception; and (d) a *retrospective phase* in which students reflect on the significance and implications of their resolution (the latter phase substitutes the *reinforcement phase* conceptualized in earlier works as consisting mainly in practice exercises geared to consolidate and further the students' newly gained understanding). Thus, conflict teaching makes explicit and positive use of errors in mathematics instruction, in ways that involve the students directly in their study. It is important to recognize, however, that its fundamental goal is still to eliminate errors and misconceptions, rather than using them as a means to encourage students to challenge the status quo and initiate more open-ended inquiry on their own.

Perhaps the closest examples to the approach of errors advocated in this book can be found in some instructional situations that have been more or less implicitly informed by an inquiry approach. When students engage in genuine mathematical problem solving and exploration, in fact, steps in the wrong direction and conflicting solutions are likely to be originated and then critically examined either by individual students in the context of small-group problem-solving activities or by the whole class in open discussions facilitated by the instructor (as illustrations in this regard see, e.g., some of the instructional episodes reported in Balacheff, 1991; Cobb, Wood & Yackel, 1991; Lampert, 1987; or Yackel et al., 1990). Indeed, although verbalizing and providing justifications to one's procedures is always an important component when engaging in genuine inquiry and mathematical debate, some authors have pointed out that doing so may be especially important when the solution proposed is somewhat incorrect—provided that the discussion thus generated is approached with an open mind on the part of all participants and led by the assumption that the solution was plausible to the students suggesting it. Thus, the students themselves can learn to monitor their own mathematical activity (rather than rely on the teacher for doing so) as well as to respect the potential contributions of each class member, as explicitly suggested by Yackel and her colleagues (1990):

When a child gives an incorrect answer, it is especially important for the teacher to assume that the child was engaged in meaningful activity. Thus, it is possible that the child will reflect on his or her solution attempt and evaluate it. . . . By allowing a child to proceed with an explanation even when the answer is wrong, the teacher fosters a belief that the teacher is not the sole authority in the classroom to whom children have to appeal to find out if their answers are right or wrong. Children are able to make such decisions for themselves. Mathematics authority does not reside solely with the teacher, but with the teacher and the children as an intellectual community. (pp. 17–18)

In most of the cases in which errors have been used constructively within inquiry lessons, however, the role played by errors per se has usually been left

implicit and, therefore, the significance and generalizability of the uses made of errors in these experiences has not been explicated fully. An important exception to this statement is presented by Balacheff's studies of the role played by counterexamples and contradictions in situations modeled after Lakatos' "proofs and refutations" approach (Balacheff, 1988, 1991). Even in this case, however, only a few specific types of errors have been considered.

In sum, whereas most of the approaches to errors discussed in this section have contributed to mathematics education in complementary ways, they have not fully exploited the potential of errors to provide a catalyst for the students' own learning and inquiry in school mathematics. I hope that the articulation of the instructional strategy of using of errors as springboards for inquiry developed in the remaining chapters of the book, while building on and/or incorporating some of these contributions, will enable other mathematics teachers to become more aware of the opportunities that mathematical errors can offer in this direction and, consequently, to take greater advantage of these opportunities in their teaching.

Chapter 4

Errors and the History of Mathematics

In this chapter, I show how mathematicians have been able to capitalize on errors in ways that contributed significantly to the growth of mathematical knowledge. Looking at the history of mathematics, I have selected four examples that well illustrate different uses of errors as springboards for inquiry made by experts in the discipline. These first four "historical" error case studies are developed in the first part of the chapter and then analyzed with the main goal of identifying how errors have been used constructively by mathematicians in the past and what outcomes such uses of errors have produced.

The examples analyzed and reported in this chapter illustrate the power of mathematical errors to initiate reflection and exploration and, thus, further contribute to the differentiation of a use of errors as springboards for inquiry from the more popular interpretation of errors as tools for diagnosis and remediation. The error case studies developed in this chapter also begin to challenge the common view of mathematics as an absolute, predetermined, and objective body of knowledge and invite a radical rethinking of the nature of mathematical knowledge, the process through which it is achieved, and even the notion of mathematical error.

SOME HISTORICAL ERROR CASE STUDIES

The idea that mathematics teachers could capitalize on errors as a means to engage their students in meaningful mathematical inquiries was suggested to me both by some personal experiences as a mathematics learner in graduate school and by my recollection of some episodes in the history of mathematics in which errors had played a very important role. Three books, in particular, had a great influence on my thinking about the role of errors in the development of mathematics as a discipline—Kline's (1980) *Mathematics: The Loss of Certainty*, Lakatos' (1976) *Proofs and Refutations*, and Dupont's (1982) *Appunti di storia dell'analisi infinitesimale*. These books report several instances when great

mathematicians made a constructive use of errors in their activities, but I focus here only on four cases that I found especially significant.

More specifically, in the first case study I first examine how a lack of rigor contributed to several errors in the early development of the calculus and consequently resulted in a radical revision of this area of mathematics and of the logical foundations of mathematics more generally ("Calculus" historical case study [B/4]). In the following error case study I look at one of the most traumatic events in the history of mathematics—the realization that Euclidean geometry did not represent the only way to describe spatial relationships—and highlight the various roles played in it by a number of "perceived errors", such as the failure to prove Euclid parallel postulate ("Non-Euclidean geometry" historical case study [C/4]). The "Infinity" historical case study [D/4] will deal with the issue of comparing the "number of elements" in infinite sets—a topic mathematicians struggled with and debated for a long time, as it involves some seemingly unavoidable contradictions. Finally, in the last case study I summarize Lakatos' historical analysis of the progressive refinements of Euler's theorem on the characteristic of polyhedra as an example of how mathematical results are (to some extent) liable to continuous refinement and, thus, errors should be considered an integral component of the very process of constructing mathematical knowledge ("Euler theorem" historical case study [E/4]).

My development of three of the error case studies reported in this section has been based essentially on the historical analysis of Dupont (for the "Calculus" historical case study [B/4]), Kline (for both the "Calculus" [B/4] and "Non-Euclidean geometry" [C/4] case studies) and Lakatos (for the "Euler theorem" historical case study [E/4]). Thus, for further details on the historical events sketched in these error case studies, I refer the reader to these authors. The "Infinity" historical case study [D/4], on the contrary, was the result of my own investigation about the historical development of the notion of infinity. This study was a component of a more comprehensive inquiry into mathematical infinity initially motivated by my interest in the errors most people make when asked to compare infinite sets (e.g., Fischbein, Tirosh, & Hess, 1979, and Fischbein, Tirosh, & Melamed, 1981). A full report of the results of this investigation can be found in Borasi (1985a).

Unlike error case studies elsewhere in the book, which deal with content covered in the K-12 mathematics curriculum, the historical case studies reported in this chapter address somewhat more advanced mathematical topics, as inevitable since the errors discussed here were made and used by professional mathematicians in the context of their research work. Yet, I believe that no sophisticated mathematical background is needed to follow at least the gist of the uses made of errors in the historical events reported—even if some of their technical details may escape some readers.

I would also like to alert the reader to the connections between some of these historical error case studies and other case studies involving mathematics edu-

cators and/or students that are reported in later chapters. More specifically: the consequences of lack of rigor in evaluating infinite expressions ("Calculus" historical case study [B/4]) are revisited in "My unrigorous proof" (H/5) and the "Teachers' unrigorous proof" (R/9) case studies; issues related to non-Euclidean geometry (see "Non-Euclidean geometry" historical case-study [C/4]) are also investigated by a mathematics teacher in the "Beyond straight lines" case study (T/9) and touched on with a group of college students in "College students' 00" (P/8); and, in the "Students' polygon theorem" case study (K/6), two secondary students engage in proving a tentative theorem about polygons along the same lines of the "proofs and refutations" process discussed here in the "Euler theorem" historical case study (E/4).

Error Case Study B: Lack of Rigor in the Early Development of Calculus and Its Positive Outcomes ("Calculus" Historical Case Study [B/4])

It may come as a surprise to many people that, far from presenting the ultimate embodiment of absolute truth and rigor, the development of several areas of mathematics has been characterized by imprecision, unjustified guesses, and lack of logical justification. The troubled history of calculus is especially revealing to this regard.

First of all, mathematicians worked for over a century with some powerful but "shaky" notions of *derivative* and *integral*. Consider for example Kline's report of how Fermat (1601–1665) would determine an instantaneous velocity (in other words, calculate a derivative):

We shall calculate the velocity at the fourth second of a ball whose fall is described by the function

$$d = 16t^2. \quad (1)$$

When $t = 4$, $d = 16 \times 4^2$ or 256. Now let h be any increment of time. In the time $4 + h$ the ball will fall 256 feet plus some incremental distance k . Then

$$256 + k = 16(4 + h)^2 = 16(16 + 8h + h^2).$$

or

$$256 + k = 256 + 128h + h^2.$$

Then by subtracting 256 from both sides

$$k = 128h + h^2$$

and the average velocity in h seconds is

$$\frac{k}{h} = \frac{128h + h^2}{h}. \quad (2)$$

Fermat was fortunate in the case of this simple function and others he considered in that he could divide the numerator and the denominator of the right side by h and obtain

$$\frac{k}{h} = 128 + h. \quad (3)$$

He then let h be 0 and obtained as the velocity at the fourth second of fall

$$\dot{d} = 128. \quad (4)$$

(The notation $\dot{d}$ is Newton's) Thus $\dot{d}$ is the derivative of $d = 16t^2$ at $t = 4$. (Kline, 1980, pp. 129–130)

Although undoubtedly creative and intuitively reasonable, the process described in this quote presents some considerable problems and, one could even say, *logical errors* (as calculus students would soon discover were they to use such a procedure in a test!). Fermat in fact started by implicitly assuming that h and k were *not* 0 (otherwise it would not have been possible to divide by h in Equation (2) and, furthermore, the expression k/h would lose meaning). Yet, at a later point Fermat considered h to be 0 in order to deduce Equation (4) from Equation (3) and thus reach a numerical result for the instantaneous velocity (or derivative) at $t = 4$. In other words, Fermat's process relies on two contradictory assumptions about the value of h .

A similar "confusion" characterized the early conception and evaluation of integrals in the context of computing areas. The first geometric notion of an *integral* was in fact based on the creative idea that the area under a curve could be *approximated* with the sum of many "thin" rectangles and that, the "thinner" the rectangles were, the closer such an approximation would be to the actual area. An unjustified leap was then made, however, when mathematicians assumed that such area could be computed *exactly* by thinking of it as the sum of infinitely many rectangles with infinitesimal width—*weird* objects that Cavalieri (1598–1647) referred to as "indivisibles."

It is important to realize that mathematicians of the 17th century were not unaware of the "logical errors" and imprecisions I have just pointed out. Rather,

many of the greatest mathematicians of the time chose to disregard these errors because the faulty procedures I have described had proved incredibly powerful and efficient for solving geometrical and physical problems of great interest and applicability, reaching results that could often be proved to the mathematicians' satisfaction—either through their application to real-world phenomena, or by deriving a mathematical proof of the same results in an alternative way (e.g., in the case of many area problems, by using the cumbersome method of exhaustion devised by the ancient Greeks).¹ Indeed, this remarkable "leap of faith" that many mathematicians of the 1600s and 1700s had been willing to make produced an incredible wealth of results that were instrumental to the advance of several branches of mathematics as well as science. As summarized by Kline (1980):

Despite the muddle, uneasiness, and some opposition, the great 18th-century mathematicians not only vastly extended the calculus but derived entirely new subjects from it: infinite series, ordinary and partial differential equations, differential geometry, the calculus of variations, and the theory of functions of a complex variable, subjects which are at the heart of mathematics today and are collectively referred to as analysis. (p. 140)

At the same time, the imprecise thinking behind some of the fundamental calculus concepts and procedures led to some unwarranted generalizations and, consequently, errors. This was especially the case in the treatment of infinite series. The example I develop in what follows, although not historical and dealing with infinite sums rather than series, is indicative of the kind of problems mathematicians encountered in this area.

Suppose we want to evaluate the following infinite numerical sum:

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots$$

This expression could also be written as:

$$1 + \frac{1}{2} \left(1 + \frac{1}{2} + \frac{1}{4} + \dots \right)$$

Because both of these expressions are identical and contain an infinite number of terms, if we indicate their value with x , we can derive the following equation:

$$x = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots = 1 + \frac{1}{2} \left(1 + \frac{1}{2} + \frac{1}{4} + \dots \right) = 1 + \frac{1}{2}x$$

¹ An explanation and various applications of the method of exhaustion can be found in Dupont (1982).

that is:

$$x = 1 + \frac{1}{2}x.$$

By solving this equation for x , we obtain $x = 2$ and, therefore, we may want to conclude that:

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots = 2.$$

This may indeed seem a reasonable result, because it is consistent with approximations of this infinite sum obtained when considering only some of its terms. This success may then suggest the possibility of further generalizing the procedure used here, so as to evaluate any infinite sum of the form:

$$1 + \frac{1}{a} + \frac{1}{a^2} + \frac{1}{a^3} + \dots$$

More specifically, we could think of setting this expression equal to x and deriving the more general equation:

$$x = 1 + \frac{1}{a} + \frac{1}{a^2} + \frac{1}{a^3} + \dots = 1 + \frac{1}{a}(1 + \frac{1}{a} + \frac{1}{a^2} + \dots) = 1 + \frac{1}{a}x.$$

that is:

$$x = 1 + \frac{1}{a}x$$

By solving this equation we would then get:

$$1 + \frac{1}{a} + \frac{1}{a^2} + \frac{1}{a^3} + \dots = \frac{a}{a-1}.$$

However, whereas this expression yields seemingly acceptable results for values of a such as 3 or 4, considerable problems are generated when we consider values such as $a = 1$, $a = 1/2$, or $a = -1$, because we would obtain absurd or meaningless results such as:

$$\begin{array}{ll} (a=1) & 1+1+1+1+\dots=1/0 \\ (a=-1) & 1-1+1-1+\dots=1/2 \\ (a=1/2) & 1+2+4+8+\dots=-1. \end{array}$$

It is important to note, however, that it was precisely "errors" of this kind that forced the mathematicians of the time to critically examine the procedures they had been using, in the hope of better defining their domain of application and providing them with a sounder foundation. Thus, several mathematicians of the 19th century, such as Cauchy (1789-1857) and Weierstrass (1815-1897), undertook the challenging task of "rigorizing" this branch of analysis. Only then the rigorous definition of limit that students now study in their very first calculus course was proposed. This definition, in turn, made a systematic reconstruction of analysis possible. Only then all the major results previously achieved by mathematicians in this area could be finally confirmed and proved to everybody's satisfaction.

It is indeed surprising and significant that for over 150 years even great mathematicians could use concepts and procedures lacking proper justification, relying merely on their intuition and the usefulness of the results obtained. Looking at this episode of the history of mathematics a posteriori, one may marvel at the risks that mathematicians were willing to take in developing analysis on such shaky grounds. Yet, such faith was vindicated in the end, and one could not even begin to conceive what mathematics and science might look like today if an extreme concern for rigor had stifled the creativity of mathematicians such as Leibnitz, Newton, or Euler.

It is also important to note that the discovery of problems made while working with infinite series and the attempts made to resolve these problems led mathematicians much further than they could have initially expected. The rigorization of analysis was in fact only the beginning of a more radical revision of the very foundations of mathematics, which ended up involving areas such as arithmetic and logic. As Kline (1980) pointed out, it was the "errors" eventually encountered in analysis as well as the optimism generated by the success in rigorizing this area that invited mathematicians to later question even more familiar and established branches of mathematics:

Though the logic of number system and of algebra was in no better shape than that of the calculus, mathematicians concentrated their attacks on the calculus and attempted to remedy looseness there. The reason for this is undoubtedly that the various types of numbers appeared familiar and more natural by 1700, whereas concepts of the calculus, still strange and mysterious, seemed less acceptable. In addition, while no contradictions arose from the use of numbers, contradictions did arise from the use of calculus and its extensions to infinite series and the other branches of analysis. (p. 145)

Error Case Study C: The Surprising Consequences of Failing to Prove the Parallel Postulate ("Non-Euclidean Geometry" *Historical Case Study* [C4])

Few events in the history of mathematics have been as challenging and disturbing for mathematicians as the creation of the first non-Euclidean geometry.

tries. In this error case study, I briefly review some of the history and major consequences of this event to highlight the role played by what might be considered "errors"—from the perspective of either the mathematicians of the time or today's mathematicians.

The very root of this event can be traced back to the discomfort felt by many mathematicians since the Greek times with Euclid's fifth postulate. This axiom is usually referred to as the "parallel postulate" and is reported in most geometry textbooks as "through a given point P not on a line l there is one and only one line in the plane of P and l which does not meet l ." Originally, however, this axiom was stated by Euclid as follows:

If a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, then the two straight lines if extended will meet on that side of the straight line on which the angles are less than two right angles. (Kline, 1980, p. 78)

As pointed out by Kline, this axiom "had bothered mathematicians somewhat, not because there was in their mind any doubt of its truth but because of its wording . . . the parallel axiom in the form stated by Euclid was thought to be somewhat too complicated" (Kline, 1980, p. 78). Thus, several mathematicians tried to either replace Euclid's original statement with an equivalent but more "self-evident" one or to deduce it as a theorem from the other nine axioms of Euclidean geometry. Whereas efforts in the first direction could be deemed worthwhile even today, attempts to "prove" the parallel postulate could instead be considered an error, since we now know that such a task is impossible. Yet, these very erroneous attempts led some mathematicians to initiate an entirely new area of research in mathematics.

Several historians of mathematics would identify Saccheri (1667–1733) as the first creator of a non-Euclidean geometry—even if he was far from recognizing the results of his work as such! Saccheri thought that he had "proved" the parallel postulate by contradiction; that is, by showing that if alternative axioms negating the parallel postulate were to be assumed, then a theorem contradicting one of the already established theorems of Euclidean geometry could be derived. As summarized by Kline (1980):

Saccheri assumed first that through [a] point P there are no lines parallel to [a given line] l . And from this axiom and the other nine that Euclid adopted Saccheri did deduce a contradiction. Saccheri tried next the second and only possible alternative, namely, that through the point P there are at least two lines p and q that no matter how far extended do not meet l . Saccheri proceeded to prove many interesting theorems until he reached one which would seem so strange and so repugnant that he decided it was contradictory to the previously established results. Saccheri therefore felt

justified in concluding that Euclid's parallel axiom was really a consequence of the other nine axioms. (p. 80)

However, other mathematicians later on found an error in his work (something that Saccheri had considered a contradiction but actually was not), thus disputing Saccheri's conclusion that the parallel postulate could be derived from the other axioms of Euclidean geometry. Yet, although failing in its original objective, Saccheri's "incorrect proof" had a greater impact on the future history of mathematics than many correct results, as it invited the consideration of geometric systems alternative to Euclidean geometry that could be derived on purely logical grounds from a selected set of axioms.

Unlike Saccheri, mathematicians such as Gauss (1777–1855), Klüger (1739–1812), Lambert (1728–1777), Lobachevsky (1793–1856), and Bolyai (1802–1860) explicitly recognized and pursued this possibility, thus developing systematically the first non-Euclidean geometries. Great debate surrounded the development of these geometries, although eventually mathematicians had to recognize not only that the systems thus created were as sound as Euclid's one from a logical standpoint, but also that some of these geometries could "be used to describe the properties of physical space as accurately as Euclidean geometry does" (Kline, 1980, p. 84). To fully appreciate the significance of these results, one has to realize that they challenged some very fundamental beliefs about "mathematical truth" that had informed mathematicians' thinking and work for centuries.

That Euclidean geometry is the geometry of physical space, that it is the truth about space, was so ingrained in people's minds that for many years any contrary thought such as Gauss's were rejected. . . . For thirty or so years after the publication of Lobachevsky's and Bolyai's works all but a few mathematicians ignored the non-Euclidean geometries. They were regarded as a curiosity. Some mathematicians did not deny their logical coherence. Others believed that they must contain contradictions and so were worthless. Almost all mathematicians maintained that the geometry of physical space, *the geometry*, must be Euclidean. . . . However, the material in Gauss's notes became available after his death in 1855 when his reputation was unexcelled and the publication in 1868 of Riemann's 1854 paper convinced many mathematicians that a non-Euclidean geometry could be the geometry of physical space and that we could no longer be sure what geometry was true. The mere fact that there can be alternative geometries was in itself a shock. But the greater shock was that one could no longer be sure which geometry was true or whether any one of them was true. It became clear that mathematicians had adopted axioms for geometry that seemed correct on the basis of limited experience and had been deluded into thinking that these were self-evident truths. (Kline, 1980, p. 88)

Thus, one could say that the creation of non-Euclidean geometries and the recognition of their validity uncovered an even more fundamental "error" that mathematicians had lived with for centuries without even recognizing it as such—that is, the assumption that Euclidean geometry was the true and only representation of physical space. The realization of this "error," in turn, caused the need for new criteria to establish the "truth" of mathematical results and made many mathematicians reconceive the very nature of mathematics as the product of the human mind rather than the result of discovering a predetermined body of absolute truths.

Error Case Study D: Dealing With Unavoidable Contradictions Within the Concept of Infinity ("Infinity" Historical Case Study [D4])

The concept of infinity, one of the most fundamental mathematical concepts, has undergone considerable debate throughout the history of mathematics. Because of our tendency to extend to the infinite our limited experience of the finite alone, our intuitive concept of infinity contains some implicit contradictions that cannot be totally eliminated. It took mathematicians a long time to come to appreciate this fact as well as some of its disturbing consequences—such as the fact that alternative resolutions of these contradictions could be equally reasonable and, therefore, which one was the "correct" one could not be decided on purely logical grounds. In this error case study, I focus on only one of the many troubling aspects of mathematical infinity—the comparison of the "number" of elements in two infinite sets—and trace how a few great mathematicians of the past dealt with this problematic issue.

To better appreciate the debate and arguments reported in what follows, let me first articulate the problem in question. When we deal with finite sets, we can always establish without doubt whether two given sets have the same number of elements by using one of the following complementary criteria:

1. If we can find a one-to-one correspondence between the elements in the two sets, then we can conclude that they have the same number of elements (*one-to-one correspondence criterion*).
2. If we can show that one set is a proper subset of the other (i.e., all its elements also belong to the other set and in addition the other set has elements that the first does not have) or, alternatively, that one set can be put into one-to-one correspondence with a proper subset of the other, then we can conclude that the two sets have a different number of elements (*part-whole principle or criterion*).

Consider, however, some of the infinite sets most commonly encountered in mathematics: $\mathbb{N}$ (the set of all the natural numbers), $\mathbb{S}$ (the set of all the squares

of natural numbers), $\mathbb{Z}$ (the set of all the positive and negative integers), $\mathbb{Q}^+$ (the set of all positive fractions), $\mathbb{R}$ (all the real numbers), the set of all the points in a segment ($\mathbb{PS}$), and the set of all the points in a line ($\mathbb{PL}$). If we try to compare some of these sets using the intuitive and reasonable criteria already described, we immediately encounter some puzzling results. For example, $\mathbb{N}$ and $\mathbb{S}$ can be put rather obviously into one-to-one correspondence (by associating each number to its square), yet $\mathbb{S}$ is a proper subset of $\mathbb{N}$. The same happens when one tries to compare $\mathbb{N}$ and $\mathbb{Q}^+$ (although in this case establishing a one-to-one correspondence requires more creativity); this result may seem even more disturbing when one considers that there are infinitely many fractions between any two consecutive natural numbers. Similarly, the points in a segment and the point in a line can be put into a one-to-one correspondence (as illustrated in Figure 4.1), yet not only could the segment be considered a proper subset of a line, but also a segment is bounded while a line extends infinitely on both directions. In sum, depending on which of the two criteria we rely on for the comparison, in each of these cases we could reach contrasting conclusions about whether the two sets have the same number of elements or not.

How did the mathematicians deal with these contradictory results? First of all, one must remember that for centuries mathematicians completely avoided the problem of comparing infinite sets because they did not even accept the concept of *actual* infinity—that is, the possibility of considering sets such as "all the natural numbers" in their entirety. Starting with the ancient Greeks, for a long time only *potential* infinity—that is, the "possibility of increasing without a bound"—was in fact accepted in mathematics. In 1831, Gauss explicitly declared: "I protest above all the use of an infinite quantity as a completed one, which in mathematics is never allowed. The infinity is only a *façon de parler*, in which one properly speaks of limits" (cited in Dauben, 1983, p. 125).

A similar view had already been expressed by Galileo Galilei (1564–1642) in one of his dialogues, where he discussed explicitly two of the "paradoxes" already mentioned—that is, (a) the fact that the natural numbers can be put into

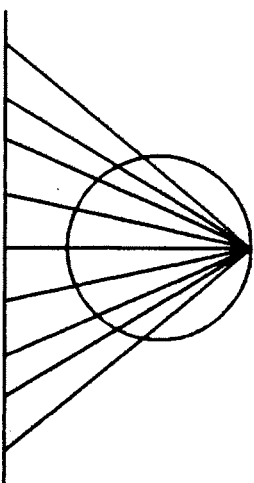


FIGURE 4.1. Illustration of how the points in a line and the points in an open segment can be put into one-to-one correspondence after the latter has been "rolled" into the shape of a circle.

a one-to-one correspondence with the square numbers, even though S is a proper subset of N ; and (b) the fact that segments of different length can be put into one-to-one correspondence. This great mathematician found these results so disturbing that he concluded that it does not make sense to compare the number of elements in infinite sets:

So far as I can see we can only infer that the totality of all numbers is infinite, that the number of squares is infinite, and that the number of their roots is infinite; neither is the number of squares less than the totality of all numbers, nor the latter greater than the former; and finally the attributes "equal," "greater," and "less," are not applicable to infinite, but only to finite quantities. (Gaillei, 1881, pp. 32-33)

It is interesting to contrast this position with the one assumed by Bolzano (1781-1848), one of the first mathematicians who considered "actual infinities" and attempted to solve the problem of their comparison. Contrary to Galileo, Bolzano accepted the fact that infinite sets may not have the same number of elements and suggested relying on the part-whole principle rather than the application of the one-to-one correspondence criterion in order to determine which of two infinite sets is "larger":

Even in the examples of the infinite so far considered, it could not escape our notice that not all infinite sets can be deemed equal with respect to the multiplicity of their members. On the contrary, many of them are greater (or smaller) than some other in the sense that the one includes the other as a part of itself (or stands to the other in the relation of part to whole). Many consider this as yet another paradox, and indeed, in the eyes of all who define the infinite as that which is incapable of increase, the idea of one infinite being greater than another must seem not merely paradoxical, but even downright contradictory. . . . Our own definition . . . does not tempt anyone to think it contradictory, or even astonishing, that one infinite be greater than another. (Bolzano, 1965, p. 95)

Bolzano's work was little known by his contemporaries and did not have much influence in the development of the notion of mathematical infinity. The rigorous theory of infinity that had the greatest influence in mathematics, and is mostly used by mathematicians today, was developed at the end of the last century by Cantor. Interestingly, Cantor himself suggested not one but a few alternative extensions for the notion of "natural number" in the infinite case. I present here the two most important ones—the cardinal and ordinal numbers.

Unlike Bolzano, Cantor decided to disregard the whole-part principle and to use instead one-to-one correspondence as the basis of establishing rigorous criteria for comparing the number of elements in two infinite sets. More specif-

cally, working within the framework of the theory of sets that he was developing, Cantor rigorously defined the notion of *cardinal number* by abstraction, that is, by giving a rigorous criterion for determining whether two sets have the same cardinal number:

We say that two aggregates M and N are "equivalent" if it is possible to put them, by some law, in such a relation to one another that to every element of each one of them corresponds one and only one element of the other. (Cantor, 1897, p. 11)

This definition enabled Cantor to rigorously establish some fundamental results—such as the fact that not only N and S have the same cardinal number, but Z and Q^+ also do. Because he was able to establish a one-to-one correspondence between the real numbers and the points of a line (think of the "number line" representation of the real numbers), and between the points of a line and those in a segment, a square, the whole plane or space, respectively, he concluded that all those sets have the same cardinal number. In addition, he also found a positive answer to the question: "Is there more than one kind of infinity?" by proving that R and N cannot be put into one-to-one correspondence (see Dauben, 1983, for an intuitive version of this proof).

For sets in which a relation of total order could be defined, however, Cantor himself also suggested the alternative notion of *ordinal number* (or *ordinal type*) again by means of a definition by abstraction. In his set theory, two sets are considered to have the same ordinal type if a one-to-one correspondence can be established between their elements, such that the order relation between corresponding elements is maintained:

We call two ordered aggregates M and N "similar" if they can be put into a biunivocal correspondence with one another in such a manner that, if m_1 and m_2 are any two elements of M and n_1 and n_2 the corresponding elements of N , then the relation of rank of m_1 to m_2 in M is the same as that of n_1 to n_2 in N . (Cantor, 1897, p. 11)

According to this definition, Cantor proved that N and S have the same ordinal type, and also R and the points of a line, respectively. The sets N , Z , Q^+ , R , and the points of a closed segment (PL), instead, all have different ordinal types. Thus, the concept of ordinal type provides again an acceptable and unambiguous criterion to compare the number of two infinite sets (and, perhaps, one that yields results a little closer to our intuition than that of cardinal number). Yet, sets that were considered to have the same "number" of elements when using the notion of cardinality may now have a different "number" of elements when using the notion of ordinal type.

Although both Bolzano's and Cantor's work provided some resolution to the apparent contradictions encountered when trying to compare the number of elements in two infinite sets, as well as rigorous criteria to achieve such a comparison, these results may still seem somewhat unsatisfactory for the following reasons:

- They offer not one, but several alternative notions of "infinite numbers," each relying on different comparison criteria and consequently leading to contrasting results; for example, our "intuitive" answer that N is smaller than Q is correct if we are talking in terms of ordinal types, but wrong if we are considering the cardinal numbers of these sets.
- The initial contradictions can be overcome only by virtue of rather arbitrary decisions about which criterion should be used and which should be disregarded; for example, Cantor decided to consider only one-to-one correspondence to evaluate the cardinal number of a set; in contrast, Bolzano suggested using the part-whole principle and disregarding one-to-one correspondence for determining which of two sets is larger.
- Each of the notions of "infinite number" proposed only partially incorporates our intuitions on the subject; for example, it is remarkable that even Cantor, after having been able to prove that the points in a segment and the points in a square had the same cardinal number because they could be put into a one-to-one correspondence, is reported to have said "I see it but I cannot believe it."

This situation may seem especially disturbing because in mathematics we have been used to unanimity about what a concept is and how one can operate with it. Furthermore, in this case it is not at all immediate to identify which of the alternative notions proposed is the "correct" or even just the "best" one and, thus, the one to be adopted by all mathematicians. In fact, given that each proposed solution only partially responds to our original intuition, we might not be able to answer this question in absolute terms. Rather, in order to decide which definition of infinite number is best in a specific case, we may need to specify and analyze the context in which we are operating and the reasons why we want to compare the given infinite sets. For instance, the success of Cantor's notion of cardinality in mathematics can be explained by observing that not only does it provide a simple yet rigorous method for comparing any pair of sets, but more importantly it helped mathematicians solve some important mathematical problems—for example, whether a function is discontinuous in a countable or an uncountable set of points in a real interval makes a difference with regard to the possibility of integrating it, or approximating it with Fourier's series (see Dauben, 1983).

In sum, this brief historical review of the development of the concept of infinite number reveals some interesting (and somewhat unexpected) develop-

ments of the initial "error"—that is, extending without justification criteria developed to compare the number of two finite sets to the new domain of infinite sets. First of all, the contradictions caused by such an unwarranted generalization made the mathematicians aware of the existence of fundamental differences between finite and infinite sets and called for an entirely new definition of "number" that would make sense in the new situation. It is also interesting to note that a "solution" to the controversy opened by such contradictory results was not only difficult to reach, but also turned out to be neither predetermined nor unique. Furthermore, the decision of which alternative definition of "infinite number" should be assumed could not be determined in absolute or purely logical terms, but rather depended on the context and purpose of application.

Error Case Study E: Progressive Refinements of Euler's Theorem on the "Characteristic" of Polyhedra ("Euler Theorem" Historical Case Study [E4])

Lakatos presented a beautiful historical example in support of his thesis that the growth of mathematical knowledge often occurs through a dialectic process of "proofs and refutations" (where initial "errors" play a crucial role) by reconstructing the development of one of the fundamental theorems of topology. This theorem was initially stated by Euler in 1758 as follows:

In a polyhedron the relationship among the number of faces (F), edges (E) and vertices (V) satisfies the relation: $V + F - E = 2$.

(Note: In what follows, I refer to the value of the expression $V + F - E$ as the *characteristic* of a three-dimensional figure.)

In a fictitious dialogue that reconstructs debates that actually occurred in the mathematical community of the 18th and 19th centuries, Lakatos showed the interplay between successive refinements of this statement, some of its tentative proofs, and the definition of polyhedron itself. In what follows, I briefly report some key points of this development that best show the role played by errors in a proofs and refutations approach. I refer the reader to Lakatos' (1976) text for more detail and precise historical references.

Lakatos' reconstruction starts with the proposal of a tentative proof of Euler's conjecture, based on an idea developed by Cauchy in 1813 and described as follows:

Step 1: Let us imagine the polyhedron to be hollow, with a surface made of thin rubber. If we cut out one of the faces, we can stretch the remaining surface flat on the blackboard, without tearing it. The faces and edges will be deformed, the edges may become curved, but V and E will not

alter, so that if and only if $V - E + F = 2$ for the original polyhedron, $V - E + F = 1$ for this flat network—remember that we have removed one face. [Fig. 4.2A shows the flat network for the case of a cube.] *Step 2:* Now we triangulate our map—it does indeed look like a geographical map. We draw (possibly curvilinear) diagonals in those (possibly curvilinear) polygons which are not already (possibly curvilinear) triangles. By drawing each diagonal we increase both E and F by one, so that the total $V - E + F$ will not be altered [Fig. 4.2B]. *Step 3:* From the triangular network we now remove the triangles one by one. To remove a triangle we either remove an edge—upon which one face and one edge disappear [Fig. 4.2C], or we remove two edges and a vertex—upon which one face, two edges and one vertex disappear [Fig. 4.2D]. Thus if $V - E + F = 1$ before a triangle is removed, it remains so after the triangle is removed. At the end of this procedure we get a single triangle. For this $V - E + F = 1$ holds true. Thus we have proved our conjecture. (Lakatos, 1976, pp. 7–8.)

As stated, this “proof” is quite tentative and sketchy. This becomes more evident when the following rather trivial counterexample is identified: If in the

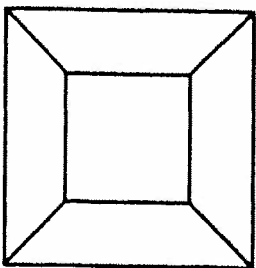


FIGURE 4.2a.

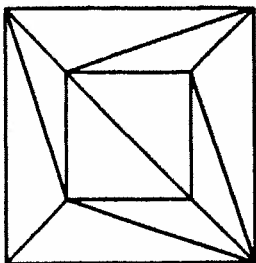


FIGURE 4.2b.

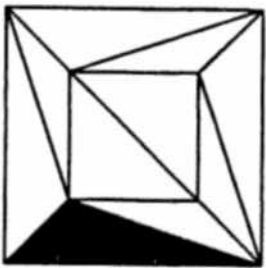


FIGURE 4.2c.

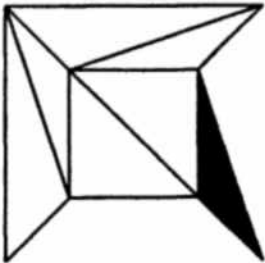


FIGURE 4.2d.

FIGURE 4.2. Cauchy's idea for proving Euler's conjecture illustrated in the case of a cube.

cube illustrated in Figure 4.2 one removes one of the internal triangles first (“as a piece of a jigsaw puzzle,” Lakatos, 1976, p. 10), then one removes a face without removing at the same time either an edge or a vertex. Although this observation identifies an error in the procedure as described in the foregoing quote, it does not necessarily call for abandoning the conjecture itself—because the characteristic of a cube is undoubtedly 2. On the contrary, the discovery of this *local counterexample* (i.e., an example that refutes a lemma in the proposed proof but not the conjecture itself) should motivate a revision of the original “proof” so as to avoid such a “misinterpretation” of the intended procedure. In this specific case, this goal can be easily achieved by specifying that in Step 3 a “boundary triangle” has to be removed at each stage (a correction to Euler's original proof suggested by Lhuillier in 1812).

A much greater challenge to Euler's conjecture is instead presented by each of the figures reproduced in Figure 4.3. Each of those figures satisfies the intuitive definition of polyhedron as “a three-dimensional figure whose surface consists of polygonal faces” (*Def. 1*), yet refutes the theorem as stated earlier—because the characteristic of these figures is, respectively, 4 for Figure 4.3A and 3 for Figures 4.3B and 4.3C, instead of 2 as suggested by Euler's theorem). Thus, they can all be considered *global counterexamples* to the conjecture.

As noted by Lakatos, these counterexamples were historically brought up by Lhuillier in 1812 (in the case of Figure 4.3A) and Hessel in 1832 (in the case of Figures 4.3A, 4.3B, and 4.3C). One might expect that the mere presence of such global counterexamples would show beyond doubt that the original conjecture and its proof were “wrong” and thus needed to be discarded. However, instead of doing so, several mathematicians attempted first to “salvage” Euler's theorem by modifying the original definition of polyhedron so as to eliminate the pathological examples reproduced in Figure 4.3. The following more restrictive definitions of polyhedron were then suggested:

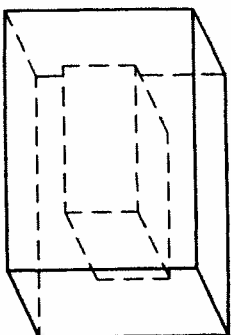


FIGURE 4.3a.

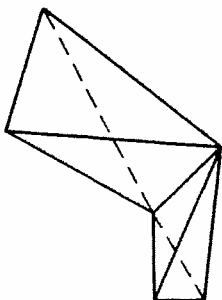


FIGURE 4.3b.

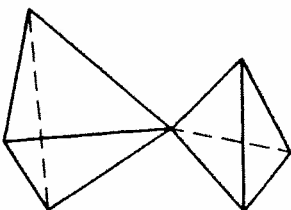


FIGURE 4.3c.

FIGURE 4.3. Examples of “pathological” polyhedra.

- "A polyhedron is a surface consisting of a system of polygons" (*Def. 2* provided by Jonquieres in 1890); this definition would exclude Figure 4.3A as an example of polyhedron (because this figure consists in two disjoint surfaces), so that such a figure would no more present a counterexample to the theorem; however, according to this definition, Figures 4.3B and 4.3C could still be considered as polyhedra and continue to present global counterexamples to Euler's conjecture;
- "A polyhedron is a system of polygons arranged in such a way that (1) exactly two polygons meet at every edge and (2) it is possible to get from the inside of any polygon to the inside of any other polygon by a route which never crosses any edge at a vertex" (*Def. 3* provided by Mobius in 1865); this definition would eliminate the threat posed by Figures 4.3B and 4.3C as well.

As shown by this specific historical event, whenever a global counterexample refutes a conjecture, one way to interpret the error thus revealed is to conclude that the counterexample (rather than the conjecture itself) is wrong—that is, it should not have been considered as a relevant example in the first place. This approach, illustrated by the previous examples and called by Lakatos rather disparagingly the "method of monster-barring," usually requires the modification of one or more existing definitions. Such an approach already has quite radical implications, because it illustrates that mathematical definitions are not "cast in concrete" as mathematics textbooks may lead us to believe, but rather may evolve with time as new developments in the discipline may invite mathematicians not only to make an existing definition more "precise" but rather to modify their conception of an establish concept and, consequently, its definition—quite a blow for anyone believing in the absolute truth of mathematics. Yet, Lakatos pointed out that such an approach may not always lead very far, as one may never be fully sure to have forestalled all possible objections and exceptions in this way.

Using this method one can eliminate any counterexample to the original conjecture by a sometimes deft but always *ad hoc* redefinition of the polyhedron, its defining terms, or the defining terms of its defining terms. We should somehow treat counterexamples with more respect, and not stubbornly exorcise them by dubbing them monsters. (Lakatos, 1976, p. 23)

Another possible reaction to the discovery of a counterexample is to interpret it as evidence that there was some "error" in the conjecture itself. Once again, this does not necessarily mean that the conjecture should be totally relinquished but rather that it needs to be re-examined and somewhat modified. Indeed, Lakatos suggests that the counterexample itself may provide a valuable lead for such a process. By analyzing the original proof with the goal of identifying which specific subconjecture (or *lemma*) the counterexample refuted,

and then turning such a lemma into a condition of the theorem itself, one can refine both the conjecture and its proof at the same time.

For example, consider the "picture frame" figure reported in Figure 4.4 (once again, this counterexample was suggested by Lhuillier in 1812).

This figure presents a global counterexample to the conjecture, because its characteristic is 0 and yet it satisfies the conditions articulated in the three definitions of polyhedron given earlier. This counterexample also refutes the first lemma in the "proof," because one cannot take away a face and then hope to "stretch" the rest of this polyhedron onto a plane. This analysis, in turn, suggests both the value of creating a new concept—that of *simple polyhedron*, defined as "a polyhedron that can be 'stretched' onto a plane after the removal of one of its faces"—and of modifying the original conjecture into the following, more restrictive conjecture: "All *simple* polyhedra have Characteristic 2." This procedure is illustrative of the proof and refutations approach advocated by Lakatos. Lakatos pointed out that such an approach may not only succeed in refining an initial conjecture by better defining its domain of application, but also suggest ways of expanding on it. For example, all the exceptions to Euler's conjecture that were brought up by various mathematicians as counterexamples motivated the classification of polyhedra with respect to their characteristic—a variable that turned out to be especially significant from a topological perspective.

A FIRST ANALYSIS OF USING ERRORS AS SPRINGBOARDS FOR INQUIRY IN MATHEMATICS

The "historical" error case studies developed in this chapter have shown that errors have played a number of important roles in the development of math-

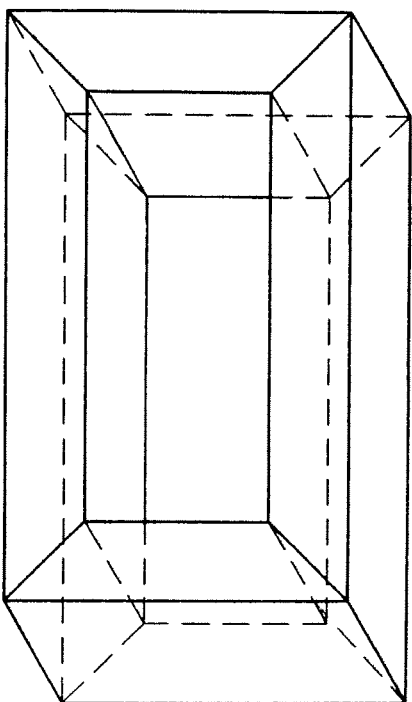


FIGURE 4.4. Another apparent counterexample to Euler's conjecture.

ematics. Building on the specific examples discussed in the previous section, it can be observed that errors could contribute to the growth of mathematical knowledge at least in the following ways:

- The presence of errors may generate the need for more rigor in the procedures and/or justifications employed—a need that may not arise as long as such procedures and/or justifications yield “acceptable” results (as illustrated by the development of calculus as discussed in the “Calculus” historical case study [B/4] and by the use of local counterexamples suggested by Lakatos in the “Euler theorem” historical case study [E/4]).
- The discovery of contradictions or counterexamples may reveal the inadequacy of initial conjectures, or even established theorems, as well as provide some concrete lead so as to refine them; more specifically, Lakatos’ analysis of what happened in the case of Euler’s conjecture about the characteristic of polyhedra (see “Euler theorem” historical case study [E/4]) suggests that a careful examination of these counterexamples and of the reasons why they refute the original conjecture may lead to:
 1. A refinement or modification of the original *proof* offered in support to the conjecture.
 2. A refinement or modification of the *definition* of some of the key concepts involved in the conjecture.
 3. A refinement or modification of the *domain of application* of the original conjecture.
 4. A refinement or modification of the *conjecture* itself.
 5. The creation of *new results and conjectures*.
- Contradictions, and some other kinds of errors as well, may enable mathematicians to identify the unwarranted application of familiar concepts and procedures to new domains, as illustrated in the case of the criteria used to compare the number of elements in two sets in the “Infinity” historical case study [D/4]: notice how a constructive use of this kind of error can have important consequences, because it may motivate an analysis of both the old and new domain, and of the concept in question, leading to:
 1. A better understanding of the characteristics of the new domain, and especially of its similarities and differences with the familiar one.
 2. A better appreciation of the implications of characteristics of the familiar domain that had been taken for granted and/or overlooked up to that point.
 3. The realization that alternative definitions of the concept in question may not be equivalent in every domain.

4. An exploration of the consequences of assuming alternative definitions of the concept, in various domains, so as to better evaluate their respective potential value.
 5. Modifications in the original concept reflecting the results obtained by addressing the previous points.
- Occasionally, errors may reveal the existence of fundamental problems that, in turn, may invite a radical re-examination of a whole area of mathematics and/or of the very foundations of the discipline (as it eventually happened as a result of the “unjustified assumptions” that were revealed by the rigorization of analysis and the discovery of non-Euclidean geometries, as briefly discussed in the “Calculus” [B/4] and “Non-Euclidean geometry” [C/4] case studies, respectively).
 - Even *perceived* errors (such as the fact that Euclid’s wording of the parallel postulate appeared too cumbersome and unintuitive—see the “Non-Euclidean geometry” historical case study [C/4]) could motivate valuable inquiry in the attempt to resolve them; it is worth noting that the results of such inquiry may often turn out to be quite different from what was initially expected.
 - Some errors may open entirely new areas of research, as they could unexpectedly show possibilities never conceived before (as illustrated by the “Non-Euclidean geometry” historical case study [C/4], where Saccheri’s faulty proof by contradiction of the parallel postulate showed the possibility of deducing a “legitimate” geometry from a set of axioms different from those articulated by Euclid).

These considerations provide further articulation of the potential of errors to provide the stimulus as well as a concrete starting point for worthwhile mathematical inquiries argued for in the previous chapter. I believe the ways of capitalizing on errors identified here are not confined to a few isolated events in the history of mathematics, but rather are characteristic of the activity of professional mathematicians. It would be difficult, however, to prove this claim, given that only the final and polished results of a mathematician’s work are usually made public. Thus, the only errors made by mathematicians that the nonspecialists are ever privileged to witness are the rare and subtle ones that have been overlooked by their author and only later identified by other mathematicians—such as those I chose for the error case studies developed in this chapter.

Looking in depth with a specific focus on errors at the historical events reported in this chapter has also revealed some unexpected aspects of both mathematics and errors that I would like to briefly comment on. First of all, these events considerably challenge the view that mathematical knowledge has been acquired through time as a result of a gradual and linear “discovery” of pre-ex-

isting "truths." Rather, the historical analyses proposed by Kline and Lakatos enable us to see how the development of several mathematical results has been characterized by elements such as:

- lack of rigor—as illustrated by the initial development of the calculus reported in the "Calculus" historical case study [B/4];
- alternative proposals—such as the contrasting criteria to "compare" the number of elements in infinite sets proposed by Galileo, Bolzano, and Cantor, discussed in the "Infinity" historical case study [D/4];
- debates—as it happened in the already mentioned cases and, even more spectacularly, when mathematicians had to face the challenge presented by the proposal of non-Euclidean geometries and some refused to accept their validity (see the "Non-Euclidean geometry" historical case study [C/4]);
- some "big" mistake shared by many mathematicians—such as assuming that Euclidean geometry would be the only "true" was to represent spatial relationships (see once again the "Non-Euclidean geometry" historical case study [C/4]).

The image of mathematics that emerges as a result is that of a much more tentative and fallible discipline, created as the result of individual efforts as well as social negotiations, and liable to continuous improvement and occasionally more radical "revolutions."

Within such a "fluid" view of mathematical knowledge, errors can never be hoped to be avoided but, rather, have to be seen as an integral part of the creation of new knowledge. At the same time, in light of the concrete examples reported in the previous error case studies, it may have become clearer why, earlier in Chapter 3, I argued that within a constructivist epistemology the definition of error itself is not straightforward. Consider for example the following questions:

- Should the procedures developed by 18th-century mathematicians to compute infinite expressions (see "Calculus" historical case study [B/4]) be considered correct or not, now that we know precisely their domain of legitimate application and have found some rigorous justification for them?
- Should Euler's original conjecture (see "Euler theorem" historical case study [E/4]) be considered incorrect or, rather, would such a decision depend on a number of factors, including the specific definition of polyhedron assumed?
- How can we be fully sure that any of the further refinements and/or elaboration of such a conjecture (or, in fact, any of the existing theorems reported in mathematical textbooks) are correct, when the history of

mathematics reports several cases in which something believed true for centuries was suddenly refuted?

- In the case of the comparison of infinite sets discussed in the "Infinity" historical case study [D/4], does the error reside in the contradiction resulting from applying the one-to-one correspondence criterion and part-whole principle for the comparison of $\mathbb{N}$ and $\mathbb{Q}^+$, or rather in the fact that mathematicians did not at first realize the fundamental differences between finite and infinite "numbers?"
- Should the attempt to prove the parallel postulate by mathematicians in the past (see "Non-Euclidean geometry" historical case study [C/4]) be considered an error, because we now know that such a task is impossible? Would the answer to the last question be different today than it was five centuries ago?
- Was it justifiable that many mathematicians of the 19th century would believe that non-Euclidean geometries must be contradictory, even if nobody had been able to find such contradiction at the time?

The considerations raised by these questions point to the fact that the notion of mathematical error is indeed more problematic than most people perceive and support the claim, made earlier in Chapter 3, that whether something is an error or not cannot be evaluated in absolute terms but rather may depend on both the mathematical and the historical context.

In sum, I hope that the examples and the considerations developed in this chapter have provided convincing evidence of how mathematicians capitalize on errors in various ways in their work. I believe that just the awareness of such uses of errors can be beneficial to mathematics students and teachers, as it may contribute to challenging some common preconceived notions about the dualistic nature of mathematics and the negative role of errors. Yet, I claim that students and teachers would benefit even more once they themselves could use errors in a similar way in the context of their mathematical activity, as I hope to show in the following chapters.

Chapter 5

Unlocking the Potential of Errors to Stimulate Inquiry Within the Mathematics Curriculum

The examples reported in the previous chapter provided compelling evidence that mathematicians have been using errors as springboards for inquiry all along. However, one could still question whether only experts in the discipline may have the expertise and ability necessary to capitalize on errors in this way. In order to dispel this doubt and further illustrate the variety of ways in which errors may stimulate mathematical inquiry, in this chapter I report and reflect on the results of my own explorations in three error case studies dealing with mathematical content relevant to the secondary school mathematics curriculum.

The case studies reported in this chapter show that no great mathematical background is needed to successfully capitalize on errors. They also implicitly begin to suggest how the proposed strategy could be employed by mathematics teachers to plan learning activities involving their students in genuine mathematical inquiries. With the goal of contributing further ideas for planning instructional activities where errors are capitalized on as springboards for inquiry (what I call *error activities* hereafter) in the context of school mathematics, I then identify and discuss both the kind of questions worth investigating that a few specific types of errors may raise and the possible sources of errors to be used as the starting point or focus of error activities.

ERROR CASE STUDIES GENERATED BY MY OWN EXPLORATION OF SPECIFIC ERRORS

The error case studies reported in this chapter can all be considered the result of my own use of errors as springboards for inquiry. At the beginning of my study of the potential of errors to stimulate exploration and reflection in the context of school mathematics, I believed that an important first step should be my own personal engagement in the kind of activities I was proposing. I thought that this experience would be important not only to generate some pre-

liminary ideas for planning similar activities for students, but also because it would enable me to experience as a learner the power and limitations of the strategy I was developing.

The three error case studies reported here are just a sample of these studies (see Borasi, 1986b, and Brown & Callahan, 1985, for more illustrations). I have selected them first of all because they well illustrate the various forms and directions that might be taken in explorations developed around errors dealing with diverse mathematical content within the secondary school curriculum. Another reason for their selection is that parts of these case studies were adapted and implemented in instructional experiences on which I report in later chapters.

The error case studies reported in this chapter are the most developed and comprehensive of those included in this book. It is my hope that they illustrate the full extent of the reflections and explorations that could be stimulated by errors, so that the reader can better appreciate the potential of this strategy for stimulating mathematical inquiry on the part of nonspecialists. For this reason, each case study also includes an explicit discussion of the potential value of the inquiries thus generated for mathematics students.

The error case studies reported in this section deal with four different areas of the mathematics curriculum—algebra, number theory, geometry, and calculus. More specifically, in the first case study I report on some interesting explorations involving the solution of unusual equations that were motivated by the puzzling fact that the simplification $\frac{16}{64} = \frac{1}{4}$ yields a correct result ("My $\frac{16}{64} = \frac{1}{4}$ " case study [F/5]). (For an illustration of how secondary school mathematics responded to the same error, see the "Students $\frac{16}{64} = \frac{1}{4}$ " case study [O/8]). The following error case study ("My definitions of circle" case study [G/5]) provides a good contrast to the more popular approach to errors as tools for diagnosis and remediation, as here I report on how the categorization and analysis of a list of over 40 incorrect definitions of circle contributed to my own understanding of the notions of circle and mathematical definition. (Activities involving a similar analysis of a list of incorrect definitions of circle in the contexts of secondary school and teacher education can be found in the "Students' definitions of circle" [U/6] and "Teachers' definitions of circle" [Q/9] case studies later in the book). Finally, in "My unrigorous proof" case study [H/5] I reconstruct in detail the explorations invited by a first creative but unrigorous proof I had devised to evaluate the infinite expression $\sqrt{2 + \sqrt{2 + \dots + \sqrt{2 + \sqrt{2}}}}$ —an experience that made me better appreciate the role played by errors in the historical development of analysis reported earlier in the "Calculus" historical case study [B/4] and led me to a deeper understanding of basic concepts, such as *limit* and *mathematical proof* and to some novel and interesting mathematical results. (See the later "Teachers' unrigorous proof" case study [R/9] for the report of an error activity based on this error, developed in the context of a teacher education course.)

Error Case Study F: $\frac{16}{64} = \frac{1}{4}$ —How Can Such a Crazy Simplification Work? ("My $\frac{16}{64} = \frac{1}{4}$ " case study [F/5])

As I was sharing my first ideas about the potential of errors to stimulate mathematical inquiries, a friend challenged me to show whether anything of interest could come out of "trivial errors" such as the following simplification:

$$\frac{16}{64} = \frac{1}{4}$$

Contrary to my friend's expectations, the fact that such an outrageous simplification could yield a correct result immediately raised my curiosity, as I began to wonder about how such a thing could happen and whether there would be other cases in which this way of simplifying fractions could work. Pursuing these questions, led me to engage in worthwhile explorations requiring the solution of simple but unusual equations which, in turn, required the use of some fundamental concepts from number theory. In this error case study, I report in detail on the surprising results of a few of these explorations and on the thinking and problem-solving processes that led me to such results, commenting in the end on the potential value of the activities I engaged in here for mathematics students.¹

Solving the Original Puzzle. As I began to seek some explanation for why such a crazy simplification could ever work, the first challenge I met was that of translating my puzzlement into specific questions that could be explored mathematically. A first step in this direction consisted of trying to better articulate what this error made me wonder about, and produced the following two questions:

1. How can it be that "canceling the sixes" in $16/64$ yields a correct result?
2. Can there be other two-digit fractions that could be correctly simplified in this way?

When translated in mathematical terms, the last question is essentially equivalent to:

- 2a. For what values of the digits a , b , and c is

¹ Some of the explorations discussed in this error case study have been reported in Borasi (1986a, 1986b). Further inquiry about possible generalizations of the "error" discussed here was also pursued by a student in one of my teacher education courses and later published (see Johnson, 1985).

$$\frac{10a+b}{10b+c} = \frac{a}{c}?$$

Or:

2b. What are the integral solutions between 1 and 9 of the equation:

$$c(10a+b) - a(10b+c) = 0? \quad (1)$$

Interestingly, these formulations of question 2 implicitly also allow us to address Question 1 somewhat. Because the triplet ($a = 1$, $b = 6$, $c = 4$) satisfies Equation 1, this could be considered a first reason why the simplification turns out to be correct in the case of $\frac{16}{64}$ —although I expect that many people, like myself, would not feel fully satisfied by such an “explanation.”

Responding to Question 2 obviously required me to solve Equation 1. This task, however, did not turn out to be as straightforward as I had initially expected, because there is no set algorithm for finding all the solutions of a linear equation in three variables. Yet, the fact that the range of the variables in question is quite limited (because a , b , and c must all be digits, given the nature of the original problem) made me hopeful of succeeding in solving this equation somehow.

The finite number of values that could be assumed by the variables suggested first of all that the problem could be solved rather trivially by checking whether Equation 1 would be satisfied or not for each possible combination of a , b , and c . Checking all these possibilities might be very tedious by hand, yet with the aid of a computer one could print out all the possible solutions in a few seconds by writing a simple BASIC program such as the following one:

```
10 FOR A = 1 TO 9
20 FOR B = 1 TO 9
30 FOR C = 1 TO 9
40 IF (10 * A + B) * C = A * (10 * B + C) THEN PRINT A, B, C
50 NEXT C
60 NEXT B
70 NEXT A
```

However, because I did not have a computer immediately available at the time and, furthermore, I was intrigued by the idea of finding the solutions to Equation 1 in a more “mathematical” way, I decided to try a different approach. Although I eventually succeeded in the task, solving such an equation turned out to be a much more challenging and rewarding problem-solving activity than I would have ever expected—as documented by the following detailed report of my activity.

First of all, the essential symmetry of Equation 1 suggested that the simplification would always work “trivially” whenever $a = b = c$. No other solution, however, suggested itself by simply “looking” at the equation. I decided therefore to try to rewrite Equation 1 in equivalent, but hopefully more suggestive, forms such as:

$$10a(b-c) = c(b-a) \quad (2)$$

$$10ab = c(9a+b) \quad (3)$$

$$9ac = b(10a-c) \quad (4)$$

Among these equivalent expressions, Equation 2 seemed the most promising one, because given that a , b , and c are all digits, the absolute value of each of its factors (i.e., a , $(b-c)$, c , and $(b-a)$) must be less than 10. Furthermore, I observed that, because 5 divides the first side and 5 is a prime number, either $c = 5$ or $|b-a| = 5$ (in the case of $\frac{16}{64}$, for example, it is $b-a = 6-1 = 5$). Thus, I decided to try to look first for possible solutions with $c = 5$. With this extra condition, Equation 2 becomes:

$$10a(b-5) = 5(b-a) \quad \text{or} \quad b = \frac{9a}{2a-1} \quad (5)$$

Using Equation 5 to compute the values of b corresponding to $a = 1, 2, \dots, 9$, I indeed found three integral values between 1 and 9, yielding two new solutions, besides a trivial one:

$$\{a = 1, b = 9, c = 5\} \quad \text{that is, } \frac{19}{95} = \frac{1}{5};$$

$$\{a = 2, b = 6, c = 5\} \quad \text{that is, } \frac{26}{65} = \frac{2}{5}.$$

Having found all the possible solutions with $c = 5$, I knew that any other nontrivial solution should meet the condition $|b-a| = 5$ —that is, either $b = a + 5$ or $a = b + 5$. Although I thought at first that checking this case would be more complicated than the previous one, it did not turn out to be so. If $b = a + 5$, then Equation 2 becomes:

$$10a(a+5-c) = 5c \quad \text{or} \quad c = \frac{2a^2 + 10a}{1+2a}. \quad (6)$$

Furthermore, this time I had to check only for $a = 1, 2, 3, 4$ in Equation 6, because of the implicit condition $b = a + 5 < 10$. This procedure revealed two nontrivial solutions, including $\frac{16}{64}$:

$$\{a = 1, b = 6, c = 4\} \quad \text{that is, } \frac{16}{64} = \frac{1}{4};$$

$$\{a = 4, b = 9, c = 8\} \quad \text{that is, } \frac{49}{98} = \frac{4}{8}.$$

Similarly, in the case of $a = b + 5$, Equation 2 becomes:

$$10(b + 5)(b - c) = -5c \quad \text{or} \quad c = \frac{2b^2 + 10b}{9 + 2b}. \quad (7)$$

Rather to my surprise, Equation 7 yielded no integral solutions for $b = 1, 2, 3, 4$, and thus no more nontrivial solutions for Equation 2.

In conclusion, the procedure just described enabled me to find all the integral solutions between 1 and 9 of Equation 1 by "checking" only 17 cases (instead of the 729—i.e., $9 \times 9 \times 9$ —verified by a computer program such as the one reported earlier). Although I felt that I had now found a neat answer to my original questions, I was still puzzled by some of the results obtained and curious to pursue some new questions such as:

3. Why did b turn out to be a multiple of 3 in all the nontrivial solutions?
4. Why did 5 play such an important role in my solution process?
5. How could I "simplify" fractions other than those with two-digit numbers at numerator and denominator in a similar way?

Pursuing New Avenues for Inquiry. The questions just listed invited further investigation, which in turn involved some interesting problem solving and problem posing. Let me briefly report on the main results of each of these activities.

Why is b a multiple of 3 in all the nontrivial solutions? The consideration of Equation 4 provided me with some justification for this unexpected result. Because 9 divides the first side of the equation, we can deduce that either $(10a - c)$ is a multiple of 9, or b is a multiple of 3. This does not mean that the condition " b is a multiple of 3" is a necessary condition for a set of solutions of Equation 4—as proved by the trivial solution $a = b = c = 1$. However, by using some divisibility considerations, it is possible to show that the only case in which $(10a - c)$ is a multiple of 9 occurs when $a = c$ (because $10a - c = 9a + (a - c)$ and 9 divides $9a$, then 9 will divide $(10a - c)$ if and only if it divides $(a - c)$; with the given restriction on the range of the variables in this problem, this is possible if and only if $a - c = 0$ —that is, in the case of trivial solutions). In conclusion, in all nontrivial solutions b must be a multiple of 3.

Why did 5 play such an important role in my solution process? The special role played by the number 5 in my solution approach was obviously connected with the presence of the number 10 in Equations 1 and 2. This, in turn, was due to the fact that the numbers I was considering were written in the usual decimal notation. This realization made me start wondering what would happen if the numbers appearing at the numerator and denominator had not been written in the usual decimal notation. If the base of numeration were not 10, but another natural number k , then the original problem should have been stated as follows: "Find the integer solutions between 1 and $(k - 1)$ of the equation:

$$c(ka + b) - a(kb + c) = 0. \quad (8)$$

Could the kind of arguments used in the previous section still be applied to solve this equation? If so, what modifications would they require? Pursuing these questions yielded some interesting results, at the same time shedding some light on the special role played by 5 in the original process.

Equation 8 could obviously still be rewritten in an equivalent but more manageable form as:

$$ka(b - c) = c(b - a). \quad (9)$$

By applying some divisibility considerations to this equation, I could still argue that because the number k divides the first side of Equation 9, it must also divide the second. Would this imply that k must divide either c or $(b - a)$? Because the property "if p divides a product mn , then p divides either m or n " applies only if p is a prime number, I decided to consider the two cases of " k is a prime number" and " k is not a prime number" separately.

If k is a prime number, because k divides $c(b - a)$, this implies that k divides either c or $(b - a)$. But if a, b , and c are nonzero digits in the numeration system with base k , it must be that $0 < c < k$ and $0 \leq (b - a) < k$. Thus, k cannot divide c nor $(b - a)$, unless $b = a$. In conclusion, if k is prime, there can be no solution besides trivial ones to our problem—in other words, no crazy simplifications can give a correct result!

If k is not a prime, then k can be written as the product of at least two primes:

$$k = p_1 p_2 \dots p_n \quad \text{with} \quad n > 1.$$

For any of its prime divisors, $q = p_i$, from Equation 9 we can deduce that either q divides c , or q divides $(b - a)$. We can then use this necessary condition to reduce the number of values of a, b , and c to be "checked" in order to find all the solutions to our problem. In fact we will only need to verify Equation 9 for:

- c equal to any multiple of q less than k .
- $kb - a$ equal to any multiple of q less than k .

From these observations it follows that the bigger we choose q amongst the prime divisors of k , the less the amount of checking required, because there will be a smaller number of multiples of q to consider. The best case will occur when $k = 2q$, because in this case the only multiple of q that is less than k is q itself. And this is exactly what happens by choosing $q = 5$ in the case of the usual decimal notation, because $10 = 2 \times 5$.

These considerations confirm that the procedure used in the previous section was indeed both justified and efficient. It also "explains" why in all the nontrivial solutions found in that case the values of c were either 5 or a multiple of 2.

How could I "simplify" fractions other than those with two-digit numbers as numerator and denominator in a similar way? The success encountered thus far invited me to restate my original question of "what other fractions could be correctly simplified in this way?" in a slightly more general form than in Question 2 and, consequently, Equations 1 through 4. In other words, I started wondering what patterns of "crazy simplifications" could be created when I did not limit myself to consider only two-digit fractions and, also, in which cases their application would yield a correct result. Consider, for example, the following possibilities involving just "three-digit" fractions:

$$\frac{728}{224} = \frac{78}{24} \quad \frac{217}{775} = \frac{21}{75} \quad \frac{484}{847} = \frac{4}{7}$$

Finding all the cases in which each of these patterns of simplification yields correct results would require first of all correct translation into an algebraic equation and then trying to "solve" it. Because this would involve in each case an equation in 4 or 5 variables, the more "mathematical" approach I was able to use in the two-digit case is no more feasible. However, the solution to these equations can still be easily found with the aid of a computer by writing simple modifications of the program reproduced earlier on. Some of the results obtained in this way indeed turn out to be rather surprising—for example, in the case of the first pattern of simplification one can find a total of 564 nontrivial solutions, whereas the second pattern of simplification yields only 6 nontrivial solutions, looking remarkably like the ones found in the two-digit case (see Johnson, 1985, for more detail on this problem).

Reflections on the Educational Value of Using This Error as a Springboard for Inquiry. Despite its apparent triviality, the error $\frac{78}{24}$ undoubtedly led me to engage in a number of challenging mathematical activities, as described in the previous sections. In order to evaluate the potential benefits of using this error

in mathematics instruction, I would like now to comment on the pedagogical value of these activities and discuss their accessibility on the part of mathematics students.

First of all, it is evident that the explorations stimulated by this error led me to engage in some genuine mathematical problem solving, especially as I tried to solve equations for which no algorithm was available. A good amount of problem posing was also involved at various stages of my inquiry. First of all, I had to translate the vague curiosity generated by this error into precise questions that could be explored mathematically. After one such formulation was achieved in terms of solving an unusual equation, the nature of this equation made me ask several unusual questions, such as: How can we eliminate some values to be checked? What values of the variables are more likely to provide solutions? Even when the original problem was solved, I felt the urge to pose new questions as I felt the need to further justify some of the unexpected results obtained (e.g., the fact that b turned out to be a multiple of 3 in all the nontrivial cases in which the simplification would yield a correct result). An entirely new set of problems was then generated as I decided to challenge the way I had stated the problem in the first place and questioned "What if the base of numeration were not 10, but another natural number?" and "What if the numerator and/or the denominator of the fraction simplified had a number of digits greater than two?"

These considerations suggest that the activities I engaged in would be especially valuable for achieving some of the general "process goals" articulated by the new *NCTM Standards* (NCTM, 1989, p. 5), especially the ones dealing with acquiring mathematical problem-solving and reasoning skills. At the same time, these activities contributed to my understanding and appreciation for some "technical" mathematical topics within algebra and number theory.

First of all, I came to better realize the power of equations as I found myself in the rare situation of having to generate as well as solve an equation in order to answer questions I was really interested in. The equations I worked with—that is, involving more than one variable but with a finite range of possible values—were also of a kind that is rarely addressed in mathematics textbooks although it may be encountered in several real-life situations. A first characteristic of an equation of this kind is that its solution set can be trivially determined by actually "checking" each possible value taken by the variables—something that can be easily accomplished with the aid of a computer.

Solving such equations more "mathematically" is still possible, yet it will require strategies and techniques that are quite different from those taught in school to solve algebraic equations in one unknown. For example, although the algorithm taught in school to solve linear and quadratic equations in one variable is based on identifying some *sufficient conditions* for the solution of the given equation, the experiences reported here made me realize for the first time the value of identifying *necessary conditions* as a heuristic to narrow down the set of poten-

tial solutions, and thus the values to be "checked." Furthermore, necessary conditions could also be used as a means to better understand the problem under study and "explain" some of the results obtained (as done when trying to understand why in all nontrivial solutions b was a multiple of 3). Yet, some of the strategies I employed in solving these equations could prove valuable when working with more standard kinds of algebraic equations as well. For instance, the procedure used to solve the original equation made me better aware of the fact that, although logically equivalent, the different ways in which an equation can be written may have specific roles in the search and analysis of solutions (as the divisibility arguments used to limit considerably the values to be checked could be based on Equation 2 alone, whereas Equation 4, on the other hand, helped provide further explanations for some of the results obtained). Overall, this experience made me better aware of the similarities and differences existing between different kinds of algebraic equations and of the importance of taking them into consideration as one approaches the solution of a novel equation.

The mathematical activities described in this error case study also drew considerably on my knowledge of number theory and thus contributed to my understanding of the potential applications of this area of mathematics. The divisibility arguments used to find all the solutions to Equations 1 and 8 also helped me find new meaning in the concepts of *prime* and *composite* number, and clarify some important differences between the usual decimal notation and numerations in other bases.

In sum, the explorations reported in this case study show how the error $\frac{1}{4}$ provided me with a real purpose as well as a meaningful context to engage in worthwhile mathematical activities that increased my understanding of some technical mathematical topics, at the same time providing opportunities for practicing mathematical problem posing and solving. Although these kinds of activities would usually not take place in a mathematics course, they would be very valuable for mathematics students for all the reasons stated. Even if some of the inquiries I engaged in may not be accessible to every student, I think that most secondary school students could find an appropriate level from which to address the questions generated by this error. For example, for younger students the task of translating the question "for what fractions could this simplification work" into an algebraic equation, and then solving it with the help of a computer, could be very challenging and worthwhile, and contribute to their understanding of the meaning of variables and equations. More advanced high school students, as well as most college students and mathematics teachers, could also engage in the challenging task of finding at least some (if not all) solutions to Equation 2 without a computer, perhaps after having been introduced to some basic divisibility concepts that they may not have explicitly encountered in their previous mathematics courses.

To conclude, I would like to observe that the fact that $\frac{1}{4}$ was an error might at first not seem so crucial, because this could easily be forgotten once one

starts engaging in specific mathematical problems such as finding all the solutions of an unusual equation and, furthermore, similar problem posing and solving activities could be generated around other problematic situations as well. Yet, I think it is important to point out that in this case it was the surprising fact that such an obviously incorrect simplification would yield a correct result that triggered my initial curiosity and, in turn, motivated and sustained the various mathematical activities described in this case study. I believe that such surprise and curiosity are most likely to occur whenever we encounter incorrect procedures yielding a correct result—as I discuss later in this chapter.

Error Case Study G: Incorrect Definitions of Circle—A Gold Mine of Opportunities for Inquiry ("My Definitions of circle" Case Study (G/5))

In this error case study I report on some explorations and insights into the notions of *circle* and *mathematical definition* that developed from my analysis of a collection of mostly incorrect definitions of circle. These definitions were collected from two different groups of people—the students in a graduate mathematics education course consisting mainly of in-service mathematics teachers, and the more "mathematically naïve" students attending a remedial college mathematics course.

All the definitions produced by each group have been reported in the following two lists. Within each list, the items have been organized in a somewhat logical order (as will become more apparent later on). In the rest of this error case study, specific definitions are referred to by the code number assigned to them in these two lists.

List of definitions of circle given by teachers/educators:

- T1. Locus of points in a plane equidistant from a given point.
- T2. Locus of points equidistant from a given point.
- T3. A line connecting a set of (infinite) points equidistant from a given point.
- T4. A set of possible points, all the same distance from a given point called the center.
- T5. Circle is a continuous curved line.
- T6. A circle is a line with ends connected.
- T7. A circle is a curved line perfectly round in shape that meets where it starts or ends where it begins. Its inner area is as barren as the area that encompasses or surrounds this curved line.
- T8. Closed curve whose points are all the same distance from a given point.
- T9. A curved line with no beginning or endpoints which at any point is equidistant from one point (center.)

- T10. A circle is a simple closed geometry figure of all points equidistant from a given center point. It is a two-dimensional figure.
- T11. Continuous set of points in a curved path, equidistant to a center point.
- T12. A circle is a continuous line in a plane that is (always the same distance away) equidistant from a fixed pt.
- T13. A curved line that intersects itself such that all points lie the same distance from a given point called the center.
- T14. The set of all points which satisfy the equation $x^2 + y^2 = C^2$ for some given C.
- T15. Take a line segment of length d with endpoints x and y . (figure) Find the midpoint c of the line segment and "spin" your line segment while keeping c where it began. The set of points that x and y take on as the segment spins is a circle.
- T16. A line with constant curvature.
- T17. A circle is a perfectly round shape or object (if split it is symmetric on both sides of the cut or split).
- T18. A collapsed straight line segment whose endpoints have fused which revolves equidistant around a center point.
- T19. A point with a perfectly symmetrical hole cut in the center which can expand away from that hole at the same rate all around its boundary.
- T20. Circle is a square (figure of a square) with no corners, or circle is a (figure of a square) with the corners pushed in (no corners).
- T21. A circle is a curved set of adjacent (touching) pts. perfectly round in shape that ends where it begins or could otherwise be a curved set of pts. that go on infinitely as long as we realize that if we start at some pt. as we venture about the circle we return to this pt. just really keep repeating or retracing our first tracks over & over & over & over, etc. Its inner area is as barren as the area that surrounds the set of adjacent pts. perfectly round in shape. If split it is symmetric on both sides of the split.
- T22. The intersection of a circular cone with a plane perpendicular to its axis.

List of definitions of circle given by naive mathematics students:

- N1. Circle is a form in which radius is equal from the center to arc. (+ figure)
- N2. A circle—a collection of points all equidistant from the center (radius).
- N3. Circle—1. a geometric form, 2. one-dimensional, 3. a bent line with one end connected to another, 4. a shape with no flat sides.
- N4. A closed, continuous, rounded line.
- N5. Circle: round, both ends meeting.
- N6. Circle: a round object which has no beginning or end, which is smooth, and which has an infinitely number of points on it!!

- N7. Circle—a geometric figure which lies on a plain that consist of a line which begins and end at the same point.
- N8. Circle—a closed round shape having the same radius throughout from the center.
- N9. Circle—has a center with a line around it, all the points on the line are an equal distance from the center, a circle is round.
- N10. Circle: $(x - h)^2 + (y - k)^2 = r$. Round
- N11. Circle = a set of coordinates falling on the plain $(x - h)^2 + (y - k)^2 = r^2$ (+ figure)
- N12. Definition Circle: Circle is a straight line that changes directions constantly.
- N13. Define 'circle'—something that is round—a round line like an orange, wheel.
- N14. Circle: Includes all the points of the circumference and all the points inside it (plane). (+ shaded figure)
- N15. Def. of a Circle. A continuous round line, Infinite—all points from center of circle are equivalent to each other. (+ figure)
- N16. A circle is a set of point with a radius. Round thing.
- N17. Circle—1. round 2. continuous—no beginning or end 3. a set of points such that when connected one gets a concoction called a circle.
- N18. Circle: consecutive points in a 360 angle when connected is round and closed.
- N19. Circle—closed line w/an angle of 360.
- N20. Round—3. 14—shape of a orange, coin, earth—Pt.
- N21. Circle—something whose area is = to πR^2 .
- N22. Definition of a circle: a perfectly round, closed figure with radius r and circumference c where r is the distance from the midpoint of the circle to any outside point and c is the distance measured around the outside.

It would certainly be quite interesting to examine these definitions from an "error analysis" or "misconceptions" perspective—that is, by questioning what these definitions could tell us about their authors' conceptions of circle or mathematical definition, and then suggesting how these notions could be better taught in school mathematics in light of this information. As I approached the study of these definitions, however, my intention was to consciously avoid a diagnosis and remediation approach and to try instead to examine these lists with the goal of clarifying and expanding *my own understanding* of circles and definitions and, possibly, of raising other mathematical questions. As I hope to show in the following report, this kind of analysis indeed proved to be very productive and worthwhile.²

² See Borasi (1986b) for a more detailed and complete report of the results of this study.

A preliminary analysis of the given lists revealed that almost all of the 44 items in them would be considered "unacceptable" as a definition of circle by a mathematician. Yet, taken as a whole, these definitions could provide a wealth of information about circles, because different people focused on different properties of circles in their attempt to characterize this mathematical object. Thus I realized that, regardless of whether the descriptions I had collected from teachers and students could be considered *good mathematical definitions*, they could help me identify many important properties of circles and, furthermore, they could suggest alternative ways to characterize circles by means of specific combinations of these properties.

These observations, in turn, suggested to me the value of categorizing the definitions collected according to two quite different criteria and goals:

- By looking at the "mathematical content" of the definition (i.e., the specific properties of circle mentioned or suggested in it), with the goal of creating as many alternative definitions of circle as possible and learning more about circles more generally.
- By looking at the "kind of error" (if any) that would make a mathematician consider the description given "unacceptable" as a mathematical definition of circle, so as to come to a better understanding of the attributes of a "good" mathematical definition.

In what follows, I report separately on the main results of engaging in these two categorization exercises, and then comment on the potential value of using similar activities in mathematics instruction.

A "Content" Analysis of the Definitions of Circle Collected. As I tried to sort the definitions collected with respect to the combination of properties of circle mentioned, I realized that each group thus identified suggested the assumption of a specific perspective on geometry—reflected in the label I chose to identify each category. Although the authors of each definition might not even have been aware of the existence of such "geometry perspectives," these characterizations were instrumental for me to gain a better understanding of circles, because they enabled me to rely on the body of mathematical knowledge connected with such a perspective in my examination of this mathematical object and its possible definitions.

Metric Definitions

- T1. Locus of points in a plane equidistant from a given point.
- T2. Locus of points equidistant from a given point.
- T3. A line connecting a set of (infinite) points equidistant from a given point.

- T4. A set of possible points, all the same distance from a given point called the center.
- N1. Circle is a form in which radius is equal from the center to arc. (+ figure)
- N2. A circle—a collection of points all equidistant from the center (radius).

Although somewhat differently worded, the definitions in this group are all based on the "metric" property that all the points on a circle are at a given distance from a given point—a property used in most elementary geometry textbooks to define circles rigorously in the context of plane Euclidean geometry. Anyone with some background in plane Euclidean geometry can appreciate the advantages of using this property to characterize circles, because it provides a simple yet precise way to verify whether a given plane figure is a circle or not, it justifies the use of a compass to draw circles, and, most importantly, it can be used to logically deduce all the other geometric properties of circles.

On the other hand, it is important to realize that a definition based on the equidistance property does nothing to highlight the more "visual" elements of circles, such as its "roundness" or continuity. Indeed, if the definition of circle as "the locus of points at a given distance from a given point" were to be interpreted in contexts other than the Euclidean plane, it could identify figures that would look quite different from what we tend to associate with the word *circle*. Consider for example what would happen if such a definition were to be interpreted in the context of *taxicab geometry*—that is, a square grid representing the idealization of a city with a regular "grid" of streets, like Manhattan. In this situation, because cars have to follow the roads in order to move from point to point, the distance between two points can no more be measured "as the crow flies," but rather should be computed as the "length of the shortest path on the grid connecting two points." Thus, all the points at distance 4 from a given point on the grid would look more like a square than a circle (see Figure 5.1).

The fact that the metric definition of circle could identify such figures may be considered a serious drawback by some people, because it may not reflect their intuitive expectation of what circles should look like. Yet, from the mathematical point of view, this could be turned into an advantage, because it invites a generalization of the Euclidean notion of circle applicable to other metric spaces. It is interesting, however, that mathematicians chose to assign a different word, that of *ball*, to this generalized notion and to reserve instead the word *circle* to indicate only the Euclidean circles we are used to.

Topological-Projective Definitions

- T5. Circle is a continuous curved line.
- T6. A circle is a line with ends connected.

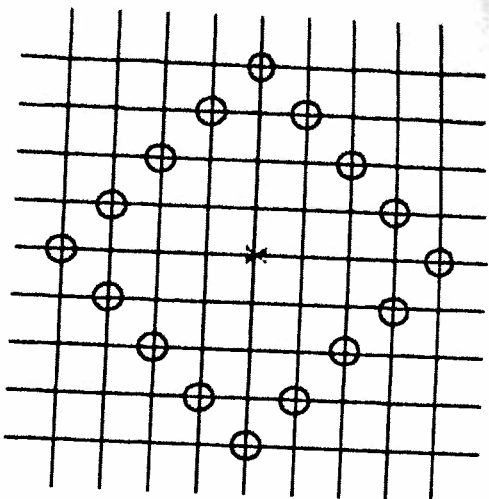


FIGURE 5.1. "Taxi-circle" of radius 4.

- T7. A circle is a curved line perfectly round in shape that meets where it starts or ends where it begins. Its inner area is as barren as the area that encompasses or surrounds this curved line.
- N3. Circle—1. a geometric form, 2. one dimensional, 3. a bent line with one end connected to another, 4. a shape with no flat sides.
- N4. A closed, continuous, rounded line.
- N5. Circle: round, both ends meeting.
- N6. Circle: a round object which has no beginning or end, which is smooth, and which has an infinitely number of points on it!
- N7. Circle—a geometric figure which lies on a plain that consist of a line which begins and ends at the same point.

I placed in this group all the definitions that pointed out mathematical properties such as continuity, closeness, being a line, being curved rather than straight, or other properties connected essentially with what circles look like. Whereas these elements are probably the most obvious and visual properties of circles, it is important to realize that they are not sufficient to distinguish circles from other figures such as ellipses or egg-shaped curves. This observation, in turn, made me curious to explore what "family of shapes" some of these properties would identify as well as what geometrical transformations could transform circles into figures in the same family, that is, preserving these fundamental properties (a key question from the perspective of transformation geometry).

As I explored this question I realized that whereas Euclidean transformations (i.e., rotations, symmetry, translations, and any of their combinations) would just change the position of a given circle, and similitudes (i.e., projections from

a point source to a parallel plane) would transform it into another circle of different size, all the other "classical" transformations would change it into another figure. Affinities (i.e., projections from an infinitely distant source) would transform circles into ellipses (with some liberty of interpretation, one could say that definitions N4, N5, and N6 describe this class of figures). Projective transformations would "deform" a circle even further, but would still retain several of its characteristics—such as being a curved (rather than straight), closed, continuous, convex, and simple (i.e., with no cavities or intersections) line. Under topological transformations (comparable to "stretching" the figure when drawn on a rubber surface) even some of these characteristics are lost, although figures equivalent to circles will still be at least closed, simple, and continuous. Hence, one could say that definitions T5 through T7 and N3 through N7 focus essentially on the topological-projective properties of circles.

Topological and Metric Definitions:

- T8. Closed curve whose points are all the same distance from a given point.
- T9. A curved line with no beginning or endpoints which at any point is equidistant from one point (center.)
- T10. A circle is a simple closed geometry figure of all points equidistant from a given center point. It is a two-dimensional figure.
- T11. Continuous set of points in a curved path, equidistant to a center point.
- T12. A circle is a continuous line in a plane that is (always the same distance away) equidistant from a fixed pt.
- T13. A curved line that intersects itself such that all points lie the same distance from a given point called the center.
- N8. Circle—a closed round shape having the same radius throughout from the center.
- N9. Circle—has a center with a line around it, all the points on the line are an equal distance from the center, a circle is round.

Definitions within this category combine the metric property of having points equidistant from the center with some of the more intuitive and visual properties discussed in the previous category. Consequently, all of the definitions in this group could be considered somewhat redundant (in the sense that they contain more than the essential elements necessary to distinguish circles from other figures). Some people, however, may find these definitions more satisfactory than purely metric definitions, as they explicitly mention characteristic properties of circle that are not "captured" by the latter.

Analytic Geometry Definitions:

- T14. The set of all points which satisfy the equation $x^2 + y^2 = C^2$ for some given C .
- N10. Circle: $(x - h)^2 + (y - k)^2 = r$. Round

N11. Circle = a set of coordinates falling on the plain $(x - h)^2 + (y - k)^2 = r^2$ (+ figure)

Definitions in this category are based on interpretations of the "equidistance" property characterizing circles within the context of analytic geometry—that is, an algebraic representation of the plane (or space) that associates each point to a pair (triple) of numbers identifying its position with respect to a given system of coordinates. Although it is not evident in the three definitions belonging to this group, found it interesting that there may be more than one way to describe circles in analytic geometry, depending on the system of coordinates chosen, so I decided to explore this issue further.

In the usual Cartesian system of coordinates, a circle in the xy plane is usually described as follows:

The curve, in rectangular Cartesian coordinates, corresponding to the equation $(x - h)^2 + (y - k)^2 = r^2$, where r is the radius of the circle and (h, k) are the coordinates of the center.

Whereas this definition has the advantage of providing explicit information about both the size of the radius and the position of the center of the circle, it is also important to realize that any equation of the form $x^2 + y^2 + 2ax + 2by + c = 0$ would represent a circle as long as it satisfies the condition $a^2 + b^2 - c > 0$. Alternatively, if we chose to use parametric equations, circles could be described by the system of equations:

$$\begin{cases} x = r \cos t + h \\ y = r \sin t + k \end{cases}$$

If we want to describe a circle in space matters complicate considerably. Circles, as any other curve in the space, can no longer be described in general by a simple equation. We could either use a set of parametric equations (three in this case) or try to describe the circle as the intersection of two three-dimensional figures—in other words, as the solution of a system of two equations. Notice that the way to do this is not unique, because a circle could be described as the intersection of a sphere and a plane, or a cylinder and a plane, or even a cylinder and a sphere.

We might also consider systems of coordinates other than rectangular Cartesian ones. Polar coordinates seem especially suitable, because circles with center in the origin can be described by the simple equation $\rho = r$ (although it would not be so easy to describe circles with a different center or positioned in the space).

In sum, any analytic definition of circle has the advantage of being extremely precise and useful in applications, as not only does it identify precisely the

"shape" of a given circle, but also its dimensions and position. It is important to appreciate, however, that these definitions are not intuitive or easy to understand for nonspecialists. Let us not forget that it took a long time and the genius of Descartes and other great mathematicians to recognize the possibility of describing curves in the plane as solutions of an algebraic equation.

"Rotation" Definitions:

T15. Take a line segment of length d with endpoints x and y . (figure) Find the midpoint c of the line segment and "spin" your line segment while keeping c where it began. The set of points that x and y take on as the segment spins is a circle.

T22. The intersection of a circular cone with a plane perpendicular to its axis.

The definitions in this category provide not so much a description of the characterizing properties of circles, but rather a method to construct such a figure, in each case based essentially on the idea of rotating an object. More specifically, in T15 the circle is characterized as the result of the rotation of a segment—a simple yet perfectly legitimate and precise way to obtain circles (as long as the rotation occurs in a plane), although I had never considered it before. By characterizing a circle as the intersection of a circular cone and a plane perpendicular to its axis, definition T22 also implicitly uses rotation—because in order to avoid circularity in the definition, one has to think of a circular cone as the result of rotating a right triangle around one of its legs. This definition also implicitly suggests a relationship between circles and other figures that can be obtained by intersecting the same circular cone with different planes—that is, ellipses, hyperbolas, and parabolas.

Differential Geometry Definitions:

T16. A line with constant curvature.

N12. Definition Circle: Circle is a straight line that changes directions constantly.

With some interpretation, both of the definitions in this category could be considered a good approximation of the definition of circle as "a plane line with constant nonzero curvature" in the context of differential geometry. Not unlike the notion of *circular cone*, *curvature* can also be defined without any direct reference to circles as "the rate of change of direction of a given line," which, in turn, can be precisely described in terms of derivatives of the line's equation.³

³ For an intuitive yet rigorous derivation of the concept of *curvature* of a line see, for example, Alexandrov, Kolmogorov, and Laurent'ev (1969).

Furthermore, circles are the only plane curve with constant curvature (different from zero), as can be proved by solving the differential equation corresponding to such a definition. Because straight lines are the only other plane lines with constant curvature (although, in this case, the curvature will always be zero), this definition reveals an unexpected commonality between circles and straight lines from a differential geometry standpoint.

Although it may seem surprising at first, the property of having constant curvature can also be related to a definition of circle as "the limit of regular n -sided polygons, when n tends to infinity." This relationship was clearly identified and justified by Papert, as he discussed the principle behind a program to draw circles in LOGO consisting essentially of the repeated command "FORWARD 1 RIGHT TURN 1":

For a student, drawing a Turtle circle is more than a "common sense" way of drawing circles. . . . The Turtle program is an intuitive analog of the differential equation. . . . Differential calculus derives much of its power from an ability to describe growth by what is happening at the growing tip. . . . In our instructions to the Turtle, FORWARD 1 RIGHT TURN 1, we referred only to the difference between where the Turtle is now and where it shall momentarily be. That is what makes the instructions "differential." There is no reference in this to any distant part of space outside the path itself. . . . In Turtle geometry a circle is defined by the fact that the Turtle keeps repeating the act: FORWARD a little, TURN a little. This repetition means that the curve it draws will have "constant curvature," where curvature means how much you turn for a given forward motion. (Papert, 1980, pp. 66–67)

Purely Visual Descriptions:

T17. A circle is a perfectly round shape or object (if split it is symmetric on both sides of the cut or split).

N13. Define 'circle'—something that is round—a round line like an orange, wheel.

Definitions in this category try to identify circles only on the basis of the most visual properties associated with this shape, or even just by indicating what objects in nature embody such a shape. It is interesting, however, to remark on the mention to symmetry made in T17. One of the most interesting and mathematically useful properties of a circle, in fact, is its symmetry with respect to any of its diameters. This, in turn, made me wonder whether the notion of circle (and, similarly, other "classical" regular figures such as squares, equilateral triangles, other regular polygons, or even ellipses) could be rigorously characterized purely on the basis of its group of symmetries.

"Weird" Definitions:

T18. A collapsed straight line segment whose endpoints have fused which revolves equidistant around a center point.

T19. A point with a perfectly symmetrical hole cut in the center which can expand away from that hole at the same rate all around its boundary.

T20. Circle is a square (figure of a square) with no corners, or circle is a (figure of a square) with the corners pushed in (no corners).

T21. A circle is a curved set of adjacent (touching) pts. perfectly round in shape that ends where it begins or could otherwise be a curved set of pts. that go on infinitely as long as we realize that if we start at some pt. as we venture about the circle we return to this pt. just really keep repeating or retracing our first tracks over & over & over & over, etc. Its inner area is as barren as the area that surrounds the set of adjacent pts. perfectly round in shape. If split it is symmetric on both sides of the split.

N14. Circle: Includes all the points of the circumference and all the points inside it (plane). (+ shaded figure)

N15. Def. of a Circle. A continuous round line, Infinite—all points from center of circle are equivalent to each other. (+ figure)

N16. A circle is a set of point with a radius. Round thing.

N17. Circle—1. round 2. continuous—no beginning or end 3. a set of points such that when connected one gets a concoction called a circle.

N18. Circle: consecutive points in a 360 angle when connected is round and closed.

N19. Circle—closed line w/an angle of 360.

N20. Round—3.14—shape of a orange, coin, earth—Pi.

N21. Circle—something whose area is $= \pi R^2$.

N22. Definition of a circle: a perfectly round, closed figure with radius r and circumference c where r is the distance from the midpoint of the circle to any outside point and c is the distance measured around the outside.

I collected in this category all the definitions that seemed strange for quite different reasons: they included weird ideas (T18–20, N17–19); combined properties of circles in unusual ways (T21, N22); or, focused on unessential elements of circles while leaving out the most important ones (N14–16, N20–21). Although these definitions are all quite interesting for what they implicitly reveal of their author's conception of what a definition should accomplish, they do not add much to a discussion of the mathematical properties of circle, with a few exceptions. For example, N14 reminds us of the important distinction between circles as lines, on the one hand, and all the points contained inside such line, on the other. N18 and N19 point to a property that circles share with many other figures—that is, the fact that the angle at the center is 360° —but that is also used extensively in most geometry textbooks to derive properties of circles such as

the relationship between angles at the center, angles at the circumference, and their corresponding arcs. An interesting connection between the circle and infinity has been drawn more than once. Several people felt the need to mention that the circle consists of infinitely many points. Others instead pointed out (more or less explicitly) the fact that, as the circle does not have a beginning or end, you can go around it "forever." It is interesting to note that this property of circles has also been captured in some drawings by Escher (consider, for example, his lithograph entitled "Reptiles," reported in Hofstadter, 1979, p. 117).

An Analysis of the Definitions Collected With Respect to Their Adequacy as Mathematical Definitions. Very few of the definitions of circle in the two lists (perhaps only T1 and T22) would be deemed acceptable by a mathematician, despite the fact that (with the exception of N14) their authors seemed to "know" the meaning of the term and tried indeed to describe circles. In what follows, I try to make explicit why this is the case by presenting and discussing a categorization of these definitions based on the "errors" perceived in them, with the ultimate goal of identifying and discussing the criteria mathematicians have established for definitions in their field.

Imprecision in Terminology:

- T4. A set of possible points, all the same distance from a given point called the center.
- T5. Circle is a continuous curved line.
- T6. A circle is a line with ends connected.
- T7. A circle is a curved line perfectly round in shape that meets where it starts or ends where it begins. Its inner area is as barren as the area that encompasses or surrounds this curved line.
- T13. A curved line that intersects itself such that all points lie the same distance from a given point called the center.
- T15. Take a line segment of length d with endpoints x and y . (figure) Find the midpoint c of the line segment and "spin" your line segment while keeping c where it began. The set of points that x and y take on as the segment spins is a circle.
- T17. A circle is a perfectly round shape or object (if split it is symmetric on both sides of the cut or split).
- T18. A collapsed straight line segment whose endpoints have fused which revolves equidistant around a center point.
- T19. A point with a perfectly symmetrical hole cut in the center which can expand away from that hole at the same rate all around its boundary.
- T20. Circle is a square (figure of a square) with no corners, or circle is a (figure of a square) with the corners pushed in (no corners).
- T21. A circle is a curved set of adjacent (touching) pts. perfectly round in shape that ends where it begins or could otherwise be a curved set of

pts. that go on infinitely as long as we realize that if we start at some pt. as we venture about the circle we return to this pt. just really keep repeating or retracing our first tracks over & over & over & over, etc. Its inner area is as barren as the area that surrounds the set of adjacent pts. perfectly round in shape. If split it is symmetric on both sides of the split.

- N1. Circle is a form in which radius is equal from the center to arc. (+ figure)
- N2. A circle—a collection of points all equidistant from the center (radius).
- N3. Circle—1. a geometric form, 2. one-dimensional, 3. a bent line with one end connected to another, 4. a shape with no flat sides.
- N4. A closed, continuous, rounded line.
- N5. Circle: round, both ends meeting.
- N6. Circle: a round object which has no beginning or end, which is smooth, and which has an infinitely number of points on it!
- N7. Circle—a geometric figure which lies on a plane that consist of a line which begins and ends at the same point.
- N8. Circle—a closed round shape having the same radius throughout from the center.
- N9. Circle—has a center with a line around it, all the points on the line are an equal distance from the center, a circle is round.
- N10. Circle: $(x - h)^2 + (y - k)^2 = r$. Round
- N11. Circle = a set of coordinates falling on the plane $(x - h)^2 + (y - k)^2 = r^2$ (+ figure)
- N13. Define 'circle'—something that is round—a round line like an orange, wheel.
- N15. Def. of a Circle. A continuous round line, Infinite—all points from center of circle are equivalent to each other. (+ figure)
- N16. A circle is a set of point with a radius. Round thing.
- N17. Circle—1. round 2. continuous—no beginning or end 3. a set of points such that when connected one gets a concoction called a circle.
- N18. Circle: consecutive points in a 360 angle when connected is round and closed.
- N19. Circle—closed line w/an angle of 360.
- N20. Round—3.14—shape of a orange, coin, earth—Pi.
- N21. Circle—something whose area is = to πR^2 .
- N22. Definition of a circle: a perfectly round, closed figure with radius r and circumference c where r is the distance from the midpoint of the circle to any outside point and c is the distance measured around the outside.

The most obvious shortcomings of the definitions in this category can be identified in the use of terms that are unclear or ambiguous and, thus, could be given different interpretations by different people. T18 through T20 and

N18 and N19 are probably the most obvious examples of this kind of error. Indeed, what do expressions like "a collapsed straight line," "a point with a perfect symmetrical hole," or "consecutive points in a 360° angle" really mean? Even in less "absurd" definitions, however, we still find very ambiguous terms—like "equivalent" (N15) or "possible" (T4) points—that could lead to various interpretations. On the contrary, we would expect mathematical definitions to use only mathematical terms that have been previously and precisely defined, so that everyone reading the definition would attribute to them the same meaning and, consequently, identify the same object as being described or not by the definition.

Although this requirement may seem quite reasonable at first, it is important to point out that its application may not always be straightforward. First of all, we have to recognize that in any deductive system there will always be at least some undefinable terms (such as point, plane, etc.) on which the definition of all the other terms will then depend. Furthermore, there is always the risk that, even when precise mathematical terminology is used by the author of a definition, it may not be understood in the same way by a more naïve reader. Think, for example, of the different meaning assumed by terms such as "simple" or "smooth" in mathematics and real life.

Non-exclusiveness:

- T2. Locus of points equidistant from a given point.
- T3. A line connecting a set of (infinite) points equidistant from a given point.
- T4. A set of possible points, all the same distance from a given point called the center.
- T5. Circle is a continuous curved line.
- T6. A circle is a line with ends connected.
- T8. Closed curve whose points are all the same distance from a given point.
- T9. A curved line with no beginning or endpoints which at any point is equidistant from one point (center.)
- T10. A circle is a simple closed geometry figure of all points equidistant from a given center point. It is a two-dimensional figure.
- T11. Continuous set of points in a curved path, equidistant to a center point.
- T13. A curved line that intersects itself such that all points lie the same distance from a given point called the center.
- T14. The set of all points which satisfy the equation $x^2 + y^2 = C^2$ for some given C .
- T16. A line with constant curvature.
- T18. A collapsed straight line segment whose endpoints have fused which revolves equidistant around a center point.
- T20. Circle is a square (figure of a square) with no corners, or circle is a (figure of a square) with the corners pushed in (no corners).

- N1. Circle is a form in which radius is equal from the center to arc. (+ figure)
- N2. A circle—a collection of points all equidistant from the center (radius).
- N3. Circle—1. a geometric form, 2. one-dimensional, 3. a bent line with one end connected to another, 4. a shape with no flat sides.
- N4. A closed, continuous, rounded line.
- N5. Circle: round, both ends meeting.
- N6. Circle: a round object which has no beginning or end, which is smooth, and which has an infinitely number of points on it!!
- N7. Circle—a geometric figure which lies on a plain that consist of a line which begins and ends at the same point.
- N8. Circle—a closed round shape having the same radius throughout from the center.
- N9. Circle—has a center with a line around it, all the points on the line are an equal distance from the center, a circle is round.
- N10. Circle: $(x - h)^2 + (y - k)^2 = r$. Round
- N13. Define 'circle'—something that is round—a round line like an orange, wheel.
- N15. Def. of a Circle. A continuous round line, Infinite—all points from center of circle are equivalent to each other. (+ figure)
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- N19. Circle—closed line w/an angle of 360.
- N20. Round—3.14—shape of a orange, coin, earth—Pi.
- N21. Circle—something whose area is $= \pi R^2$.
- N22. Definition of a circle: a perfectly round, closed figure with radius r and circumference c where r is the distance from the midpoint of the circle to any outside point and c is the distance measured around the outside.

Another problem shared by most of the definitions collected is that they do not succeed in isolating circles from all other figures—one of the main purposes of a mathematical definition. For example, by their very nature all the definitions in the topological-projective category identify circles as well as ellipses and similar kinds of curves. Even most of the metric definitions fail to isolate circles, because they do not explicitly mention the fact that all the points considered should belong to a plane—thus allowing for spheres to satisfy the definition as well.

Once again, it seems rather obvious that we would like a mathematical definition to list a combination of properties that can be satisfied only by instances of the mathematical object it tries to identify. Yet, the implementation of this

requirement may turn out to be problematic in practice whenever the object we are trying to define is not so intuitively clear to us a priori as circle is. The progressive modifications that occurred historically in the development of the definition of polyhedron, as discussed earlier in the "Euler theorem" historical case study [E/4], are a good illustration of how some mathematical concepts, and consequently their definitions, may change in time as mathematicians find new applications for such concepts. The previous discussion of what happens when the metric definition of circle is interpreted in contexts other than the Euclidean plane also reveals that a given definition may no more identify the same set of objects when interpreted in a different mathematical context.

Non-inclusiveness:

- T14. The set of all points which satisfy the equation $x^2 + y^2 = C^2$ for some given C .
 N10. Circle: $(x - h)^2 + (y - k)^2 = r$. Round
 N11. Circle = a set of coordinates falling on the plain $(x - h)^2 + (y - k)^2 = r^2$ (+ figure)

An error opposite to the one just described would consist in not including all the possible circles in the given definition. It is interesting to note that such an error was not very common among the definitions collected (perhaps because of the nature of the concept in question). In fact, only the analytic geometry definitions could be said to present this shortcoming, because T14 identifies only circles with center at the origin, whereas N10 and N11 include only circles lying in the xy plane. Yet, even these criticisms could be countered by the argument that given a specific circle one can always find a suitable system of coordinates in which such a circle would be characterized by the equations contained in these definitions.

Redundancy:

- T3. A line connecting a set of (infinite) points equidistant from a given point.
 T4. A set of possible points, all the same distance from a given point called the center.
 T7. A circle is a curved line perfectly round in shape that meets where it starts or ends where it begins. Its inner area is as barren as the area that encompasses or surrounds this curved line.
 T8. Closed curve whose points are all the same distance from a given point.
 T9. A curved line with no beginning or endpoints which at any point is equidistant from one point (center.)
 T10. A circle is a simple closed geometry figure of all points equidistant from a given center point. It is a two-dimensional figure.
 T11. Continuous set of points in a curved path, equidistant to a center point.

- T12. A circle is a continuous line in a plane that is (always the same distance away) equidistant from a fixed pt.
 T21. A circle is a curved set of adjacent (touching) pts. perfectly round in shape that ends where it begins or could otherwise be a curved set of pts. that go on infinitely as long as we realize that if we start at some pt. as we venture about the circle we return to this pt. just really keep repeating or retracing our first tracks over & over & over & over, etc. Its inner area is as barren as the area that surrounds the set of adjacent pts. perfectly round in shape. If split it is symmetric on both sides of the split.
 N8. Circle—a closed round shape having the same radius throughout from the center.
 N9. Circle—has a center with a line around it, all the points on the line are an equal distance from the center, a circle is round.
 N10. Circle: $(x - h)^2 + (y - k)^2 = r$. Round
 N20. Round—3.14—shape of a orange, coin, earth—Pi.

Few people would disagree about considering "overkill" definitions such as T21 and N22 inappropriate. Yet, it may be more difficult to justify why visual and metric definitions such as T8 and T9 should also be discarded on the basis that they include some unnecessary properties of circle along with characterizing ones. Wouldn't such a definition add to a purely metric one by highlighting some visual properties of circles and, thus, make the concept more understandable for naive mathematics students? I was not able to address this question satisfactorily until I realized that mathematicians' concern to reduce the number of properties listed in a definition to a minimum was based on the hope of minimizing the risk of creating *contradictory definitions* (i.e., definitions that would identify only an empty set because the properties they include cannot all coexist) especially when the original definition is interpreted in new domains. For example, it is interesting to note that T8 would be meaningless when interpreted in the context of taxicab geometry mentioned earlier in this case study, because in taxicab geometry the locus of points equidistant from a given point is neither "curved" nor "closed."

Circularity:

- N20. Round—3.14—shape of a orange, coin, earth—Pi.

A first reading of T16 ("A line with constant curvature") and T22 ("The intersection of a circular cone with a plane perpendicular to its axis") made me identify noncircularity as an important attribute of mathematical definitions, because a definition would be meaningless if it uses the same term it tries to define as part of the definition itself. A closer look at those two definitions, however, made me realize that both the concept of *curvature* and *circular cone*

may actually be defined prior to and independently from *circle* (as discussed in greater detail earlier). Consequently, T16 and T22 cannot truly be criticized as circular—a realization that made me aware of the importance of looking not so much at the words employed in a definition, but rather at their use as well as possible definition. The consideration of recursive definitions, which occur frequently in computer programming, also required me to further clarify this criteria. Definition N20, on the contrary, could be considered truly circular, because I cannot see a way of defining P1 meaningfully without invoking the notion of circle in the first place.

Concluding Observations on the Pedagogical Value of the Previous Categorizations and Analyses. The study of a list of mostly incorrect definitions of circle, as described in this error case study, led me much farther than I could have ever expected at the beginning. In what follows I try to identify more precisely what I learned as a result of the inquiries motivated by these errors, so as to better evaluate their potential value for mathematics students.

First of all, my analysis of these definitions of circle increased my knowledge of mathematical content. I gained a better understanding of circles in that specific definitions highlighted different properties of this geometric figure and their relationships. I also came to realize the possibility of defining circle rigorously in a number of different ways. My mathematical analysis of the definitions also produced some unexpected outcomes, such as the identification of commonalities that I had not previously realized existed between circles and a number of other geometric figures. Consequently, I reached a better understanding of these figures, as well as a better appreciation of the implications of assuming different geometric perspectives (such as Euclidean geometry, transformation geometry, analytic geometry, or differential geometry). In sum, despite their inadequacy as definitions of circle, the items collected in the two lists enriched my own image of circles in ways that I had not previously achieved.

The critical study of these definitions also required me to exercise some mathematical problem posing and problem solving, as my analysis often involved the formulation of questions and problems worth exploring (What other figures can be described by this definition? What is their relationship with circles? Is there a mathematical context in which this definition would be considered acceptable?). My inquiry also required me to retrieve and use relevant mathematical facts that I had learned in quite different mathematics courses—an exercise I find especially important for mathematics learners at all levels.

The fact that most of the definitions of circle I had collected were somewhat incorrect also motivated an inquiry into the notion of mathematical definition. As a result, I not only became more aware of the nature and rationale of the criteria usually imposed on definitions in mathematics, but also was led to realize some of their limitations. As a result, I became especially aware of the dy-

namic nature of mathematical definitions and of the need to take into consideration the mathematical context, purpose, and intended audience of a given definition of its evaluation. More implicitly, this initial analysis of the notion of definition also suggested the value of discussing the difference between *definition* and the related but distinct notions of name, symbol, description, model, and example of a given mathematical concept. I would like to point out that such an abstract inquiry into a metamathematical notion was made much more concrete and accessible by the fact that I could examine some concrete examples of definitions of a mathematical object I was very familiar with, as this enabled me to easily pinpoint errors and shortcomings and, by contrast, identify what I would have liked a mathematical definition to achieve.

All the outcomes described here should be considered very desirable for mathematics students and teachers at all levels. Thus, I believe that other mathematics teachers as well as mathematics students should be encouraged to engage in similar exercises, although I would expect that the extent of their mathematical analysis of specific definitions would depend on their prior knowledge of Euclidean geometry as well as more advanced areas of mathematics such as analytic geometry, transformation geometry, and differential geometry (as confirmed later in the "Students' definitions of circle" [U/6] and "Teachers' definitions of circle" [Q/9] case studies).

Error Case Study H: The Unexpected Value of an Unrigorous Proof ("My Unrigorous Proof" Case Study [H/5])

I first came across the surprising result $\sqrt{2 + \sqrt{2 + \dots + \sqrt{2 + \sqrt{2}}}} = 2$ in a geometric proof given by Viète. Although intrigued by this result, I felt uncomfortable with Viète's rather complex proof, especially because I could think of an apparently more direct and immediate way of evaluating the infinite expres-

sion $\sqrt{2 + \sqrt{2 + \dots + \sqrt{2 + \sqrt{2}}}}$ (abbreviated as R[2] in what follows), as described here:

Let us call x the expression we want to evaluate:

$$\sqrt{2 + \sqrt{2 + \dots + \sqrt{2 + \sqrt{2}}}} = x$$

squaring both sides, we get:

$$2 + \sqrt{2 + \dots + \sqrt{2 + \sqrt{2}}} = x^2$$

The radical on the left side is again x , because it also contains infinitely many terms. Therefore, that expression is equivalent to the second degree equation

$$2 + x = x^2$$

Solving this equation for x yields the solutions $x = -1$ and $x = 2$. Because the radical is certainly positive, the root $x = -1$ has to be excluded. Therefore, it must be that:

$$\sqrt{2 + \sqrt{2 + \dots + \sqrt{2 + \sqrt{2}}}} = 2$$

Having been a calculus student not too long before, I was well aware that this procedure, at least in the form stated here, could not be considered "mathematically acceptable." Yet its simplicity, combined with the knowledge that it did yield a correct result (as confirmed by Viète's rigorous proof), made this procedure look very appealing to me and suggested the value of exploring whether it could be made rigorous. In other words, although I recognized that my original derivation of the result $R[2] = 2$ was incorrect—or, more precisely, it was inadequate as a mathematical proof—I believed that the idea behind my reasoning was good and, if only I could identify and justify the assumptions I had implicitly made in my procedure, I might be able to turn it into an elegant and rigorous proof.

This indeed turned out to be the case. Furthermore, the activity of critically analyzing my intuitive procedure and trying to rigorize it proved to be very interesting from both a mathematical and a pedagogical viewpoint—as it resulted in new insights into several fundamental concepts of calculus as well as the nature of mathematical proofs and also provided the stimulus and means to evaluate other infinite expressions.⁴ In what follows, I will try to reconstruct, as precisely as possible, my thinking process as I first worked toward rigorizing an alternative proof for the result $R[2] = 2$ and then used modifications of this procedure for the evaluation of the more general expression $\sqrt{a + \sqrt{a + \dots + \sqrt{a + \sqrt{a}}}}$ (abbreviated with $R[a]$ in what follows).

Constructing a Rigorous Alternative Proof for $R[2]$. As I started to analyze my intuitive procedure, I could first of all identify the act of setting the infinite expression $R[2]$ equal to x as the first critical point in my reasoning. Initially, I did it casually, as we usually do in algebra. We start by giving a name to the ob-

ject we want to evaluate. However, as I looked critically at my work, I soon realized that it was this simple act that naturally led me to the following steps (squaring both sides, and so forth) and consequently to the solution. I then started to ask myself what it really meant to set $R[2] = x$ and whether I was entitled to do so. It took me a while to realize that, in order to use algebraic manipulations to operate with x (such as "squaring" or solving an equation) I had implicitly assumed that $R[2]$ was a real number. In order to justify my procedure, therefore, I would first need to establish whether this was really the case. This realization made me raise, for the first time, this very basic question:

What is the meaning of an infinite expression like $\sqrt{2 + \sqrt{2 + \dots + \sqrt{2 + \sqrt{2}}}}$? Up to then I had just accepted it as a strange mathematical notation that *must* have a numerical value. Yet, thinking back at some examples of "meaningless" infinite expressions encountered in my previous calculus courses (e.g., the infinite sum $1 - 1 + 1 - 1 + 1 \dots$), I had to admit that such an assumption could have been unwarranted.

This issue left me puzzled for a while. Finally, resorting to my previous calculus experiences, I thought that because $R[2]$ involved infinitely many terms, the only way of making sense of it (and, in fact, its rigorous definition) was to consider it as the limit, if it existed, of the following sequence of real numbers:

$$\begin{aligned} x_1 &= \sqrt{2} \\ x_2 &= \sqrt{2 + \sqrt{2}} \\ x_3 &= \sqrt{2 + \sqrt{2 + \sqrt{2}}} \\ x_n &= \underbrace{\sqrt{2 + \dots + \sqrt{2 + \sqrt{2}}}}_{n \text{ terms}} \end{aligned}$$

This realization came to me almost as a surprise and produced a real enlightenment on the nature of sequences and infinite expressions. Up to then I had experienced practically no use for sequences and even less had occasion to create one for some real purpose. I could now conclude that my implicit assumption that $R[2]$ represented a real number was justified if and only if I could prove that the sequence $\{x_n\}$ converged to a finite value.

Again, at first I was at a loss about how to prove this result. I knew, though, that if the convergence of the sequence $\{x_n\}$ could be established, then I could easily jus-

tify the other critical step in my derivation (i.e., $\sqrt{2 + \sqrt{2 + \dots + \sqrt{2 + \sqrt{2}}}} = x$ therefore $2 + x = x^2$) by applying some basic properties of limits to the sequence $\{x_n\}$:

⁴ A first and more thorough report of this error case study can be found in Borasi (1985b, 1986b).

For every value of n , we can state that $x_n^2 = 2 + x_{n-1}$. For every convergent sequence it is true that:

$$\lim_{n \rightarrow \infty} x_n^2 = (\lim_{n \rightarrow \infty} x_n)^2, \quad \lim_{n \rightarrow \infty} (x_n + a) = \lim_{n \rightarrow \infty} x_n + a, \text{ and} \\ \lim_{n \rightarrow \infty} x_{n-1} = \lim_{n \rightarrow \infty} x_n.$$

Therefore, if $\{x_n\}$ is convergent, we can write:

$$x^2 = (\lim_{n \rightarrow \infty} x_n)^2 = \lim_{n \rightarrow \infty} (x_n^2) = \lim_{n \rightarrow \infty} (2 + x_{n-1}) = 2 + \lim_{n \rightarrow \infty} (x_{n-1}) \\ 2 + \lim_{n \rightarrow \infty} x_n = 2 + x.$$

Skimming through my notes from courses in mathematical analysis, I finally found a first rather cumbersome method to establish the convergence of $\{x_n\}$.⁵ Although at the time I was perfectly satisfied with such a proof, and left the problem aside as solved, later I hit on another method for proving the convergence of $\{x_n\}$ that was both conceptually simpler than the previous one and presented the attractive advantage (to me) of being closer to my original intuition. This alternative proof for the convergence of the sequence $\{x_n\}$ was based on the realization that, because the sequence in question is ever increasing, in order to prove that it converges it suffices to find an upper bound for it. Because my intuitive derivation of the value of $R[2]$ suggested 2 as a possible solution, I thought of trying to prove that 2 would provide such an upper bound to the sequence. Indeed, I was able to do so by induction, as shown here:

- In the sequence $\{x_n\}$, for every value of n it is true that $x_n < 2$, because
- $x_1 = \sqrt{2} < 2$
 - If we assume that $x_{i-1} < 2$ (inductive hypothesis), then we can show that $x_i < 2$, because: $x_i^2 = 2 + x_{i-1} < 2 + 2 = 4$.

Now, not only was I convinced of the soundness of my original argument, but I was also ready to write a rigorous mathematical proof based on it. At this point, however, it also struck me that the specific value of 2 did not seem to have a special role in my proof, and thus I became intrigued by the possibility of using my "method" to evaluate the more general expression $R[a] = \sqrt{a + \sqrt{a + \dots + \sqrt{a + \sqrt{a}}}}$.

Exploring the More General Expression R[a]. At first, I was confident that the (now rigorous) procedure I had followed to derive and prove the result $R[2] = 2$ could be easily adapted to evaluate the expression $R[a]$ for any positive value of a . This, however, turned to be only partially the case.

⁵ See Bornsai (1985b) for a report of this method.

I already knew that, provided that I could prove that the sequence $\{y_n\}$ defined as

$$y_1 = \sqrt{a} \\ y_2 = \sqrt{a + \sqrt{a}} \\ y_3 = \sqrt{a + \sqrt{a + \sqrt{a}}} \\ \dots \\ y_n = \underbrace{\sqrt{a + \dots + \sqrt{a + \sqrt{a}}}}_{n \text{ terms}}$$

converged, I could then conclude that the value of $R[a]$ would be the nonnegative solution of the equation $y^2 = a + y$, that is:

$$\sqrt{a + \sqrt{a + \dots + \sqrt{a + \sqrt{a}}}} = \frac{1 + \sqrt{1 + 4a}}{2}.$$

However, as I tried (in analogy to what I had done in the case of $R[2]$) to prove that the increasing sequence $\{y_n\}$ was convergent because it was bounded by a , I met with some unexpected results. A proof by induction that, for every value of n , $y_n < a$, "works" only if $a \geq 2$, because:

$$y_{n+1}^2 = a + y_{n+1} < a + a = 2a \\ \text{and} \\ 2a \leq a^2 \text{ if and only if } a \geq 2.$$

The fact that this proof breaks down, however, does not mean that $\{y_n\}$ does not converge for values of a between 0 and 2. Rather, an alternative approach to prove the convergence of the sequence will need to be found for these cases. One such approach presented itself to me as I observed that if $0 < a < 2$, then for any value of n it will be $y_n < x_n < 2$.

In what follows, I report the organized and rigorous proof for the result $R[a] = (1 + \sqrt{1 + 4a})/2$, which I finally produced following this reasoning:

- Suppose that the sequence $\{y_n\}$, defined as:

$$y_1 = \sqrt{a}$$

$$y_2 = \sqrt{a + \sqrt{a}}$$

$$y_3 = \sqrt{a + \sqrt{a + \sqrt{a}}}$$

...

$$y_n = \underbrace{\sqrt{a + \dots + \sqrt{a + \sqrt{a}}}}_{n \text{ terms}}$$

(where $a > 0$) converges to a finite value y (this will be proved in part 2 of the proof). Then, using basic properties of limits and the relation $y_n^2 = a + y_{n-1}$, we can write:

$$y^2 = \lim_{n \rightarrow \infty} y_n^2 = \lim_{n \rightarrow \infty} a + y_{n-1} = a + y.$$

Therefore, $y = \lim_{n \rightarrow \infty} y_n$ must satisfy the condition:

$$y^2 - y - a = 0.$$

Because the solutions of this quadratic equation are:

$$y' = \frac{1 - \sqrt{1 + 4a}}{2} \quad \text{and} \quad y'' = \frac{1 + \sqrt{1 + 4a}}{2}$$

and $y_n > 0$ for any n , we must exclude the solution $y = y'$. Therefore:

$$y = y'' = \frac{1 + \sqrt{1 + 4a}}{2}.$$

2. We now have to prove that, for any positive a , $\{y_n\}$ converges to a finite value. We treat separately the two cases: (a) $a \geq 2$; and (b) $0 < a < 2$.

(a) $a \geq 2$:

$\{y_n\}$ is an increasing sequence, because:

$$y_n = \underbrace{\sqrt{a + \dots + \sqrt{a + \sqrt{a}}}}_{n \text{ terms}} > \underbrace{\sqrt{a + \dots + \sqrt{a + \sqrt{0}}}}_{n \text{ terms}} = y_{n-1}.$$

Therefore $\{y_n\}$ admits a limit. Such limit, however, can be either finite or infinite. To prove that the limit is finite, it suffices to prove that $\{y_n\}$ admits an upper bound, that is, to find a constant c such that, for any n , $y_n < c$.

We can prove that $c = a$ is a possible choice, because we can prove by induction that $y_n < a$, as follows:

1. $y_1 = \sqrt{a} < a$ because $a > 1$.

2. Suppose $y_{n-1} < a$ (inductive hypothesis); then, because $a \geq 2$:

$$y_n^2 = a + y_{n-1} < 2a \leq a^2.$$

(b) $0 < a < 2$:

In this case, we can prove the convergence of $\{y_n\}$ using the comparison test

with respect to the sequence $\{x_n\}$ (where $x_n = \sqrt{2 + \sqrt{2 + \dots + \sqrt{2 + \sqrt{2}}}}$):

1. Because $a < 2$, for any n , then $y_n < x_n$.

2. $\{x_n\}$ converges, as proved earlier in Case (a); therefore, also $\{y_n\}$ must converge.

QED.

At the same time, now that I had found a way to evaluate the expression $R[a]$, I became curious to know more about this strange mathematical object. I started my exploration by systematically evaluating $R[a]$ for some integer values for a , thus obtaining the following results:

$$R[1] = (1 + \sqrt{5})/2 \quad (\text{the golden ratio})$$

$$R[3] = (1 + \sqrt{13})/2$$

$$R[4] = (1 + \sqrt{17})/2$$

$$R[5] = (1 + \sqrt{21})/2$$

$$R[6] = (1 + \sqrt{25})/2 = 3 \quad (\text{again an integer!})$$

This last result surprised me: It seemed such a strange coincidence to be able to express a natural number by means of an infinite nested radical, yet it happened twice just in the first few cases! This result made me wonder for what other values of a in $\mathbb{N}$, $R[a]$ would turn out to be a natural number.

Without much hope of finding a complete solution to this problem, I started my search for other "lucky cases" by looking for necessary conditions for $R[a]$ to be a natural number. This, in turn, led me to solve an interesting integral equation, following the reasoning summarized here:

In order for $R[a] = (1 + \sqrt{1 + 4a})/2$ to be a natural number, $\sqrt{1 + 4a}$ must be an odd number and, furthermore, $(1 + 4a)$ must be a perfect square. In other words, there must be another natural number b such that:

$$1 + 4a = b^2.$$

Trying to rewrite this integral equation in a more "inspiring" form I got the equivalent expression:

$$4a = (b - 1)(b + 1).$$

It is easy to see that this equation is satisfied for integer values of the variables whenever b is an odd number, that is, $b = 2n - 1$ with $n = 1, 2, 3, \dots$. In other words, for any natural number $n \geq 1$ we have:

$$a = \frac{(b-1)(b+1)}{4} = \frac{(2n-2)2n}{4} = n(n-1)$$

and

$$R[a] = R[n(n-1)] = \frac{1 + \sqrt{1 + 4a}}{2} = \frac{1 + (2n-1)}{2} = n.$$

Notice how, by switching my attention from the search of suitable a s to that of suitable b s, I was able to find a rather unexpected result: Any natural number n can be expressed as an infinite radical $R[a]$, with a being a natural number (precisely, $a = n(n-1)$).

I soon realized, however, that this statement needed to be somewhat refined, because for $n = 1$ the equation $R[n(n-1)] = n$ would suggest that

$$\sqrt{0 + \sqrt{0 + \dots + \sqrt{0 + \sqrt{0}}}} = 1 \text{—a result that is obviously incorrect. This unexpected breakdown of my result called for an explanation. As I looked back at the original equation } (y^2 - y + a = 0), \text{ I realized that for } a = 0 \text{ the value of the nested radical } \sqrt{0 + \sqrt{0 + \dots + \sqrt{0 + \sqrt{0}}}} = 0 \text{ coincides with the "smaller" solution to the equation (i.e., } y' = \frac{1 - \sqrt{1 + 4a}}{2} \text{). Further "explanation" for this result can also be found by looking at what happens to the square root function in the neighborhood of 0. Whereas } 0 = 0, \text{ any value between 0 and 1 is increased by taking its square root, the more so the closer it is to zero (in fact, the limit of the derivative of } y = \sqrt{x}, \text{ for } x \rightarrow 0, \text{ is infinite). This effect will be emphasized by taking square roots repeatedly, as shown by the well-known, yet surprising result:}$$

$$\lim_{n \rightarrow \infty} \sqrt[n]{\frac{1}{n}} = 1.$$

As a result of these observations, my previous result then needed to be slightly modified as follows: "Every natural number $n > 1$ can be expressed by an infinite nested radical $R[a]$, where $a = n(n-1)$ "—which is still an unexpected and impressive result! Furthermore, these interesting results invited generalizations of my initial procedure so as to evaluate infinite expressions⁶ such as:

$$E[\sqrt{2}] = \sqrt{2} \sqrt{2} \sqrt{5} \sqrt{3} \dots$$

or:

$$1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \dots}}}$$

Reflections on the Nature of Mathematical Proofs. The mathematical explorations described in the previous subsections involved a lot of proofs, but in a way that was quite different from my previous experiences with this important element of mathematical thinking. As a mathematics student, I had been expected mostly to be able to understand and reproduce proofs proposed by other mathematicians for some of the results studied. Sometimes, as a sort of exercise for homework or in a test, I had been asked to prove a given result in such a way that it would be considered acceptable by the instructor. In both cases, the proof felt somewhat unnecessary, as icing on the cake, because the result was already available and there was no doubt in my mind about its validity.

At first, my initial attempt to prove the result $R[2] = 2$ in an alternative way may seem no different, because I was already familiar with a rigorous proof of this result (the geometric one proposed by Viète). Yet, in this case I felt a real investment in producing a convincing alternative proof—perhaps because I felt that my intuitive procedure was "simpler" than Viète's and, furthermore, I was genuinely intrigued by the fact that it led to a correct result despite the fact of being obviously unacceptable in its initial form.

Another novel element of this experience was the fact that my attempts to make the proof more rigorous were motivated not by the need to satisfy some external criteria set by authority (as it is often the case when one proves results in the context of a mathematical course) but rather by the desire to convince

⁶ See Borasi (1985b) for a discussion of $E[\sqrt{2}]$.

myself of the validity of my original intuition. This, in turn, made it easier for me to decide when I had reached a "satisfactory" proof—something that I often found more problematic when asked to produce proofs in the context of a graduate mathematics course.

Looking at it a posteriori, I found it interesting that even my final rigorous proof for the evaluation of the general expression $R[a]$, reported in this case study, contains some "leaps" and "holes"—although there is no doubt in my mind that such a proof is a perfectly acceptable one, whereas my first intuitive one is not. Thinking of the proofs reported in most mathematics textbooks (with the exception perhaps of logic texts), I realized that indeed those proofs, too, left many steps and their justification implicit. This, in turn, made me question what makes a mathematical proof acceptable or not. Contrary to my initial expectations, I reached the conclusion that this question cannot be answered in absolute terms—in other words, there are no foolproof procedures or sets of rules that, when carefully followed, guarantee an unimpeachable proof. Rather, in order to be "understandable" as well as mathematically acceptable, any proof of a certain complexity must report enough steps and justification to be fully "convincing," at the same time leaving implicit those steps and justifications that can be easily filled in by the reader on the basis of his or her own prior knowledge of mathematical facts and logic rules. Interestingly, such a criterion would depend essentially on the reader's background, so that a proof that may appear perfectly acceptable and complete to a professional mathematician may not be sufficiently convincing for a more naïve student.

A comparison between the formal proof for the evaluation of $R[a]$ I finally produced and the reasoning that led to my construction of it (as described prior to the formal proof in my report) was also thought provoking. Although using the same ideas, it is remarkable how these two justifications look different, both in the level of details and in the order in which steps are presented. The formal proofs presented in most textbooks or professional articles are indeed elaborate reconstructions (rather than reports) of the thinking process that led to them and, thus, may look quite different from the way mathematicians actually "think" and "prove" results. It should be no surprise, therefore, that many mathematics students are puzzled by the presentation of such "polished" final products and may not understand how they could ever be able to produce something similar on their own.

Another element I would like to comment on was the fact that, contrary to what happens when students engage in proofs in the context of a course assignment, my activity did not end when I found a "satisfactory" proof for $R[2] = 2$ based on my initial intuitive idea. Rather, as soon as I was convinced of the validity of my approach, I became interested in generalizing it so as to be able to evaluate other infinite expressions for which I did not know the result a priori. In this case, my development of a proof went hand in hand with the discovery of the result—although, once again, I first "guessed" the result by ap-

plying the procedure unrigorously and only then went back to "fill in the gaps." This experience made me better appreciate the generative role of proofs in the development of mathematical knowledge, as suggested by Lakatos (1976).

An analysis of the process I had followed, somewhat spontaneously, to rigorize my original proof for $R[2] = 2$ and generalize it for evaluating other cases, also revealed some heuristics that could be valuable when developing other proofs as well. First of all, it is interesting to note that, although unacceptable in its original form, my first intuitive proof was a necessary first step in developing a more rigorous proof. Not only did it provide an initial framework for the proof that I could then refine by identifying hidden assumptions that needed to be verified and/or justified, it also provided information that became essential in rigorizing the proof. For example, by "guessing" the result to be 2, I was able to choose 2 as a suitable candidate as an upper bound for the sequence that I was trying to prove was convergent. The fact that my proof for the result $R[2] = 2$ suggested a procedure applicable to evaluating the more general expression $R[a]$ also made me realize the value of considering special cases first, even when the task assigned may involve more general expressions. Working with specific values may in fact be easier and at the same time reveal how some of the results and/or procedures used are independent from the specific value considered.

Finally, I would like to remark that having two alternative proofs for the same result (i.e., Viète's and my derivation for $R[2] = 2$) naturally invited their comparison and, more generally, the consideration of the implications of having alternative proofs for the same result. Although both proofs, in their final form, can be considered equally rigorous and, thus, essentially equivalent from a logical viewpoint, their potential contribution to developing further mathematical results are quite different. My proof, as shown by the experiences reported here, has the advantage of being easily generalizable for the evaluation of other infinite expressions. This is not true, instead, in the case of Viète's proof, although his proof has the advantage of suggesting some geometrical implications of this result—such as the fact that the limit of regular polygons with increasingly large number of sides is a circle—that are not at all evident from my proof. As a result, one can conclude that trying to develop new proofs for a result that has already been justified otherwise is not a futile exercise, as it can lead to new insights about the problem in question and suggest ways to expand on the result obtained along different directions.

Commentary on the Experience and its Pedagogical Value. The activity of debugging my initial incorrect proof for the result $R[2] = 2$ indeed provided the stimulus and context for considerable mathematical problem posing and solving. My attempts to rigorize my original proof also invited me to revisit several concepts and procedures learned in my calculus courses and resulted in a much deeper understanding of the meaning and potential applications of the concepts of limit and convergence of sequences. The critical analysis and pro-

gressive refinements of a first intuitive proof motivated some reflection about mathematical proofs that resulted in some new insights about this important element of mathematical thinking, as summarized in the previous subsection.

It is worth pointing out that in this case the initial error—that is, my first unacceptable proof of $R[2] = 2$ —not only generated the curiosity that invited further inquiry, but was also instrumental for the inquiry itself. As discussed earlier, in fact, my incorrect proof provided some partial results and suggested valuable directions for developing a rigorous proof of the original result as well as its generalization—playing a role somewhat similar to that of a first draft in a philosophical essay.

This experience made me aware that errors—in the form of unjustified guesses, partial results, and refutable conjectures—should be considered an integral part of mathematical activity and one that professional mathematicians and more naïve students of the discipline should learn to capitalize on. It also enabled me to better appreciate what must have happened when mathematicians first developed calculus—as discussed earlier in the “Calculus” historical case study [B/4]—and the significance of their choice to trust their initial intuitions even when they could not immediately find a rigorous justification for them.

I also believe that a similar experience could be very valuable for calculus students and mathematics teachers (for many of whom calculus still represents a challenge) to make them better appreciate both the complexity of the rigorization of analysis and the power of the intuitions that made the whole field possible in the first place. In addition, engaging in activities such as the ones described in this error case study could also motivate valuable reflections and inquiry on the notion of mathematical proof—a fundamental element of mathematics that very few students, and even teachers, usually come to appreciate.

FIRST THOUGHTS ABOUT USING ERRORS AS SPRINGBOARDS FOR INQUIRY AS AN INSTRUCTIONAL STRATEGY

The error case studies reported in the previous section have suggested both the possibility and the educational value of engaging mathematics students in inquiries developed around the study of some specific mathematical error. Although they may already have suggested specific ideas for organizing some error activities in the context of school mathematics, I would like now to explicitly examine these examples with the goal of beginning to articulate some elements worth taking into consideration when thinking of using errors as springboards for inquiry as an instructional strategy. More specifically, based on an analysis of the variety of errors used in the error case studies discussed thus far and of the kinds of inquiry they generated, in this section I try to identify and discuss the possible sources of errors that teachers could take into consid-

eration when planning error activities for their students, some types of mathematical errors and the kinds of questions worth investigating that they may suggest and, more generally, the levels of mathematical discourse at which the inquiry invited by an error can develop.

Potential Sources of the Errors to Be Studied

Although student errors have been the only object of study for researchers and teachers in the error analysis and misconceptions literatures, it is worth noting that the error case studies discussed thus far were developed around errors coming from a wider variety of sources. Because teachers interested in capitalizing on errors in their classes may question where they could find appropriate errors around which to plan worthwhile error activities for their students, it seems important to identify these sources more explicitly.

Building and expanding on the examples discussed in the book thus far, I would like to point out that the error(s) used as the starting point and/or focus of an error activity could have been made by:

- The person who is now engaging in the error activity, as it happened in “My unrigorous proof” case study [H/5], where I became interested in analyzing and rigorizing an unacceptable proof that I had previously created.
- “Mathematically naïve” students. This was the case for about half of the definitions of circle I examined in “My definitions of circle” case study [G/5]; although I do not know who genuinely made the error of simplifying $\frac{1}{2}$ by crossing out the sixes, discussed in “My $\frac{1}{2} = \frac{1}{2}$ ” case study [F/5], I imagine it must have been a student not very familiar with fractions.
- Mathematics teachers, as in the case of some of the incorrect definitions of circle examined in “My definitions of circle” case study [G/5], because several in-service teachers attended one of the courses in which these definitions were collected.
- Experts in the discipline, as it happened in the error case studies reported in Chapter 4, where the errors discussed were mostly made by great mathematicians.
- No one in particular, because the error considered is inherent to mathematics itself. I would like to consider in this category the contradictions encountered when trying to compare the number of elements in infinite sets (see “Infinity” historical case study [D/4]) and the mathematicians’ failure to prove the parallel postulate (see “Non-Euclidean geometry” historical case study [C/4]), because these “errors” (or perceived errors) were the inevitable consequence of some limitations in the mathematical

systems one was operating, rather than a "mistake" made by a specific person.

This list does more than simply widen the pool of potential choices for the starting point of error activities. Not limiting one's attention only to student errors may in fact have some important psychological implications as well. Especially when introducing the strategy of capitalizing on errors in an instructional context, one could expect that working explicitly with errors made by persons perceived to be experts in mathematics, or even more with errors inherent to mathematics itself, may help some mathematics students (and even teachers) look at errors in a more positive light. Conversely, occasionally choosing errors made by more mathematically naive persons as the focus of an error activity may turn out to be less intimidating for the students (or teachers), because dealing with more familiar mathematical areas may make them feel more confident in generating questions and pursuing inquiry (as it happened for me, e.g., in the studies reported in the "Ratios" [A/1] and "My definitions of circle" [G/5] case studies).

Some Types of Errors and the Questions for Exploration They Are Likely to Raise

The errors discussed in error case studies A through H presented a considerable variety not only with respect to their origin, but in other ways as well. The starting point of "My $\frac{1}{4} = \frac{1}{2}$ " [F/5] and "My unrigorous proof" [H/5] case studies, for example, was in each case a somewhat incorrect procedure that had yielded a correct result. Other error case studies, instead, dealt with plainly incorrect results, such as some of the absurd values proposed for divergent series discussed in the "Calculus" historical case study [B/4] or proposing the result $\frac{7}{10}$ as the sum of $\frac{1}{2}$ and $\frac{1}{5}$ in the "Ratios" case study [A/1]. A list of incorrect definitions was instead the starting point of "My definitions of circle" case study [G/5]. Real or apparent contradictions encountered when trying to compare the number of elements in certain infinite sets were the focus of the "Infinity" historical case study [D/4]. The successive refinements of Euler's theorem on the characteristic of polyhedra, discussed in the "Euler theorem" historical case study [E/4], resulted instead from the analysis of conjectures refuted by specific counterexamples. The errors made by mathematicians in the context of failing to prove Euclid's parallel postulate (see "Non-Euclidean geometry" historical case study [C/4]) are even more difficult to classify, because in most cases they had to do with inappropriate expectations and/or assumptions that were ultimately revealed by what was at the time perceived as an error (but may not be considered as such today). Although this is certainly far from being a comprehensive list of all the types of mathematical errors that could be capitalized on

as springboards for inquiry, in what follows I try to examine each of the types of error thus identified with the goal of highlighting questions worth investigating they are likely to raise.⁷

Wrong Procedures Yielding a Correct Result. As mentioned in the context of the report of my own inquiry in "My $\frac{1}{4} = \frac{1}{2}$ " [F/5] and "My unrigorous proof" [H/5] case studies, errors of this kind are especially likely to stimulate curiosity. It may seem especially puzzling when a "shaky" or "plain wrong" procedure produces results that prove to be correct (because they can be obtained by means of a more reliable method or are confirmed by some other source) and, thus, we may want to find out how and why this could have happened. More specifically, this situation may naturally invite the following questions:

- Why did we get a right result in this specific case?
- Is the procedure just imprecise? If so, how could it be made more rigorous?
- In what other cases would the procedure yield a correct result? In what cases would it fail? What assumptions are necessary in order to ensure that the procedure would yield a correct result?

Pursuing all of these questions may contribute to refining the original procedure and/or better defining its domain of application (although this may not always be the case, as shown in the case of the "crazy" simplification explored in "My $\frac{1}{4} = \frac{1}{2}$ " case study [F/5]). At the same time, engaging in this kind of activity can motivate further inquiry that may go beyond the initial concern to explain the phenomenon, by suggesting questions such as:

- What are alternative procedures to obtain the same result and how would each compare with our (eventually refined) one?
- How could this (eventually refined) procedure be further generalized and/or adapted?
- How can we decide whether a given procedure is sufficiently rigorous? In mathematics, who would make such a decision and on what basis? Could such a decision be different depending on the mathematical and/or historical context?

Wrong Results. A similar set of questions could also be stimulated when a procedure yields a result that is recognized to be wrong. More specifically, if our goal is essentially that of trying to modify our product in order to meet the

⁷ Some of the questions for inquiry discussed in this section were initially identified and reported in Bortol (1987a).

original expectations (i.e., correct the error), the following questions could be generated and pursued:

- In what sense is the result wrong?
- Where did the procedure fail? Could it be fixed somehow, and thus lead to different results?
- What were our assumptions, and are they justified? In what cases?
- Is our result right at least in some case? If so, how can it be generalized?

However, errors of this kind can also stimulate interesting explorations when we conceive of the possibility of challenging the result we were originally expecting—as I did for example in the “Ratios” case study [A/1], when questioning whether indeed it would make sense in some case to add fractions by adding numerators and denominators separately, or as it happened in the history of mathematics when mathematicians reacted to their failure to prove the parallel postulate by questioning whether there could indeed be legitimate geometric systems based on an alternative postulate (see the “Non-Euclidean geometry” historical case study [C/4]). When approached in this way, what has been perceived to be a wrong result thus far may suggest questions such as:

- In what circumstances could such a result be considered “right?”
- Are there mathematical systems allowing for such a result to be correct?
- What would be the consequences of accepting such a result?
- Could the same result be both “right” and “wrong?” How could such a thing ever happen in mathematics? What does it tell us about the nature of mathematics?

Incorrect Definitions. Mathematical concepts are not easy to grasp and use correctly. One way to become more aware of our image of a mathematical concept is to analyze the definition(s) we think are appropriate to describe it. More often than we imagine, such definition(s) turns out to be different from the standard one agreed on by the community of mathematicians. A tentative or even incorrect definition can be the starting point for an exploration of the properties of the concept in question and the identification of its differences from other concepts, because it can stimulate the following questions:

- What properties can be derived from this definition? Which ones fit our image of the concept? Which ones do not?
- What other mathematical objects could be described by this definition? What is their relationship with the concept in question?
- What instances of the concept are not described by this definition?
- What is the meaning of all the terms used?

- Are all the properties stated essential? Could any be eliminated?
- How could we improve on this definition?

Even more than wrong results, incorrect definitions can also be the stimulus of real open-ended explorations. In fact, once we challenge the standard characterization of a concept, we open up to the study of new, nonstandard areas of mathematics. Very creative mathematical activities can be motivated by questions such as:

- What would the concept itself be if we were to accept this definition? How would it compare with the standard one?
- What would be the consequences of accepting this definition in mathematics?
- How could this definition be further modified? What alternative notions could then be created?
- Why did mathematicians choose the standard definition?

In addition, as shown in “My definitions of circle” case study [G/5], an analysis of incorrect definitions could also provide a concrete starting point and lead into an inquiry into the nature of mathematical definitions, through questions such as:

- What characteristics would we like mathematical definitions not to have? Why?
- What should a mathematical definition accomplish? What do we use definitions for in mathematics?
- How can we evaluate and choose among alternative definitions for a given concept?

Contradictions. Contradictions (i.e., two conflicting results achieved by applying apparently sound methods) are usually perceived as among the most unacceptable things in mathematics, something that may challenge the very existence of the system in which they are discovered and, thus, requires to be resolved somehow. As a result, finding a contradiction in mathematics corresponds to a very strong call for reflection and action, which could be led by questions such as:

- Where did the contradiction originate from? Is it the consequence of applying two alternative procedures, definitions, or criteria that were considered equivalent up to that point? Is it the consequence of applying a familiar procedure, definition, or property in a new domain?
- What are the consequences of this contradiction? Does it indicate the need for restricting the domain of application of a definition, procedure, or

property? Does it challenge the validity of results obtained up to this point? Does it require a redefinition of some of the concepts in question in a new domain? Does it call for a redefinition of some of the concepts in question in the old domain as well?

Depending on the answer given to some of these questions, the analysis of a contradiction may invite more general questions such as:

- What are the uses and purposes of the procedures, concepts, or criteria in question?
- What are the characteristics of the new domain? How do they compare with the more familiar domain?

Conjectures Refuted by Counterexamples. Following the lead provided by Lakatos' discussion of his proofs and refutations approach (see the "Euler theorem" historical case study [E/4]), whenever an initial conjecture is refuted by a counterexample we can try to improve it by asking and pursuing the following questions:

- Does the counterexample reveal a "careless error" in the initial conjecture that needs to be "fixed"? If so, how would the conjecture need to be modified to account for the counterexample? (See the "Students' polygon theorem" case study [K/6], reported in a later chapter, for an example in this direction.)
- Does the counterexample refute only the proof or the conjecture itself? In the first case, what step of the proof does it refute and why? Could the proof be fixed so as to avoid it?
- If the counterexample refutes the conjecture itself, what step(s) in the proof would it refute as well? What lemmas could be incorporated in the conjecture so as to avoid this problem? Are there different ways of doing so, each resulting in a different, complementary conjecture?
- Is the counterexample really an acceptable example or a pathological one that we would like to eliminate? If so, how could some definitions be modified? What would be the consequences of modifying such definitions?

Engaging in this kind of activity, in turn, may go much beyond remediating the original procedure, as it may also open new areas for investigation (as shown e.g., in the "Euler theorem" historical case study [E/4]) and also raise some questions about the nature of mathematical activity and results:

- How can we be assured that we have reached a "correct" conjecture? What is needed to turn a conjecture into an accepted theorem?

- Can we ever be sure that we have reached a correct conjecture?
- How do mathematicians decide whether it is more appropriate to modify a definition or a conjecture in order to account for a given counterexample?

Level of Mathematical Discourse at Which the Inquiry Occurs

Despite the different questions and directions for inquiry that could be invited by the various types of errors previously discussed, it is also possible to recognize some important commonalities and differences cutting across these categories in the way errors were used in error case studies A through H. One of these elements—the levels of mathematical discourse at which the inquiry developed around the chosen error—seems especially important when considering the planning of error activities in mathematics instruction. In order to better illustrate how the level of abstraction or focus of the inquiry often changed even within the same error activity, I would like to briefly examine as a significant example the various activities that were developed in "My unrigorous proof" case study [H/5].

The inquiry that generated from the analysis of my first unacceptable proof for the result $R[2] = 2$ indeed occurred at a few different levels of abstraction and, consequently, contributed to different kinds of mathematical learning, all valuable and complementary to each other. First of all, when I initially tried to identify what was "wrong" in my original proof so as to make it more rigorous, or later on questioned whether any natural number could be expressed by means of a nested infinite radical, my use of errors was informed by the goal of completing a specific task I had set for myself. At the same time, these activities also led me to revisit some properties of infinite sequences and limits and resulted in some reflections about these notions that ultimately resulted in a more sophisticated understanding of some technical mathematical content, that is, some fundamental concepts of calculus. Finally, the attempt to characterize what made my original proof unacceptable, and to identify some of the heuristics that I had employed to refine and generalize it, led me to examine issues regarding the nature of mathematics as a discipline and, more specifically, to gain a deeper understanding of the mathematical notion of "proof."

Generalizing from this specific example, it seems important to appreciate that the mathematical inquiry that could be stimulated by an error could take on different forms, depending which of the levels of mathematical discourse identified here one is operating on:

- *Performing a specific mathematical task*—that is, solving a problem, performing a computation, attempting to prove a result, producing an acceptable definition for a given concept, and so on.

- *Understanding some technical mathematical content*—be it a concept, rule, or topic, such as limit, limit operations, or calculus, respectively.
- *Understanding the nature of mathematics*—this could involve understanding metamathematical notions such as definition, proof, or algorithm; becoming aware of helpful heuristics as well as of their domain of application and limitations; appreciating what characterizes mathematical thinking and mathematics as a discipline; and so on.

It is important to point out that these three levels of mathematical discourse are all complementary to each other and valuable for mathematics students, yet they may contribute to achieving different goals. As shown by the results of my own studies of errors reported in this chapter, when informed by the attempt to perform a specific task, a use of errors as springboards for inquiry is especially likely to provide students with valuable opportunities for problem posing and solving, as well as for experiencing and practicing other important processes such as mathematical reasoning and communication. For example, debugging one's errors may indeed prove to be a challenging problem-solving activity—as any computer programmer well knows—and one that would require the learner first of all to ask appropriate questions so as to identify the causes of the error and eliminate them. Once an error is challenged, and the concern for remedialing it set aside for more open-ended explorations, one can even get involved in more creative and rewarding activities (as shown in—"My $\frac{1}{4} = \frac{1}{4}$ " [F/5], "My definitions of circle" [G/5], and "My unrigorous proof" [H/5] case studies). It is interesting to compare the role played by errors in these cases with the "What-If-Not" problem-posing strategy suggested by Brown and Walter (1969, 1970, 1990, 1993). In order to help mathematics students generate questions for investigation around a mathematical theorem, object, or situation, these authors proposed to first identify some of the attributes involved and, then, to try to change some of them and explore the consequences. Although there are five distinct phases of the "What-If-Not" strategy, its essence is captured by an awareness of attributes and a consequent modification of those attributes. An error, too, may sometimes be the result of an involuntary change in attributes (or assumptions), and as such can provide a natural contrast for the standard result and thus spontaneously suggest questions for exploration.

Especially when used at the second level of mathematical discourse, errors can contribute to a better understanding of specific mathematical topics, as a result of the insights gained by engaging in the open-ended and creative activities discussed earlier—as demonstrated again by the insights I gained as a result of my use of errors in the previous "My $\frac{1}{4} = \frac{1}{4}$ " [F/5], "My definitions of circle" [G/5], and "My unrigorous proof" [H/5] case studies. The reflections motivated by the "conflict" created by an error—as was the case, for example, when encountering contradictions within the concept of infinite number (see "Infinity" historical case study [D/4])—can make us aware for the first time that

we had been missing some main points in a topic without even realizing it or, alternatively, that we had subconsciously been holding inappropriate beliefs and images about some specific mathematical concepts or areas. Finally, when used at the most abstract level of mathematical discourse, errors can provide both the stimulus and a concrete means to critically examine our conception of mathematics and of some fundamental metamathematical notions. This, in turn, is especially likely to challenge the dualistic and deterministic views of this discipline shared by most people and may lead to an appreciation of the more humanistic aspects of mathematics, thus contributing to decreasing math anxiety or avoidance and developing attitudes and behaviors toward the study of mathematics that could be more conducive to academic success.

Chapter 6

Capitalizing on Errors in Mathematics Instruction: A First Analysis

The error case studies discussed in the previous chapter have suggested that no great technical background or ability is necessary to take advantage of the potential of errors to stimulate worthwhile mathematical inquiries. In the next three chapters, I examine how the use of errors may play out in mathematics instruction, building on the results of a school-based research project designed to study the proposed strategy in action in a variety of instructional contexts (mainly at the secondary school level).¹

With the goal of providing an image of how secondary school students can capitalize on errors, in this chapter I first report four episodes that I believe well illustrate the variety of ways in which mathematics students can use errors constructively and benefit from them. These examples are then used as a concrete background and reference point for a more analytical discussion of some important variations within the proposed strategy as well as of the learning opportunities and outcomes that a use of errors as springboards for inquiry can offer to mathematics students.² The categories thus generated are used in the rest of the book as a conceptual tool to analyze and evaluate the proposed strategy in different contexts.

¹ This project, entitled "Using errors as springboards for inquiry in mathematics instruction" and supported by a grant from the National Science Foundation (award #MDR-8651582), involved the development and study of 11 instructional experiences where the teachers made a conscious effort to capitalize on errors as learning opportunities. More information about some of these experiences is provided in later chapters.

² The examples and analysis reported in this chapter are derived from the research study summarized in Borasi (1994); I refer to this publication for a detailed discussion of the qualitative methods employed to derive the results discussed here and in the next chapter.

ERROR CASE STUDIES REPORTING ON ERROR ACTIVITIES EXPERIENCED BY SECONDARY STUDENTS WITHIN A TEACHING EXPERIMENT

The error case studies reported in this section consist of four vignettes developed within the same instructional context—a teaching experiment on mathematical definitions that I designed and taught, making a conscious effort to capitalize on errors. As this teaching experiment as a whole is discussed in more depth in Chapter 7, I refer the reader to that chapter for more information on the context of the experiences reported here.³ For the moment, in order to understand the setting in which these vignettes took place, it will suffice to say that the teaching experiment involved two 16-year-old students, Katya and Mary, and it consisted of 10 lessons in which the students engaged in a series of activities about specific mathematical definitions designed to stimulate and support the students' own inquiry about the nature of mathematical definitions, with the ultimate goals of enhancing the students' understanding of this complex mathematical notion and of helping them reconceive their views of school mathematics more generally.

The episodes reported in this section have been selected mainly because, despite their brevity, they illustrate several important categories that are identified and analyzed later in the chapter. The first vignette ("Students' definitions of circle" case study [L/6]), which occurred at the very beginning of the teaching experiment, sees the students engaged in the analysis of a list of incorrect definitions of circle, along the lines of the study reported earlier in "My definitions of circle" case study [G/5], although developed here to a lesser extent due both to the students' less extensive mathematics background and the time constraints imposed by the instructional context. In the following error case study I report instead on an error activity developed a few sessions later, when one of the students asked me to review her unsuccessful attempts to complete the assigned homework—an event that unexpectedly led us not only to find out what had gone wrong in her procedure, but more importantly to pose and solve some new problems ("Students' homework" case study [J/6]). The next vignette focuses on various constructive uses of errors made by the students, toward the middle of the teaching experiment, in the context of solving a mathematical problem about polygons that was both novel and challenging for them ("Students' polygon theorem" case study [K/6]). Finally, the students' final struggle to come to grips with the impossibility of defining 0°, as well as the reflections about the

nature of mathematics that this experience motivated, are reported in the "High school students' 0°" case study [L/6]. It is my hope that these vignettes will provide a flavor of the kind of valuable mathematical activities that can be developed around errors and of students' reactions to those activities.

Error Case Study I: Students' Analysis of Incorrect Definitions of Circle ("Students' Definitions of Circle" Case Study [L/6])

After a brief introductory and diagnostic activity, the students' inquiry into the nature of mathematical definitions began with a concrete task aimed at helping them recognize and appreciate the standard requirements for mathematical definitions. As a start, the students were asked to analyze the following list of incorrect definitions of circle:

- A. All the possible series of points equidistant from a single point (A) (Mary)
- B. πr^2 circumference formula, = radius, an exact center, 360°. (Katya)
- C. Round—3.14—shape of an orange, coin, earth—Pi.
- D. Circle = something whose area is = to πr^2 .
- E. Circle: $(x^2 + y^2 = r^2)$. Round.
- F. A circle is a geometric figure that lies in a two-dimensional plane. It contains 360 and there is a point called the center that lies precisely in the middle. A line passing through the center is called the diameter. 1/2 of the diameter is the radius. I don't like circles too much any more because they look like big fat zeros but they can be fun because you can make cute little smiley faces with mohawks out of them.
- G. A closed, continuous, rounded line.
- H. I sometimes find myself going around in them . . .

I had compiled this list of definitions by putting together the two definitions of circle previously produced by Mary and Katya themselves (Definitions A and B, respectively) with other six definitions collected from other students in their school. The activity was designed with the hope that by trying to understand why a specific definition was not "good," the students would become explicitly aware of the role of criteria such as isolation of the concept, precision, and essentiality in mathematical definitions.

The two students' familiarity with the notion of circle indeed allowed them to quickly decide that all the definitions in this list were somehow incorrect. Trying to make explicit the rationale behind their evaluation, however, was not always immediate and forced the students to make explicit some of the requirements they might have used only subconsciously before when dealing with mathematical definitions.

³ The teaching experiment discussed here and in Chapter 7 has been reported and analyzed in depth in my book *Learning Mathematics Through Inquiry* (Borasi, 1992) as an example of mathematical instruction informed by an inquiry approach. I refer the interested reader to this publication for a report of components of the experience not reported in this book as well as for more information about the context of the experience and its evaluation.

Consider for example how the analysis of some of the incorrect definitions of circle in the list led the students to realize the value of essentially in a mathematical definition—a requirement that is not easily appreciated by mathematics students:⁴

- T: All the things you wrote down are correct, right? (referring to a revision of Kaya's definition of circle, which now read: " $2\pi r$ circumference formula, = radius, an exact center, 360 degrees")
- K: Yes. But I was not able to put down a round answer, I just put what came to my mind . . .
- T: That's also what this Definition F does. Why do you think we may not want to have a long list of properties?
- K: I am not saying that it would not be good, but . . .
- T: Oh, you would like to have put even more?
- K: I just did not remember . . .
- M: But for a definition . . . it should be stated as simply as possible . . .
- T: So, we want a definition to be able to identify only circles. And a long list of properties would probably do that even better. Then why you would not like it?
- M: Why? Because a definition is something you have to remember . . . you don't want to remember all the little things . . . the whole list . . .

The analysis of some definitions in the list also raised new questions that I had not originally planned. For example, in the case of Definition E, the students did not immediately see that the equation proposed ($x^2 + y^2 = r^2$) would describe only circles with center in the origin, and was thus too restrictive—which was the primary reason why I had included this definition in the list! Rather, the students' analysis of Definition E developed in other, even more interesting, directions as shown in the following dialogue:

- K: I don't know what x and y are . . .
- T: You are right, we have to say what x and y are, or it does not make any sense . . .
- K: Like, in mine, if I should do it over, I should say what r means . . .
- T: Let's say we were using graph paper . . .
- K: Oh, that makes sense!
- M: But this is not the full sense of what a circle is . . . because you do not always have graph paper . . .

⁴ All the dialogues reported in this book are verbatim transcripts from instructional sessions or interviews. In all the error case studies reported in this chapter, the following abbreviations will be used to identify each speaker: K = Kaya, M = Mary, T = Teacher. To further distinguish the students from the teacher's voice, italics has also been used to identify the latter.

- T: That's a good point. But we can say . . . if you have a circle, you can put on it graph paper . . .
- M: With some work . . . This is a good definition, though, because it will only give circles . . .
- T: But, how can you check if it does?
- (On my suggestion, a specific value (of 5) is given to the radius, and the students start to complete an x - y chart, in order to plot some points on the graph paper. To check whether the points we plotted really belong to a circle, I use a compass to draw a circle with center in the origin and radius 5—our points are compass on the circle! This demonstration seems to convince the students, but also raises a new question.)
- M: How would you figure out if something is a circle, if there is no measurement for the radius?
- T: Ah! This is a good point!
- M: What if they just say "circle," "draw a circle" and you are . . . what's its r ?

Thus, the analysis of Definition E unexpectedly generated valuable questions and reflections regarding analytic geometry, the representation of circles in that context, and what we should expect from mathematical definitions, which in turn resulted in new insights for the students about these important mathematical concepts. In addition, in the effort to pursue some of these questions, the students became engaged in the meaningful performance of a concrete mathematical task—the derivation of the equation of circle in Cartesian coordinates.

In the process of examining the incorrect definitions I had provided, the students also engaged in the creation of a "correct" definition of circle. One such definition was produced by the continuation of the previously described activity, and led to the characterization of circles in the context of analytic geometry by means of the more general equation $(x - h)^2 + (y - k)^2 = r^2$. Another acceptable definition of circle resulted from an improvement of Definition A ("All the possible series of points equidistant from a single point"). This activity was initiated by Mary herself (the author of this definition), in response to my comment that spheres were also described by this definition. By adding the condition that "all the points should be on a two-dimensional plane," Mary produced a version of the metric definition of circle reported in most geometry textbooks—thus showing her ability to put into practice her understanding of the criteria of isolation of the concept. Both definitions of circle were used by the students in later activities.

Notice how the various activities invited by the analysis of a list of incorrect definitions of circle engaged the students in valuable reasoning, mathematical thinking, and problem solving, led to new insights into the notion of circle and the principles of analytic geometry, and finally provided a concrete starting point and means to examine an abstract issue such as the characteristics of good mathematical definitions. Because I had selected the errors the students were

working on (i.e., the list of incorrect definitions of circle) with the hope of achieving these very results, throughout this activity I felt very much in control of the instructional agenda, even if the students engaged actively in their own analysis of these errors. However, even in this case there were some minor unexpected deviations from my original lesson plan—for example, when the students' lack of background in analytic geometry suggested the value of deriving the equation of a generic circle in that context, and when Mary's observations about needing to know the measure of the radius in order to draw a circle implicitly raised some interesting questions about how specific a definition could and should be in mathematics.

Error Case Study J: Debugging an Unsuccessful Homework Assignment ("Students' Homework" Case Study [J/6])

As the students proceeded in the teaching experiment, I assigned to them a few problems whose solution required the use of a rigorous definition of circle, with the goal of helping them appreciate the need for "good" definitions in mathematics. One of these problems—"Finding the circle passing through three given points" (which I had marked as (7,0), (5,4), and (6,-3) on graph paper)—had been first assigned for homework, and then solved in two alternative ways, with my help, in our third lessons. At the beginning of the following session, one of the students asked me to revisit this activity so that she could better understand what she had done wrong when trying to solve the problem at home using an analytic geometry approach.

This activity was initiated by Kaya's overwhelming concern with understanding and remedying her errors, as implicitly expressed in the way she presented her request:

K: It didn't seem right to me . . . was *any* of this right?

At first, the student had tried to use the "simpler" formula $x^2 + y^2 = r^2$, obtaining the following results:

$$\begin{array}{lll} x^2 + y^2 = r^2 & & \\ 5^2 + 4^2 = r^2; & 25 + 16 = r^2; & r^2 = 41 \\ 6^2 + 9^2 = r^2; & 36 + 9 = r^2; & r^2 = 45 \end{array}$$

The fact that each equation yielded different values for the radius must have been a warning to the student herself that something was not working. After these attempts, Kaya had in fact abandoned this initial approach and, instead, set up the following set of two equations:

However, her attempts to solve this system of equations, reproduced here, left her puzzled, as they apparently yielded no solution:

$$\begin{cases} (7-h)^2 + (0-k)^2 = r^2 \\ (5-h)^2 + (4-k)^2 = r^2 \end{cases}$$

$$\begin{cases} (7-h)^2 + k^2 = r^2 \\ (5-h)^2 + (4-k)^2 = r^2 \end{cases}$$

$$49 + h^2 - 14h + k^2 = 25 + h^2 - 10h + 16 + k^2 - 8k$$

$$24 - 4h = 16 - 8k$$

$$8 - 4h = -8k$$

$$\frac{8-4h}{-8} = k$$

$$\frac{-2+h}{2} = k$$

$$k = h - 2$$

To help Kaya realize the value, as well as the limitations, of her solution approach, I first suggested that we compare her work with the solution I had presented in the previous session and recorded in a handout (this approach involved the traditional analytic method of setting up and solving a system of three equations using the coordinates of all the three given points.) As a result of this explicit comparison, it was easy to recognize that Kaya's system of equations used the information provided by only two of the three given points, and thus would not be sufficient to solve the problem. We were also able to detect and correct a computational mistake Kaya had inadvertently made in one of her last steps (so that Kaya's final line now read " $2k = h - 2$ ").

As I was trying to point out that Kaya had been on the right track and had reached what could be considered a "partial result" that could be used as a step toward the correct solution, I also suddenly realized that Kaya's approach could provide new insights into the problem under study. I tried to communicate this new awareness to the students in the following conversation:

- T: If you use only [the information from] two points, what this tells you is that there are really infinitely many circles that pass through those two points. Do you understand what I am saying?
- M: Many circles are passing through those two points just because they can change all the time?
- T: Right, many values of h and k satisfy this equation . . .

M: k and h are the [coordinates of the] center, right?

T: Yes!

M: Wait, wait! No!

T: I might be wrong!

M: It can't be a bigger circle . . . because just as you drew it goes like that (she shows on the figure that this circle would not pass through the third point) . . . if you proved, the way she got this . . . did she use two of these points?

T: Yes, she used only two of them . . . these two . . .

M: If she pulled in the third point, then it would make it definite, because . . . there is no other circle than that one that would pass through all three.

T: So if we have three points, only one circle passes. If we have two—which is what Kalya started to do—then we find many circles that pass through those points. But only one of those will pass also through [the third point].

(pause)

T: Can you notice anything special about [the circles passing through the two given points]?

(a few values for the coordinates of the center of those circles are computed once again, and plotted on the graph)

M: It's going to be a straight line! Because I remember when we were doing the line equation [in her previous mathematics course] it always ended up like that (pointing to the equation $2k = h - 2$).

Notice how the discussion reported in this conversation went beyond Kalya's initial concern to diagnose and remediate her error. Her incorrect procedure to solve the original problem of "finding the circle passing through three given points" indeed proved useful to pose and solve yet another problem I had not even considered before—"finding the circle(s) passing through two given points." It also provided some unexpected information about the solution to the new problem—the fact that all the centers of the circles would be on a straight line. Paradoxically, all these results would have been missed had Kalya not made an error!

It is also important to remark that because the whole activity was initiated by the request of one of the students, and focused on her errors, it proved to be highly motivating for the students. Finding out that one of their errors had led the instructor to a new discovery also sent a powerful message to the students about the potential value of errors.

This proved to be a very rewarding, but also a little threatening, experience for me as the teacher. Given the unexpected nature of the error, the explorations that followed it were totally unplanned and challenged me to think on my feet. At the same time, this had the positive implication of putting me in the position of genuinely inquiring along with my students, and modeled for them the

need to take intellectual risks and not be afraid to voice tentative hypotheses that may not always turn out to be correct or even worthwhile.

Error Case Study K: Students Using Errors Constructively When Developing a Theorem About Polygons ("Students' Polygon Theorem" Case Study [K6])

Toward the middle of the teaching experiment, Kalya and Mary worked with the more unfamiliar definition of polygon in activities designed to enable them to appreciate some important limitations in the application of the criteria of isolation of the concept, precision, and essentiality to mathematical definitions. One such activity involved verifying and refining the following (incorrect) theorem about polygons: "In any polygon the sum of the interior angles is 180° times the number of sides." I suggested this activity to provide the students with a concrete means to refine their shaky understanding of what figures should or should not be considered as examples of polygons (taking inspiration from Lakatos' proofs and refutations process, as illustrated by Lakatos, 1976, and Lakatos' proofs and refutations process, as illustrated by Lakatos, 1976). Errors summarized earlier in the "Euler theorem" historical case study [E4]). Errors played a variety of important roles in this process.

First of all, because proving the tentative theorem presented a genuine problem for the students, a number of *faux pas* naturally alternated with good ideas as they engaged in this activity. Notice, for example, how they initially approached the task:

T: How do you think we can prove something like this?

M: I don't know. Take a polygon as an example. (I immediately draw one, an "almost regular" pentagon.)

M: We never really got to the definition of a polygon. We think this is a polygon.

T: Right, so for the moment we think it's a polygon. How [can we] figure out the sum of the interior angles?

M: Put a circle around it that meets all the points.

T: (I draw a very "skinny" polygon) What if the polygon is like this?

K: Make it into triangles.

M: Take a center point. But if it's really weird shaped you can't do it . . . Oh yes, you could do it.

T: It seems that whenever I have a polygon I can pick a point in the center, more or less. Does it matter which point I pick? Maybe we should ask Kalya why it is that you wanted to break it down into triangles like this?

Notice how, in this beginning stage of the process, various suggestions and ideas were raised, some of which could be considered errors because they led

nowhere (as Mary's idea of drawing a circle around the polygon in this case). In the dialogue just reported, I played the roles of moderator and leader, helping the students select which ideas to follow among the many suggested and also explicitly suggesting the consideration of some potential counterexamples. Later on, although I continued to play the role of moderator to a certain extent, the students themselves showed more initiative and control in evaluating alternative options and identifying and correcting potential errors. An example of this can be found in the conversation that developed after Katya had decided to pursue her idea of breaking up the original polygon in triangles to use the property that the sum of the angles in a triangle is 180° and had produced Figure 6.1.

- T: Now that we broke [the pentagon] into triangles, what can we say about the angles of these triangles?
 M: That is $3 \times 180^\circ$ altogether.
 T: Okay. Correct? Then is my theorem correct?
 K: (to Mary) Why do you say this is $3 \times 180^\circ$?
 M: (pointing to the 2 triangles in the "square" part of the pentagon) there is two triangles in there.
 K: But you could make this (and she draws another diagonal in the square, thus producing 5 triangles in all—see Figure 6.2).
 T: Okay, interesting. Then you would have triangles.
 M: Good point . . . (almost believing she has been proven wrong).

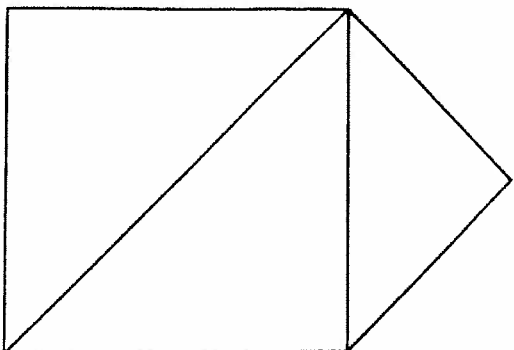


FIGURE 6.1. First pentagon drawn by Katya.

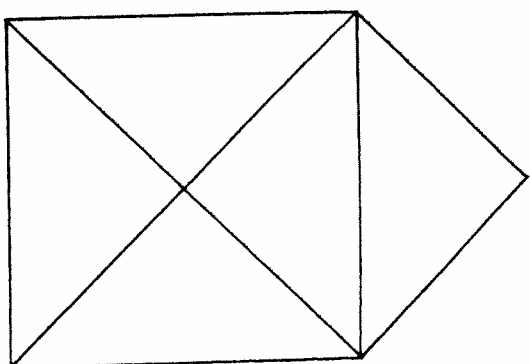


FIGURE 6.2. Second pentagon drawn by Katya.

- T: Well, the sum of all the angles of these triangles is $5 \times 180^\circ$. We are sure of that. But is it the sum of the angles of the original polygon? Or are we having something extra here?
 (Katya immediately points to the four central angles).
 T: So if you want to use this idea, you should take away these extra. How much are these extra angles?
 M: All together?
 K: . . . okay, these are all 90°
 M: Wow! 90° , these are $180^\circ \times 4$. . . (she has the right idea, but misspeaks, so I quickly correct her.)
 M: (doubting that such a conclusion can be drawn) But you haven't proven that's a square yet.
 T: No, that's correct. Can we say this is 360° even without knowing if this is a square?
 M: Yes, you can, because it's a circle! All the angles put together make one big angle all the way around! Wow! We are the "discovery channel" today!!
 T: So we have 360° , which is like $2 \times 180^\circ$. So this is like $5 \times 180^\circ - 2 \times 180^\circ$, which is like $3 \times 180^\circ$. So now the two [methods] have given us the same result. Does it fit with the theorem I stated? How many sides did the polygon have here?
 M: Five.
 T: So, instead here we have 3 (pointing to the $3 \times 180^\circ$) so do you trust more my theorem, or do you trust this?

M: (without hesitation) Trust ourselves.

T: So the theorem was wrong, so I suggest a correction. (I modify the original theorem to read: "In any polygon, the sum of the interior angles is 180° times the number of sides -2 ".)

Notice how the students showed considerable creativity and mathematical intuition in this activity, and used their own errors—such as not dividing the pentagon in the minimal number of triangles possible—constructively. What is most remarkable in this dialogue is that the *students themselves* showed a very critical and independent stance as mathematics learners. More than once, it was a student, not the instructor, who raised questions about a proposed result, calling for a justification or suggesting possible alternatives—see for example Katya's question to Mary "Why do you say this is $3 \times 180^\circ$?" when I had already implicitly accepted this result, a request followed by Mary's explanation and then Katya's alternative suggestion to divide the pentagon in 5 triangles; also, later on, see how Mary voiced her concern for justification through the implicit question "you haven't proven that's a square yet." In this activity, the students seemed alert to and ready to deal with their own errors, as if they perceived this to be an integral part of solving a mathematical problem of this kind—an attitude that is unfortunately quite uncommon among mathematics students in a traditional classroom.

Although in this case I had not intentionally planned the error, starting with a statement of the theorem that later on was found to be incorrect turned out to be quite valuable, because it required the students to decide between contrasting results, one derived through their own logical reasoning and the other proposed by authority. It is interesting to note that the students had no hesitation in choosing their own result in this case! Furthermore, having started with a statement that needed to be corrected may have contributed to making the students more cautious in their activity. This may also have influenced, later on, Mary's spontaneous suggestion to verify whether the corrected theorem would hold with other, stranger examples of polygons.

Indeed, after having verified that the "modified" theorem (i.e., "In any polygon the sum of the interior angles is $180^\circ(n-2)$ ") held for another figure I had proposed (a convex hexagon), Mary exclaimed "It's too neat" and set out on her own initiative to discover some exceptions. She first drew a 5-pointed star with lines crossing over (see Figure 6.3) but then commented "Did we dismiss polygons that look like that? I think we have." As I supported her comment by observing that indeed in this case it would be difficult even to determine which angles were the interior angles of the figure, Mary then creatively suggested a modification to her original star (see Figure 6.4), observing "Then we just count the outside lines!"

Again, on her own initiative Mary set out to verify the property stated in the theorem on her 10-sided star. She was both surprised and thrilled when she suc-

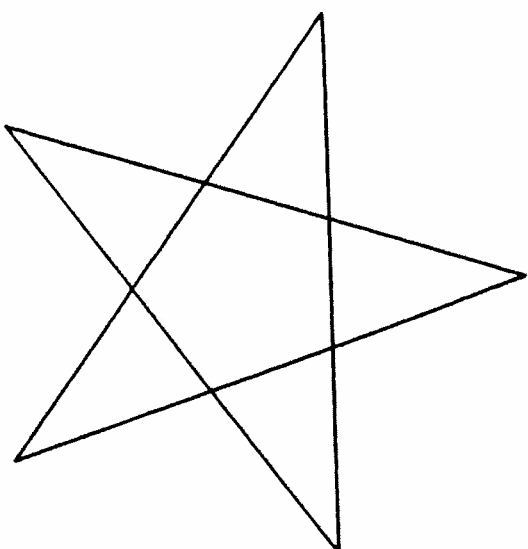


FIGURE 6.3. First star drawn by Mary.

ceeded in her task. And when, on my request, the two students went back to re-examine the tentative definition of polygon they had started with (i.e., "polygon: a closed geometric figure of straight lines"), Mary was quick to propose adding the condition "in which no lines cross over." Here Mary's activity of refining both a theorem about polygons and the definition of polygon itself

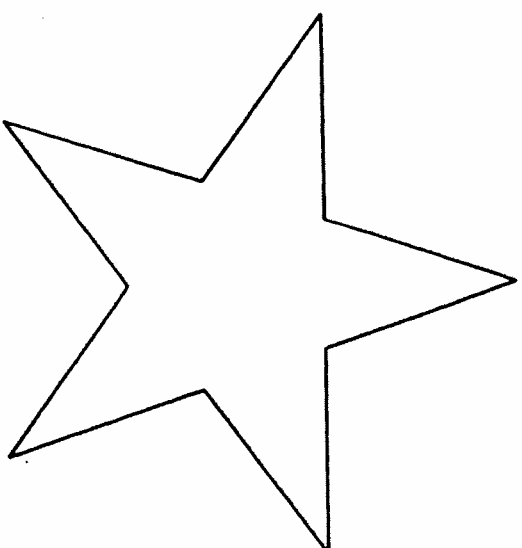


FIGURE 6.4. Second star drawn by Mary.

closely resembles some of the approaches to refine a tentative conjecture discussed earlier in the "Euler theorem" historical case study [E/4].

This episode also constituted an important turning point in the students' inquiry about the nature of mathematical definitions, because it enabled them to recognize and appreciate an element of *tentativeness* in mathematical definitions. This attitude is evident in their responses to the following questions in a later "review sheet":

When trying to find a definition of "polygon," we had some trouble to start with because we did not really know precisely what we wanted a "polygon" to be—except that it was going to be a concept generalizing triangles, squares, pentagons, etc. So we started with a tentative definition, and "refined it" so that polygons would have some interesting properties.

We ended up with the following definition: A polygon is a closed geometric figure, with straight edges that do not cross.

Are you satisfied with this definition? How could we ever know if it is correct? Explain your answer.

M: Not quite. Play w/ polygons. Test the theory. Use examples of figures that meet the requirements for the def. but don't meet properties. OR, just Trial and Error.

K: I am satisfied with this for the time being but I think possibly as I start to use them more and more, I may desire something more exact¹

What is the value, then, of a preliminary, tentative definition?

M: It's like a theory that hasn't been fully tested yet and therefore is open to modification

K: getting acquainted with a new concept—have something to base further defining on. you have to start with something

Error Case Study I: Students' Dealing With an Unresolvable

Contradiction: The Case of 0⁰ ("High School Students' 0⁰" case study [L/6])

Toward the end of the teaching experiment, after the students had successfully developed new definitions and rules to deal with negative and fractional exponents, I decided to challenge them with some of the limitations that are inevitably encountered when extending exponentiation. One of these activities involved the definition of 0⁰.

First of all, I set up two alternative patterns that would lead the students to consider the plausibility of either 1 or 0 as possible values for 0⁰, and then

feed them with the resulting contradiction (which in turn could make one consider either solution as an "error"):

T: I'll show you another thing that gave me a problem. What do you think is 0⁰?

K: Zero?

T: And how do you think ... why do you think that?

M: It's not ... it's undefined.

T: How would you justify that it is undefined?

M: Because ... well, I can see how she's going to say zero times zero equal zero, but I would say undefined because you can't raise something to the power of zero; that makes it totally imaginary.

K: (emphatically) You can!

T: You can, right? Do you remember [how we computed] 20⁰ (I briefly remind them of how we had previously created a pattern that justified the definition 20⁰ = 1, and invite the students to repeat the procedure to evaluate 30 and 40). As a result, the following pattern is produced: 50 = 1, 40 = 1, 30 = 1, 20 = 1, 10 = 1, 1.

T: So a pattern would give you that. All these numbers [with exponent 0] will give you 1. So it seems to be reasonable that [0⁰] gives you 1 too (I write 0⁰ = 1).

(The students seem convinced, but at this point I show the possibility of an alternative pattern)

T: What about this pattern? 0⁵ = 0, 0⁴ = 0, 0³ = 0, 0² = 0, 0¹ = 0.

K: Equal zero.

T: So in this case we might even argue that this [0⁰] should be zero. (writing 0⁰ = 0)

K: Makes more sense to me that that would be zero.

T: But this is really the kind of situation that I think makes mathematicians mad. I mean, you have two patterns. [Patterns] seemed to work so nicely before. We got all this nice extension of the definition [of exponentiation]. And this time one pattern gives you one result, and the other pattern gives you the other result. So, how do you think we can deal with situations like this? (long pause)

K: I don't know ... I don't know with this one ...

M: I ... that zero ... negative zero, or something like that. I don't like it.

T: (agreeing) That's probably what the mathematicians would have said too: I don't like it!

M: Pretend we never saw it ...

As shown in the preceding dialogue, the students were at first both puzzled and disturbed by this contradiction, as it presented them with a mathematical error that apparently could not be eliminated and, furthermore, they had never

¹ The lack of capitalization in the text is Kaye's.

before encountered a similar situation in mathematics nor had they recognized other limitations in the discipline. This became even more evident later on, as Mary remarked:

M: The thing about this confusion is, I sort of doubt, like, I don't know, if we can't figure that out, who's to say the rest of these are right? . . . And that's relatively simple, and that's just an exponent. I'm wondering about all this stuff that we are learning, ten years from now we'll find out, you're wrong!

Yet, despite this immediate reaction of frustration, the students did not give up their attempts to "make sense" of this situation. This brought Mary, in particular, to some very creative thinking that led to a re-examination of her view of mathematics. At first, she still tried to fix the error somehow, by finding a different pattern that would work better. Because this approach did not lead her very far, Mary then suggested the idea of trying to change system, so as to overcome the problem—a quite creative approach and one of which any mathematician could be proud! Although this idea was not fully pursued, it is interesting to see in the following dialogue how the very fact of conceiving this possibility had an impact on this student's conception of mathematics:

M: I was thinking about, you know, when we come to an equation like that [00], when it just cannot be figured out by me, or by the next person, and then it just reminds me that this was all invented by people. It's not something like we are born and there is a tree and it has been there forever. It's like we invented this, out of our minds. And we invented zero to ten, and the whole number system and all the other number systems, and, and, so . . .

T: And it could have been invented in a slightly different way.

M: Yeah, that too. And then I was thinking about, what if we had decided that it was one through ten, and there is no zero (laughter), then our problems would be solved!

T: Maybe some problems would be solved, but others would remain. And in fact some other would be so big that that's why they invented zero.

M: It would be neat though if some people just decided that that's the way their numbers system is going to be so the whole world uses it, umm, I don't know.

Mary's observations here could have provided a wonderful opening for a historical inquiry about when and why zero was invented and how mathematicians dealt without it before (see "Numbers without zero" case study [S/9], later in the book, for an example of possible inquiry in this direction). Unfortunately,

because at this point the time allotted to our teaching experiment was coming to its end, we were unable to follow this intriguing investigation—something I especially regretted because it would have enabled me to observe the students' reactions to an inquiry into the history of mathematics motivated by the consideration of an error.

Despite this limitation, this vignette suggests that some mathematical errors—especially those that cannot be totally eliminated—have the potential to stimulate students' reflections on the nature of mathematics and make them realize that mathematics, too, is the product of human activity and thus not as "perfect" as they might have expected.

IMPORTANT VARIATIONS WITHIN THE INSTRUCTIONAL STRATEGY OF CAPITALIZING ON ERRORS

The diversity among the error activities developed within the examples reported in the previous section is remarkable and suggests that capitalizing on errors should not be perceived or evaluated as a monolithic instructional strategy. Rather, in order to better understand and take full advantage of the potential of this strategy, it is important to identify some important variations within it and to discuss how these variations may affect both the uses made of errors and the potential outcomes of such activities.

The categories identified and discussed in this section emerged from the analysis of all 24 error activities developed within the teaching experiment on "Mathematical definitions" (see chapter 7 for a complete list of these error activities). However, for the reader's convenience, in the discussion that follows I refer for illustration only to the four examples reported in the previous error case studies I through L.

A Taxonomy of Complementary Uses of Errors as Springboards for Inquiry

A first important difference among the examples reported earlier in this chapter has to do with the specific uses made of errors within these instructional episodes. In Chapter 5, I already suggested that the mathematical inquiry stimulated by an error could take on different forms depending on the level of mathematical discourse at which one is operating. Even just within the activities described in the "Students' definitions of circle" case study [L/6] it is possible to identify situations where the analysis of incorrect definitions of circle led the students to:

⁶ See Borasi (1994) for a discussion of the research methodology employed for this analysis.

- *Perform a specific mathematical task*—as when they engaged in the creation of some acceptable definitions of circle by trying to “fix” the most promising items in the list (*task level*).
- *Learn some technical mathematical content*—as they analyzed and became more aware of various properties of circles and their relationship as a by-product of analyzing specific definitions in the list (*content level*).
- *Learn about the nature of mathematics*—as they engaged in an analysis of what constitutes a “good mathematical definition” based on the identification of the shortcomings of specific definitions in their list (*math level*).

At the same time another important factor, independent of the level of mathematical discourse just discussed, seems also to have affected the students' attitudes and goals in the study of specific errors. For example, when the students analyzed obviously incorrect definitions of circle in the “Students' definitions of circle” case study [I/6], or when one of the students asked the instructor to go over her work to see “what had gone wrong” in the first part of the “Students' homework” case study [J/6], the activity was essentially one of debugging, where a result was clearly recognized as incorrect and analyzed in the belief that it could provide valuable information to help one reach the original objective. The situation and use made of errors was quite different when the students were engaged in a genuine problem-solving or discovery activity, as it happened for example when the students tried to prove the instructor's incorrect statement in the “Students' polygon theorem” case study [K/6]. In this case, because the students did not know a priori what results to expect, it was difficult for them to recognize when something was an error or not. Nonetheless, it was essential for them to critically evaluate tentative results and steps in order to reach a solution to the set task. Finally, some of the activities reported in order the “Students' homework” [J/6] and the “High school students' 00” [L/6] case studies have shown that errors can also generate new questions that may not directly contribute to reaching the goal originally set for the activity and, yet, stimulate inquiry and learning in valuable directions. Consider, for example, how in the “Students' homework” case study [J/6] the unsatisfactory result reached by the student with respect to the task of “finding the circle passing through the *three* given points” provided valuable information for yet a different problem (“finding the circle passing through *two* given points”) and how in the “High school students' 00” case study [L/6] the contradictions encountered when trying to define O , instead of being resolved, led to unexpected insights about the limitations of mathematics as a discipline.

The differences highlighted in the preceding paragraph are somewhat independent of the mathematical content of the error activity and rather influenced by the nature of the learning context and the degree of “open-endedness” of the lesson—something that I could not have fully appreciated in the atypical situation in which I conducted the studies of errors reported in Chapter 5. These dif-

ferences suggest the identification of the following *stances of learning* as another element affecting the instructional use of specific errors:

- *Remediation stance*: Here both the question and the answer informing the student activity are predetermined and known by authority, and furthermore the student is aware that his or her result is not correct (although he or she may or may not know what the correct result is); the expectation is that by analyzing the error one could identify what went wrong and correct it.
- *Discovery stance*: This stance occurs when the student is learning something new or trying to solve a genuine problem, although both the question and the answer informing the activity are still predetermined and perceived as known by authority; because the student is not expected to already know the answer, steps in the wrong direction are seen as a natural occurrence (although they may not always be immediately recognized as errors) and there is the expectation that any result needs to be examined critically so as to determine whether it is correct or not.
- *Inquiry stance*: Here neither the answers nor the questions directing the student's mathematical activity are perceived as necessarily predetermined, and detours as well as redefinitions of the original task are encouraged; questions raised by errors may thus initiate exploration and reflection in totally new directions, and even invite students to challenge the status quo.

By combining these three categories with the three levels of mathematical discourse identified earlier, a 3 x 3 matrix of complementary ways to capitalize on errors in mathematics instruction can be generated, as illustrated in Table 6.1.

The taxonomy represented in Table 6.1 highlights some important differences in the way errors can be capitalized on in mathematics instruction. As such, it provides a valuable conceptual tool for planning and analyzing instructional uses of the proposed strategy as well as for revising its initial conception. First of all, the uses of errors identified in Table 6.1 suggest that my initial conception of the strategy (as reflected in some of my first error case studies, especially the “Ratios” [A/1], “Non-Euclidean geometry” [C/4], “Infinity” [D/4], and “My definitions of circle” [G/5] case studies) was somewhat limited, insofar as it mainly focused on the three uses of errors now identified by the boxes belonging to the inquiry stance (i.e., *inquiry/task*, *inquiry/content*, and *inquiry/math* uses). Although some of the most interesting instances of error activities described earlier in this chapter involved initiating new explorations and uncovering aspects of mathematics never before appreciated by the students (see, for example, the later segments of the “Students' homework” [J/6] and “High school students' 00” [L/6] case studies), valuable learning opportunities were provided in error activities informed by different stances of learning as well. In particular, the vignettes reported in this chapter show the need for re-

TABLE 6.1 A Taxonomy of Uses of Errors as Springboards for Inquiry

Stance of Learning	Level of Math Discourse		
	Performing a Specific Math Task	Understanding Some Technical Math Content	Understanding the Nature of Mathematics
Remediation	Analysis of recognized errors to understand what went wrong and correct it, so as to perform the set task successfully. (<i>Remediation/task</i>)	Analysis of recognized errors to clarify misunderstanding of technical mathematical content. (<i>Remediation/content</i>)	Analysis of recognized errors to clarify misunderstandings regarding the nature of mathematics or general mathematical issues. (<i>Remediation/math</i>)
Discovery	Errors and uncertain results are used constructively in the process of solving a novel problem or task, monitoring one's work to identify potential mistakes. (<i>Discovery/task</i>)	Errors and uncertain results are used constructively as one learns about a new concept, rule, topic, etc. (<i>Discovery/content</i>)	Errors and uncertain results are used constructively as one learns about the nature of mathematics or some general mathematical issues. (<i>Discovery/math</i>)
Inquiry	Errors and puzzling results motivate questions that may generate inquiry in new directions and new mathematical tasks to be performed. (<i>Inquiry/task</i>)	Errors and puzzling results motivate questions that may lead to new perspectives and insights on a concept, rule, topic, etc., not addressed in the original lesson plan. (<i>Inquiry/content</i>)	Errors and puzzling results motivate questions that may lead to unexpected perspectives and insights on the nature of mathematics or some general mathematics issues. (<i>Inquiry/math</i>)

organizing the potential benefits of engaging students themselves in the debugging of their (or other people's) errors—that is, capitalizing on errors within a remediation stance, a variation of the strategy that I had initially downplayed in my efforts to differentiate it from a diagnosis and remediation approach to errors (see Borasi, 1985c, 1986b, 1987a). It is also interesting to point out that, occasionally, an error actively initiated by a student with a concern for remediation might later develop into more open-ended inquiry (as shown in the "Students' homework" [J/6] and "My unrigorous proof" [H/5] case studies).

The categories employed in the construction of this taxonomy suggest that each of the uses of errors thus identified may have different implications in terms of the learning opportunities it may offer to students and the instructional goals it may

contribute to support. First of all, as already argued in chapter 5, it is obvious that the level of mathematical discourse at which the analysis of an error occurs will affect what the students may learn from the experience—whether it is a better conceptual understanding of technical concepts and rules (*remediation/content*, *discovery/content*, and *inquiry/content* uses), a better appreciation of mathematics as a discipline (*remediation/math*, *discovery/math*, and *inquiry/math* uses), or learning how to solve a specific problem or task and/or developing more generally some problem posing and solving skills (*remediation/task*, *discovery/task*, and *inquiry/task* uses), respectively. More indirectly, the learning stance assumed in the study of errors may also influence a student's approach to mathematical activity and, consequently, his or her performance in this subject. In particular, taking advantage of errors within an inquiry stance may have an important role in challenging a dualistic view of mathematics and, consequently, enabling students to develop attitudes toward the discipline that are more conducive to success (an issue I revisit in more depth later in this chapter).

These considerations suggest that maximum benefits from engaging in the study of errors can be achieved when all the different uses of errors as springboards for inquiry identified in Table 6.1 are taken into consideration and employed, as appropriate, in mathematics instruction. At the same time, mathematics teachers also need to realize that the current reality of school mathematics practice may not equally support all the variations identified in Table 6.1—an issue that I discuss in more depth in Chapter 8, after having reported on a few implementations of the proposed strategy in the context of regular mathematics classrooms.

It is also worth pointing out that what could be considered as the starting point of an error activity (i.e., the "error") may change depending on the stance of learning assumed. Implicit in a remediation stance is in fact the assumption that the result being considered is *not* correct (although one may not know why, or what the "correct result" would look like). In contrast, as one moves from the top-left to the bottom-right corner of Table 6.1, deciding whether something is correct or not a priori may no longer be obvious—at least for the students if not for the teacher. Yet, the benefit of engaging in error activities organized in these situations proved to be essentially the same regardless of whether the students concluded that the result in question was correct or not—as shown both in the "Students' homework" [J/6] and the "Students' polygon theorem" [K/6] case studies.

Levels of Student Involvement in the Study of Errors

One of the main characteristics of the proposed strategy of capitalizing on errors is the expectation that the *students themselves* engage in the mathematical activities that could be stimulated by specific errors. Yet, an analysis of the error

activities reported earlier in this chapter, conducted with respect to the extent to which the two students engaged in them, reveals at least the following possible levels of student involvement in the study of an error:

- The inquiry stimulated by the error is mostly conducted by the instructor and later shared with the students, as it happened in the "Students' homework" case study [J/6], when I suddenly realized that Katya's system of two equations could be used to shed light on a *different* problem than the one originally assigned and tried to communicate this realization to the students (*teacher modeling—1*).
- The students engage actively in an error activity organized and led by the instructor, as illustrated at various points in all the four vignettes reported previously and especially throughout the analysis of incorrect definitions reported in the "Students' definitions of circle" case study [J/6] (*teacher-led student inquiry—1*).
- The error activity is initiated and developed by the students themselves, with some (or no) participation on the part of the instructor, as illustrated in the "Students' polygon theorem" case study [K/6], when Mary decided to verify whether the revised theorem would hold for some "borderline cases of polygon" and initiated and pursued this task essentially on her own (*independent student inquiry—1*).

It is worth pointing out that the levels of student involvement thus identified, although mutually exclusive, may yet occur at different points in time within the same error activity—as illustrated for example by the "Students' polygon theorem" case study [K/6], where the students engaged at first in a constructive use of their errors under the instructor's guidance (teacher-led student inquiry—level 1) as they tried to prove the original incorrect statement, although this very experience motivated Mary later on to initiate new questions and lead in their exploration (independent student inquiry—level 1).

The experiences reported earlier in this chapter show that all these levels of student involvement are possible and valuable. Indeed, they suggest that even level 1 (i.e., teacher modeling), which I had initially considered as a borderline and not totally satisfactory implementation of the strategy because it would not engage the students directly in the inquiry stimulated by the error, should not be discounted. Mathematics students could in fact benefit from seeing all the valuable explorations and results that a study of errors could produce, and thus come to appreciate the potential benefits and expectations of capitalizing on errors in a nonthreatening way, especially when this happens before they are expected to (and perhaps able to) engage directly in similar activities. These considerations seem in line with the sequencing of *modeling*, *coaching*, and *exploration* (or independent performance) suggested in the cognitive apprenticeship literature (e.g., Collins, Brown, & Newman, 1989). At the same time, it is

important to keep in mind that experiencing error activities at level 1 is not a necessary prerequisite for engaging more actively in a constructive use of errors, as shown by the sequencing of the experiences reported earlier.

Sources of the Error Studied

When planning the error activities reported earlier in the chapter, I took into consideration several of the sources of errors identified earlier in Chapter 5. Implementing a use of errors as springboards for inquiry in an actual instructional context, however, led me to further revise the list of potential sources of errors reported there, because I came to realize the importance not so much of *who* made the error to be studied, but rather his or her *relative position* with respect to the person now examining this error. In light of this consideration, the categories generated in Chapter 5 can now be further expanded and related as follows:

- *Same student error*—that is, when the error is made by the same person now engaging in the error activity (as in the "Students' definitions of circle" case study [J/6], where Katya and Mary's definitions were included in the list discussed and in the "Students' polygon theorem" case study [K/6], when the students' own tentative steps and hypotheses were critically examined as they attempted to verify a given statement about polygons).
- *Other classroom error*—that is, when the error is made by another member of the group now engaging in the error activity (as illustrated for Mary in the "Students' homework" case study [J/6], where Katya's error was analyzed).
- *Teacher error*—that is, when the error is made by the classroom teacher (as illustrated in the "Students' polygon theorem" case study [K/6], where the initial incorrect statement about polygons examined had been proposed to the students by the instructor).
- *Outside peer error*—that is, when the error is made by a peer not belonging to the group now engaging in the error activity (this case happened in the "Students' definitions of circle" case study [J/6], where six of the definitions of circle analyzed had been collected from students of the same age but attending a different class).
- *Naïve person error*—that is, when the error is made by a person more mathematically naïve than the one now engaging in the error activity (a category illustrated earlier in "My definitions of circle" case study [G/5], but that I did not happen to use in this teaching experiment).
- *Expert error*—that is, when the error is made by a person more "expert in mathematics" than the one now engaging in the error activity (once again, in the teaching experiment on mathematical definitions I did not use this

category of errors, although it has been illustrated earlier in the "Infinity" historical case study [D/4]).

- *Math-inherent error*—that is, when the error is due to the limitations of mathematics itself, rather than the cause of a person's mistake (as in the case of contradictions created when trying to define 00, discussed in the "High school students' 00" case study [L/6]).

Somewhat to my surprise, in the error activities reported in this chapter, Katya and Mary did not seem to react very differently to errors belonging to these different categories. At most, they occasionally showed a greater eagerness to analyze the errors they themselves had made and felt very proud when some valuable outcome was produced by this activity (as shown especially by Katya in the "Students' homework" case study [J/6]).

Instead, the analysis of the error activities developed in the teaching experiment suggested another way in which the source of the error employed may affect how the proposed strategy is used in mathematics instruction. The teacher's control of the error activity and his or her level of comfort in it may in fact be influenced by whether the error studied was:

- *A planned error*—that is, an error previously selected by the teacher and presented to the student within a planned activity (as it was the case in the "Students' definitions of circle" case study [J/6], where I put together the list of incorrect definitions the students were asked to analyze, and in the "High school students' 00" case study [L/6], where I proposed the consideration of two contradictory values for 00).
- *An unexpected error*—that is, an error made by a student and used in the same lesson, but in a situation where the teacher had expected (and perhaps even prompted) a similar error to occur (as illustrated in the "Students' polygon theorem" case study [K/6], where I had expected that the students would take steps in the wrong direction in solving the problem and, although I could not precisely predict the errors these students would actually make, I had some ideas about what some of these errors could be).
- *An unexpected error*—that is, an error made unexpectedly by a student or the teacher and immediately pursued in class in an impromptu error activity (as illustrated especially in the "Students' homework" case study [J/6], where I had no opportunity to examine Katya's "mistake" before being asked to analyze it in the students' presence).

As already commented in the context of the "Students' homework" case study [J/6], error activities falling within this last category are probably the most powerful ones to show students the possibility of capitalizing on errors on their own, although they may also prove to be quite challenging for the teacher.

POTENTIAL BENEFITS OF CAPITALIZING ON ERRORS IN MATHEMATICS INSTRUCTION

The analysis conducted in the previous section, along with the illustrations reported earlier in this chapter, suggest that using errors as springboards for inquiry could contribute to students' mathematical learning and growth in more than one way. In this section, I try to discuss more explicitly what the benefits of implementing the proposed strategy in mathematics instruction could be. I do so by trying to identify as working hypotheses both the desirable learning opportunities that some kinds of error activities can offer students and the potential outcomes that the students could derive when they take advantage of such opportunities.

Let me start, then, by examining the learning opportunities that capitalizing on errors could offer to mathematics students, using the four examples reported in the first section of this chapter to illustrate some of my points. Before I proceed in this task, however, it is important to clarify that my approach to this issue has been informed by my position about what kinds of learning experiences are most valuable for mathematics students. Thus, in what follows I specifically examine the extent to which the proposed strategy could facilitate engaging students in the following learning experiences, which I have identified as especially important and "desirable" within the inquiry approach to mathematics instruction articulated in Chapter 2:

- *Experiencing constructive doubt and conflict regarding mathematical issues*: Even just within the error activities reported earlier, it is possible to notice several instances when, because of perceived errors, Katya and Mary encountered contrasting solutions (e.g., in the "Students' polygon theorem" [K/6] and the "High school students' 00" [L/6] case studies), results that did not make sense (e.g., an equation that could not be "solved" in the "Students' homework" case study [J/6]), or solutions that contrasted with what they had initially expected (as it happened in a few occasions in the "Students' polygon theorem" case study [K/6]). These errors, in turn, generated some sort of "doubt" or "cognitive dissonance" that made the students question their activity up to that point and/or initiate new inquiries. The variations within uses of errors identified earlier suggest that this learning opportunity is most likely to be offered to students when a discovery or inquiry stance of learning is assumed. (*Experiencing doubt learning opportunity*)

- *Pursuing mathematical explorations*: The constructive doubt created by an error is likely to generate some mathematical questions worth exploring. Sometimes the explorations thus initiated may turn out to be quite extensive (as shown especially by the challenging questions Mary raised and explored in the "Students' polygon theorem" case study [K/6] after the initial statement about polygons was proved incorrect and revised) and/or may lead into unexpected di-

rections (as it was the case in the second part of the "Students' homework" case study [J/6], when Katya's incorrect procedure suggested some interesting results about the circles passing through two given points.) These more extensive and open-ended explorations, however, are most likely to occur when the error is used assuming a discovery or inquiry stance of learning and the student is actively involved in the activities thus generated (levels I_2 and I_3 of student involvement). (*pursuing explorations learning opportunity*)

- *Engaging in challenging mathematical problem solving.* As part of the mathematical explorations stimulated by an error, students will often engage in posing and solving some specific mathematical problem (as shown at more than one point within the "Students' homework" [J/6] and the "Students' polygon theorem" [K/6] case studies). These problems may occasionally turn out to be quite challenging and enticing for the students, more so than the often contrived word problems found in most textbooks. Most importantly, these problems are likely to be perceived by the students as more meaningful and relevant because they were generated in a more "real" context. (*problem-solving learning opportunity*)

- *Experiencing the need for monitoring and justifying their mathematical work.* Such a need will be a natural consequence of the doubt and conflict caused by specific errors, especially when a student is called to support a result that contrasts with another one produced by a classmate (or even by an authority)—as seen, for example, in Katya and Mary's questioning each other in the "Students' polygon theorem" case study [K/6]. In addition, error activities conducted within a remediation stance of learning may concretely show students the consequences of errors due to a superficial understanding of a mathematical topic and also enable students to appreciate how an attention to justifying one's mathematical results may lead to avoiding such circumstances (an opportunity partially offered to the students in the activity of analyzing a list of incorrect definitions, in the "Students' definitions of circle" case study [J/6]). (*monitoring/justifying learning opportunity*)

- *Experiencing initiative and ownership in their learning of mathematics.* The genuine doubt generated by some errors may invite the students themselves to generate their own mathematical questions or problems that they are interested in pursuing, instead of always engaging in mathematical activities dictated by the teacher or the textbook. Especially when, within an inquiry stance of learning, one challenges the status quo by asking questions such as "in what circumstances could this 'error' be considered correct?" or even "what would happen if this 'incorrect' definition or result were accepted?", errors may naturally initiate problem posing in a way that mirrors the "What-If-Not?" strategy proposed by Brown and Walter (1990). Although students engaging in an error activity at an independent student inquiry (I_3) level of involvement are by definition posing their own questions, ample opportunities for student problem posing and initiative are also offered when the level of student involvement is

I_2 (teacher-led student inquiry) (as illustrated in the "Students' polygon theorem" case study [K/6]). Analyzing and correcting their errors on their own when assuming a remediation stance of learning is also likely to provide students with a greater feeling of ownership as mathematics learners. Finally, when explicit attention is consistently paid to errors in mathematics instruction, students may be invited to assume a more critical stance toward their mathematical activity, that translates into a greater willingness to question each other's results and their justification (as explicitly observed, once again, in the "Students' polygon theorem" case study [K/6]). (*Experiencing initiative learning opportunity*)

- *Reflecting on the nature of mathematics.* The constructive doubt generated by errors may also often provide a concrete context and starting point for reflecting and discussing mathematics as a discipline. For example, the activity of analyzing a list of incorrect definitions in the "Students' definitions of circle" case study [J/6] enabled the students to address the more abstract issue of what the characteristics of mathematical definitions should be. In the "High school students' 00" case study [L/6], instead, we have seen Mary question the truth of mathematical results taken for granted up to that point as a result of encountering an unresolvable contradiction in the case of 00. By definition, this opportunity to examine issues related to the nature of mathematics will require a use of errors at the math level of discourse. (*reflecting on math learning opportunity*)

- *Recognizing the more humanistic aspects of mathematics.* The analysis of some math-inherent error (such as the contradiction encountered in the "High school students' 00" case study [L/6] when trying to evaluate 00), as well as the discussion of mathematics as a discipline that can be generated in the context of uses of errors at a math level of discourse, may offer mathematics students with an opportunity to see that ambiguity and uncertainty may exist even in mathematics, and appreciate them as a positive element (as shown by the students' reflections about the potential limitations of their definition of polygon reported at the end of the "Students' polygon theorem" case study [K/6]). A teacher's account of the explorations invited by some errors in the history of mathematics (such as those discussed earlier in Chapter 4) may also help students see mathematics as the product of human activity. Finally, the experience of personally engaging in problem posing and explorations motivated by some errors (as in the "Students' polygon theorem" case study [K/6]) may also enable students to appreciate that mathematics could offer them rich opportunity for creativity and personal judgment. (*humanistic math learning opportunity*)

- *Verbalizing their mathematical ideas and communicating them.* Evidence of Katya and Mary engaging in mathematical communication—between themselves as well as with the instructor—can be found in all the vignettes reported earlier in this chapter. Although one may argue that this dialogue was facilitated by the very context of a teaching experiment involving just two students and a teacher, the focus on analyzing errors also provided a genuine need for the stu-

dents to explain how they had achieved certain results (see "Students' polygon theorem" case study [K/6]), to justify their reasoning (see "Students' definitions of circle" case study [U/6]), to develop an argument in support of a suggested solution (see "Students' polygon theorem" case study [K/6], or even to articulate their views of mathematics as they tried to express why they were finding certain errors particularly disturbing (see "High school students' 00" case study [L/6]). (*communication learning opportunity*)

To complete this analysis of the learning opportunities that can be offered by capitalizing on errors in mathematics instruction, it is also important to keep in mind that the learning opportunities potentially offered by a specific error may not all be exploited within a specific instructional experience for various reasons. This was already mentioned in my discussion of the "High school students' 00" case study [L/6], where Mary's idea that we might try to "change" the number system by eliminating "zero" suggested a historical inquiry about "zero" that I chose not to pursue due to time constraints. The contrast between the activities around incorrect definitions of circle developed by the students in the "Students' definitions of circle" case study [U/6] and those I had earlier engaged in, as reported in "My definitions of circle" case study [G/5], also well illustrate this point.

Providing mathematics students with learning experiences such as the ones discussed here should be considered a benefit *per se* within an inquiry approach, provided that the students are able to actually take advantage of the opportunities thus offered to them—as Katya and Mary undoubtedly did in the experiences reported in this chapter. This is especially important if we consider that these learning opportunities are rarely offered to mathematics students in traditional mathematics instruction. At the same time, it is worth complementing the previous analysis with a more explicit discussion of how the proposed strategy could support the achievement of some fundamental goals for school mathematics. I have identified such goals as follows:

- *Enable students to gain a better understanding of mathematics as a discipline.* In today's mathematics classes, issues regarding the nature of mathematics or metamathematical notions (such as definition, proof, and model) are rarely explicitly addressed. This omission could be due in part to the perceived complexity and abstraction of these issues. As discussed earlier, once the appropriate level of mathematical discourse is assumed, errors may instead contribute to making these issues more concrete and accessible to students and/or enable students to recognize some humanistic aspects of mathematics, thus challenging the common view of mathematics as cut and dried and impersonal (Borasi, 1990; Schoenfeld, 1989). Finally, engaging in the explorations that can be motivated by an error may enable mathematics students to personally experience the kind of activities that characterize the work of professional math-

ematicians, and thus come to better appreciate what "doing mathematics" is really about.

- *Enable students to gain a better understanding of mathematical content.* A study of errors can help students realize and resolve some of their misconceptions (remediation/content use of errors) and explore new properties and uses of a given concept, rule or mathematical situation (discovery/content and inquiry/content uses). These experiences, in turn, can result in improved knowledge of some technical mathematical content (as it happened for example for Katya and Mary in the cases of the concepts of circle, polygon, or exponent in the error case studies discussed in this chapter).

• *Enable students to become more proficient in "doing" mathematics.* Recent reports on the future mathematical needs of our society have stressed the importance that students become proficient in processes such as problem posing and solving, communication, and mathematical reasoning (see, e.g., NCTM, 1989, p. 5). As argued earlier, a use of errors as springboards for inquiry can provide students with valuable opportunities to practice such processes and, therefore, improve their performance in them. In addition, specific uses of errors suggested by this strategy could provide students with yet another set of heuristics to support their independent mathematical activity, such as using errors constructively as a potential source of information when solving a problem (especially when working at a task level of discourse). Furthermore, by becoming more cautious and critical of their own mathematics activity as a result of getting used to paying more explicit attention to errors may enable students to become more independent from authority for the verification of their work.

- *Enable students to become more confident in their ability to learn and use mathematics.* This desirable outcome is likely to be a by-product of achieving several of the other goals discussed thus far—in particular, experiencing success in the learning of some challenging mathematical content, acquiring effective strategies to "do mathematics" and appreciating some of the more humanistic aspects of the mathematics that could help one perceive this discipline as more accessible. In addition, positive experiences with the proposed strategy can also help students develop a more healthy attitude about making mistakes and, thus, make them more willing to take intellectual risks and engage in open-ended inquiry in the context of school mathematics. As a consequence of all this, one can also hope that the students would develop more positive attitudes toward mathematics and be encouraged to engage in mathematical activities both in and outside of school.

Obviously, all the potential benefits discussed so far are not going to be guaranteed by the mere use of the proposed strategy. Rather, their attainment will depend on a number of factors, including not only the characteristics of the specific instructional context in which the strategy is implemented, but also what variations of the strategy will be employed, how extensively errors will be

used as springboards for inquiry, how the strategy is first introduced, and the students' willingness to actually engage in the error activities planned by the teacher. The more extensive illustrations of how errors could be capitalized on in various instructional contexts, reported and discussed in the next two chapters, provide further evidence in support of this observation as well as of the working hypotheses generated in this chapter.

Chapter 7

Capitalizing on Errors in Mathematics Instruction: A Teaching Experiment

The vignettes included in the previous chapter have been valuable to illustrate how mathematics students could capitalize on errors in a variety of ways and, also, to come to the identification of some conceptual categories that can now be used to evaluate the educational potential of the proposed strategy. However, because the error case studies reported in Chapter 6 described isolated instructional events, they could not show how the proposed strategy could be integrated in the planning of a whole instructional unit or what learning outcomes a use of errors as springboards for inquiry could facilitate when employed consistently in school mathematics. To achieve these goals, in this chapter I now report and discuss the results of the whole teaching experiment on mathematical definitions within which the error case studies reported in the previous chapter took place.¹

I have selected to discuss this specific teaching experiment for several reasons. First of all, the greater freedom from curricular constraints characteristic of teaching experiments enabled me to create learning situations that fully reflected an inquiry approach, at the same time taking advantage consistently of opportunities to capitalize on errors. As a result, within a relatively short span of instructional time (equivalent to about 2 weeks of regular mathematics instruction) I was able to generate a rich collection of error activities. Second, the small number of participating students made it possible to carefully monitor the instructional experience and the students' responses to it. This enabled me to evaluate both what the students learned from the experience and how a consistent attempt to capitalize on errors may have contributed to these results.

In what follows, I describe how the proposed strategy was implemented consistently in this teaching experiment and what the students gained from the experience. This is achieved by briefly describing the 24 error activities developed in this experience and listing the variations of the strategy used in each, as well

¹ See Borasi (1994) for a more thorough discussion of this research study.

as by evaluating the major outcomes of the experience and how capitalizing on errors contributed to achieving them. A brief overview of the teaching experiment precedes this analysis to provide information about the context of the experience.

OVERVIEW OF THE TEACHING EXPERIMENT

The teaching experiment discussed in this chapter was designed and taught by the author at the School Without Walls, an urban alternative high school that is part of the Rochester City School District. The students participating in the experience, Kaya and Mary, were two 16-year-old students who, although interested in the arts and humanities, had not been very successful in school mathematics. Mary, in particular, had failed a statewide exam in geometry just the previous year. Both students had declared a dislike for mathematics because of its "cut-and-dried" and impersonal nature. Indeed, their participation in an experience involving "extra" mathematics instruction was endured only by their need for making up for numerous absences in a previous mathematics course.

My choice of mathematical definitions as the focus of the instructional sessions was mainly motivated by the consideration that this topic was likely to provide a good context for the students to engage in genuine mathematical inquiries and to experience some of the more humanistic aspects of mathematics, yet without requiring much technical knowledge.¹ An inquiry approach to mathematics instruction characterized both the design and the implementation of this teaching experiment. My initial plan for the instructional unit consisted of a series of concrete activities about the definitions of circle, polygon, exponent, and variable. These activities were designed to support the students' own inquiry into mathematical definitions by helping them identify the various roles played by definitions in mathematics, appreciate the rationale and the limitations of the criteria traditionally imposed by mathematicians on mathematical definitions, and recognize that mathematical definitions are not always as "perfect" as most people would expect. I hoped that these experiences would challenge the students' previous view of mathematics as a dualistic discipline and enable them to appreciate that, even in mathematics, there are issues on which no agreement can be reached and alternative solutions are possible, results are not always absolutely right or wrong but rather depend on the context in which one is operating, and personal judgment and creativity can be exercised.

¹ I refer the reader once again to my book *Learning mathematics through inquiry* (Borasi, 1992) for more information on this experience.

² Several other topics could have served these goals equally well. For example, in other instructional experiences I have been able to witness the potential of the topics of infinity, probability, tessellation, or alternative geometries to provide students with powerful mathematical experiences and the appreciation of the more humanistic aspects of mathematics (see, e.g., Borasi, 1984, and Borasi and Siegel, 1990).

The teaching experiment evolved over 10 sessions of about 40 minutes each, followed by a take-home project. After the project was completed, a series of interviews was conducted to collect the students' impressions and reactions about the experience.

With the exception of the very first introduction lesson, errors were used to initiate and sustain inquiry in more than one occasion in all the 10 instructional sessions. Overall, a total of 24 error activities were identified (including the four episodes reported in Chapter 6 as error case studies I through L). Each of these error activities was then analyzed using the categories developed in Chapter 6. A thorough analysis of the transcripts of all instructional sessions, of the final project produced by each student, and of the interviews conducted at the end of the experience, was also conducted to evaluate what the students learned from participating in the teaching experiment and the influence that using errors as springboards for inquiry had on these results.⁴

BRIEF DESCRIPTION AND ANALYSIS OF ALL THE ERROR ACTIVITIES DEVELOPED IN THE TEACHING EXPERIMENT

Telegraphic Description and Categorization of Error Activities

The brief description of each of the error activities developed in the teaching experiment provided in this section is intended to document the consistent use of the proposed strategy made throughout this instructional experience and the variety of ways in which errors were used. Each error activity is preceded by a brief description of the instructional episode in which it occurred, as a way to provide some information on the context in which the use of error described developed. Although I realize that a few lines of description for each error activity can do little more than characterize each of these events, I hope that case studies I, J, K and L helped the reader to get a flavor of the kind of learning experienced by Kaya and Mary in the teaching experiment and, thus, interpret these telegraphic descriptions.

To highlight the variations within the strategy of capitalizing on errors employed, I follow the description of each error activity with the identification of:

- The type of error used.
- The specific use(s) of errors made in the error activity (among those identified in the 3×3 matrix reported in Table 6.1).
- The level(s) of student involvement in it.

⁴ More information about the research design of this study can be found in Borasi (1994).

- The source(s) of the errors studied (with respect to both who made the error, and the level of teacher input in the choice of such an error as the focus of the activity).
- The learning opportunities the error activity offered the students.

With the exception of the error type case, a description of all these categories and of the abbreviations used to identify their possible attributes can be found in Chapter 6.

Instructional Episode 1. As a diagnostic activity, the students are asked to write a definition for the concepts of circle, square, polygon, variable, exponent, equation, cat, purple, and crazy (Day 1—20 minutes).

Error activities. None (but note that some of the incorrect definitions that the students produced in this occasion were used later on in the experience—see Error Activities 2A, 6A, and 8B)

Instructional Episode 2. The students try to create a “good” definition of circle and to identify important characteristics of mathematical definitions through the analysis of a given list of incorrect definitions of circle (Day 2—30 minutes).

Error activity 2A.³ The analysis of the eight incorrect definitions of circle proposed by the teacher enables the students to better appreciate the role of isolation of the concept and precision in terminology in mathematical definitions, and also to generate two acceptable definitions of circle (reported only in part in the previous “Students’ definitions of circle” case study [1/6]).

- **Error Type(s):** incorrect/tentative definition.
- **Use(s):** remediation/task, remediation/content, remediation/math, inquiry/content.
- **Involvement:** teacher-led student inquiry.
- **Source(s):** same student/classmate, outside peer; planned.
- **Learning Opportunities:** experiencing doubt, problem solving, pursuing explorations, reflecting on math, monitoring/justifying, experiencing initiative, communication.

Error activity 2B. One student’s confusion about the difference between the expressions $x^2 + y^2 = r^2$ and $a^2 + b^2 = c^2$ (as used in the context of the analytic equation of a circle and the Pythagorean theorem, respectively) leads unexpectedly to a first discussion about the difference between variable and constant.

³ For easy reference, each error-activity has been assigned a code consisting of the number of the instructional episode in which it occurred (2 in this case) followed by a letter (A in this case, since it is the first error-activity occurring in the instructional episode).

- **Error type(s):** thing that does not make sense, ambiguous expression.
- **Use(s):** remediation/task, inquiry/content, inquiry/math.
- **Involvement:** teacher modeling.
- **Source(s):** same student/classmate; unexpected.
- **Learning opportunities:** experiencing doubt, experiencing initiative, humanistic math, communication.

Instructional Episode 3. The students solve the problem of “Finding the circle passing through three given points,” using the two definitions of circle previously generated in Instructional Episode 2 (Day 3—15 minutes; Day 4—15 minutes).

Error activity 3A. Errors are made and used constructively by the students in the process of solving the problem using a synthetic geometry approach (i.e., by finding the center of the circle as the intersection of the perpendicular bisectors of two different pairs of the given points).

- **Error type(s):** tentative steps.
- **Use(s):** discovery/task.
- **Involvement:** teacher-led student inquiry.
- **Source(s):** same student/classmate; expected.
- **Learning opportunities:** experiencing doubt, problem solving, monitoring/justifying, experiencing initiative, communication.

Error activity 3B. Once the problem has been solved, the eight incorrect definitions of circle discussed in the previous episode are briefly revisited, both to evaluate which of them could have enabled us to solve the problem and to further derive some properties of circle from them.

- **Error type(s):** incorrect/tentative definition.
- **Use(s):** remediation/content, remediation/math.
- **Involvement:** teacher-led student inquiry.
- **Source(s):** same student/classmate, outside peer; planned.
- **Learning opportunities:** experiencing doubt, monitoring/justifying, communication.

Error activity 3C. Upon a student’s request, her own incorrect solution to the problem is analyzed; this not only leads to the identification and correction of the errors she had made, but also reveals the possibility of a new alternative approach to solving the original problem (see “Students’ homework” case study [1/6]).

- **Error type(s):** computational mistake, incorrect/tentative procedure.

- *Use(s)*: remediation/task, remediation/math, inquiry/task, inquiry/math.
- *Involvement*: teacher modeling, teacher-led student inquiry.
- *Source(s)*: same student/classmate; unexpected.
- *Learning opportunities*: experiencing doubt, pursuing explorations, problem solving, monitoring/justifying, experiencing initiative, communication.

Instructional Episode 4. The students solve the problem of "Finding the interior angle of a regular pentagon inscribed in a circle," using alternative definitions of circle and isosceles triangle (Day 3—15 minutes).

Error Activity 4A. Errors are made and used constructively by the students in the process of solving the problem; one of these errors also leads into the unplanned discussion of the definition of isosceles triangle, which in turn provides some alternative definitions for this figure as well as an appreciation of the value of avoiding nonessential properties in a definition.

- *Error type(s)*: incorrect/tentative definition, incorrect/tentative hypothesis, incorrect/insufficient explanation.
- *Use(s)*: discovery/task, discovery/content, inquiry/task, inquiry/content, inquiry/math.
- *Involvement*: teacher-led student inquiry, independent student inquiry.
- *Source(s)*: same student/classmate; expected, unexpected.
- *Learning opportunities*: experiencing doubt, pursuing explorations, problem solving, monitoring/justifying, experiencing initiative, reflecting on math, humanistic math, communication.

Instructional Episode 5. To further explore the distinction between properties and definitions of a mathematical concept, the students are asked to debug my erroneous proof for the (correct) theorem "Any triangle inscribed in a semicircle is a right triangle," and also end up proving the theorem using an alternative approach (Day 4—15 minutes).

Error activity 5A. In analyzing my erroneous proof, the students realize that a property of right triangle had been used instead of its definitions, and they propose some creative ways to "fix" the proof.

- *Error type(s)*: incorrect/tentative proof.
- *Use(s)*: remediation/task, remediation/math.
- *Involvement*: teacher-led student inquiry.
- *Source(s)*: teacher; planned.
- *Learning opportunities*: experiencing doubt, problem solving, monitoring/justifying, reflecting on math, communication.

Error activity 5B. Errors are made and used constructively by the students as they try to prove the theorem in an alternative way.

- *Error type(s)*: tentative steps.
- *Use(s)*: discovery/task.
- *Involvement*: teacher-led student inquiry.
- *Source(s)*: same student/classmate; expected.
- *Learning opportunities*: experiencing doubt, problem solving, monitoring/justifying, communication.

Instructional Episode 6. The students try to create an acceptable definition for the unfamiliar notion of polygon, using the tentative definitions they created in Instructional Episode 1 as a starting point (Day 4—10 minutes).

Error activity 6A. As the students analyze the definitions of polygon they had previously created against various examples and counterexamples of polygon, they recognize some of their errors and try to fix them. This process, however, also reveals to them how limited their own understanding of polygon actually is, and how consequently this procedure would not lead them very far in refining the definition of polygon.

- *Error type(s)*: incorrect/tentative definitions.
- *Use(s)*: discovery/content, discovery/math, inquiry/math.
- *Involvement*: teacher-led student inquiry.
- *Source(s)*: same student/classmate; planned.
- *Learning opportunities*: experiencing doubt, pursuing explorations, monitoring/justifying, communication.

Instructional Episode 7. Recognizing their limited understanding of polygon, the students continue to work toward the creation of an acceptable definition of this concept but following an alternative approach inspired by Lakatos' proofs and refutations approach (Lakatos, 1976), as they try to prove the following tentative theorem proposed by the teacher: "In an n -sided polygon the sum of the interior angles is $180^\circ \times n$ " (Day 5—40 minutes).

Error activity 7A. As the students try to prove (and revise) the proposed theorem—a challenging activity that engages them in genuine problem solving—a number of errors are made and used constructively. This process also helps them refine their understanding of the concept of polygon as well as its definition (see "Students' polygon theorem" case study [K/6]).

- *Error type(s)*: incorrect/tentative definition, incorrect/tentative hypotheses, tentative steps, contrasting results.

- *Use(s)*: discovery/task, discovery/content, discovery/math, inquiry/math.
- *Involvement*: teacher-led student inquiry.
- *Source(s)*: same student/classmate, teacher, planned, expected, unexpected.
- *Learning opportunities*: experiencing doubt, pursuing explorations, problem solving, monitoring/justifying, experiencing initiative, reflecting on math, humanistic math, communication.

Instructional Episode 8. The students explore the concept and definition of variable, as an example of a mathematical concept that can be used without a rigorous definition (Day 6–30 minutes).

Error activity 8A. The confusion between the expressions $x^2 + y^2 = r^2$ and $a^2 + b^2 = c^2$, raised earlier on in Error Activity 2B, is briefly revisited to remind the students of the difference between variable and constant.

- *Error type(s)*: ambiguous expression.
- *Use(s)*: discovery/content.
- *Involvement*: teacher-led student inquiry.
- *Source(s)*: same student/classmate, planned.
- *Learning opportunities*: experiencing doubt, reflecting on math, humanistic math, communication.

Error activity 8B. In the attempt to create a good definition of this concept, the students examine critically and try to refine the definitions of variable they initially wrote in Instructional Episode 1. A discussion of their lack of success in this task leads them to realize the difficulty of creating definitions for certain mathematical concepts even when we can understand and use them. This, in turn, raises some questions about whether such definitions are always needed.

- *Error type(s)*: incorrect/tentative definition.
- *Use(s)*: remediation/task, remediation/content, inquiry/math.
- *Involvement*: teacher-led student inquiry.
- *Source(s)*: same student/classmate, planned.
- *Learning opportunities*: experiencing doubt, pursuing explorations, monitoring/justifying, reflecting on math, humanistic math, communication.

Instructional Episode 9. In preparation for extending the definition of exponentiation beyond whole numbers, the extension of multiplication to negative numbers and fractions is revisited (Day 7–15 minutes).

Error activity 9A. In the process of examining how multiplication of negative and fractional numbers can be defined, the students come to realize that

the original definition of multiplication as repeated addition would not make sense within these new number systems and therefore needs to be modified.

- *Error type(s)*: thing that does not make sense; unjustified “transfer” to a different context.
- *Use(s)*: remediation/task, remediation/content, remediation/math.
- *Involvement*: teacher-led student inquiry.
- *Source(s)*: math-inherent, planned.
- *Learning opportunities*: experiencing doubt, pursuing explorations, problem solving, monitoring/justifying, reflecting on math, humanistic math, communication.

Instructional Episode 10. Using procedures similar to the ones illustrated in the case of multiplication, the students extend the definition of exponentiation to negative integer exponents and start exploring some properties of the extended operation (Day 7–25 minutes).

Error activity 10A. An initial computational mistake, leading to the incorrect result $2-6 = \frac{1}{2}$, suggests the value of using the rule $a^b + c = a^b \times a^c$ to verify the value of 26 (i.e., $26 = 23 \times 23 = 8 \times 8 = 64$); the validity of this rule with the extended operation is then verified, and the initial mistake corrected.

- *Error type(s)*: computational mistake, contrasting results.
- *Use(s)*: discovery/task, discovery/content.
- *Involvement*: teacher-led student inquiry.
- *Source(s)*: same student/classmate, unexpected.
- *Learning opportunities*: experiencing doubt, pursuing explorations, problem solving, monitoring/justifying, experiencing initiative, humanistic math, communication.

Error activity 10B. The previous activity invites one of the students to explore what happens when you multiply numbers with different bases and to compute: $32 \times 12-1 = 32 \times 4 \times 3-1 = 32-1 \times 4 = 12$; the analysis of this result leads to correct the result and discover that the distributive property of exponentiation over multiplication still holds when the operation is extended to negative integers.

- *Error type(s)*: incorrect/tentative result.
- *Use(s)*: discovery/task, discovery/content.
- *Involvement*: teacher-led student inquiry.
- *Source(s)*: same student/classmate, unexpected.

- *Learning opportunities*: experiencing doubt, pursuing explorations, problem solving, monitoring/justifying, experiencing initiative, humanistic math, communication.

Instructional Episode 11. The extension of exponentiation to fractional exponents is undertaken by the students (Day 8—10 minutes).

Error activity 11A. The pattern initially suggested by a student to justify the definition $2^{1/2} = \sqrt{2}$ (i.e., $12^{1/2} = \sqrt{12}$; $8^{1/2} = \sqrt{8}$; $4^{1/2} = \sqrt{4}$; $2^{1/2} = \sqrt{2}$) is considered inappropriate by the instructor, because here the student had used the very result she wanted to derive (i.e., $n^{1/2} = \sqrt{n}$) in the construction of the pattern itself. The analysis of this inappropriate pattern leads to a discussion of when and how patterns can be used effectively as a heuristic to extend a definition, as well as to developing an alternative pattern to accomplish the original task.

- *Error type(s)*: incorrect/insufficient explanation.
- *Use(s)*: remediation/task, remediation/content, remediation/math.
- *Involvement*: teacher modeling, teacher-led student inquiry.
- *Source(s)*: same student/classmate; unexpected.
- *Learning opportunities*: experiencing doubt, problem solving, monitoring/justifying, communication.

Instructional Episode 12. The students are introduced to the mathematical context of "taxicab geometry"—that is, the geometrical idealization of a city with a regular grid of streets—and are asked to interpret the familiar metric definition of circle in this new context (Day 8—30 minutes).

Error activity 12A. As the students are introduced to taxicab geometry, their initial disregard and challenge for some of the constraints assumed in this situation leads to a clarification of the basic rules of taxicab geometry, as well as to the awareness that different definitions of distance could be used depending on the constraints one wants to consider.

- *Error type(s)*: unjustified "transfer" to a different context.
- *Use(s)*: remediation/content, inquiry/content.
- *Involvement*: teacher-led student inquiry.
- *Source(s)*: same student/classmate; unexpected.
- *Learning opportunities*: experiencing doubt, pursuing explorations, experiencing initiative, reflecting on math, humanistic math, communication.

Error activity 12B. The two students propose contrasting solutions (i.e., a Euclidean circle, and a taxicircle or "diamond") to the problem of "Finding all

the points in the city at distance 5 from a given point." This spontaneously leads the students to a debate about which solution is correct, a debate that is finally resolved by referring to the rules set for taxicab geometry. This activity also raises the question of whether the metric definition of circle generated in Instructional Episode 2 can still be considered appropriate—a question initiating the following error activity.

- *Error type(s)*: contrasting results.
- *Use(s)*: discovery/task, discovery/content, inquiry/task, inquiry/math.
- *Involvement*: teacher-led student inquiry.
- *Source(s)*: same student/classmate; unexpected.
- *Learning opportunities*: experiencing doubt, pursuing explorations, monitoring/justifying, experiencing initiative, reflecting on math, humanistic math, communication.

Error activity 12C. One of the students decides that, because "diamonds" do not share some important properties of circle, the definition of circle used so far might be faulty and, thus, sets out to re-examine and "fix" it. In the process, some new insights about circles, diamonds, definitions, and the differences between taxicab geometry and Euclidean geometry are gained. This activity also inspires the student to generate the new question of what would be the definition of "diamond."

- *Error type(s)*: thing that does not make sense; tentative/incorrect definition.
- *Use(s)*: remediation/task, remediation/content, remediation/math, inquiry/task, inquiry/content.
- *Involvement*: independent student inquiry.
- *Source(s)*: same student/classmate; unexpected.
- *Learning opportunities*: experiencing doubt, pursuing explorations, monitoring/justifying, experiencing initiative, reflecting on math, humanistic math, communication.

Error activity 12D. The students engage on their own in the activity of creating a good definition of diamond, following the process of generating tentative definitions that are critically examined and modified against various examples and counterexamples of "diamonds."

- *Error type(s)*: incorrect/tentative definitions.
- *Use(s)*: discovery/task, discovery/content.
- *Involvement*: independent student inquiry.
- *Source(s)*: same student/classmate; unexpected.
- *Learning opportunities*: experiencing doubt, problem solving, monitoring/justifying, experiencing initiative, humanistic math, communication.

Instructional Episode 13. Some problems inherent in the extension of exponentiation are explicitly examined, so as to enable the students to appreciate the existence of limitations even in mathematics (Day 9—30 minutes).

Error activity 13A. The fact that $(-2)^{1/2}$ would give an imaginary number leads to a discussion of the alternative suggestions of (a) accepting complex numbers as legitimate results of exponentiation or (b) limiting the extension of exponentiation to positive bases; this experience opens the unexpected possibility that there may be unavoidable limitations in extending operations.

- **Error type(s):** contrasting results, things that do not make sense.
- **Use(s):** discovery/task, discovery/content, inquiry/math.
- **Involvement:** teacher-led student inquiry.
- **Source(s):** same student/classmate, teacher, math-inherent; planned, unexpected.
- **Learning opportunities:** experiencing doubt, pursuing explorations, reflecting on math, humanistic math, communication.

Error activity 13B. A contradiction is generated in the attempt to define 00 by using patterns. The analysis of this situation leads to the realization that it is impossible to resolve such a contradiction and opens a discussion about limitations in mathematics and the nature of mathematics itself (see "High school students' 00" case study [L/6]).

- **Error type(s):** contrasting results, contradiction.
- **Use(s):** discovery/task, discovery/content, inquiry/math.
- **Involvement:** teacher-led student inquiry.
- **Source(s):** math-inherent; planned.
- **Learning opportunities:** experiencing doubt, pursuing explorations, monitoring/justifying, experiencing initiative, reflecting on math, humanistic math, communication.

Instructional Episode 14. The activity of writing an appropriate definition for the concepts of circle, square, polygon, variable, exponent, equation, cat, purple, and crazy is revisited, with the main goal of comparing mathematical definitions with definitions in other fields (Note: only Mary was present) (Day 10—30 minutes).

Error activity 14A. The difficulty experienced by one of the students as she tries to write a definition for cat leads to an interesting discussion that touches on the role and uses of definitions, the differences between definitions in mathematics and definitions in the common language, and also the variety existing among mathematical definitions.

- **Error type(s):** nonmathematical.
- **Use(s):** inquiry/math.
- **Involvement:** teacher-led student inquiry.
- **Source(s):** same student; expected.
- **Learning opportunities:** experiencing doubt, experiencing initiative, reflecting on math, humanistic math, communication.

Instructional Episode 15. A take-home project is assigned to the students as their final evaluation; this project involves first the interpretation of three given definitions (equivalent definitions of perpendicular bisector in plane Euclidean geometry) and the discussion of their relationship, then the interpretation of the metric definitions of circle, perpendicular bisector, ellipse, and parabola in both the usual Euclidean context and in taxicab geometry, and finally a more artistic task.

Error Activity 15A. One of the students notices the (unintended) ambiguity inherent in the definitions of perpendicular bisector given in Part I of the project, and exploits it by investigating the possibility of two alternative interpretations of these definitions in Euclidean and taxicab geometry, respectively.

- **Error type(s):** ambiguous instructions.
- **Use(s):** inquiry/task, inquiry/content.
- **Involvement:** independent student inquiry.
- **Source(s):** teacher, unexpected.
- **Learning opportunities:** experiencing doubt, pursuing explorations, problem solving, monitoring/justifying, experiencing initiative, humanistic math, communication.

Analysis of the Uses of Errors Made Throughout the Teaching Experiment

The previous analysis of each of the error activities developed in the teaching experiment enables us to identify, even just within this short instructional experience, several examples for each of the nine uses of errors identified by the taxonomy developed in Chapter 6—thus suggesting the accessibility of each of these variations of the strategy for secondary school students. It is important to note that a *combination* of uses of error was employed in many error activities, as already evident in the vignettes reported in Chapter 6. Table 7.1 summarizes the number of occasions in the teaching experiment (i.e., the total number of error activities) when each of the nine complementary uses of errors as springboards for inquiry actually took place.

TABLE 7.1. Occurrence of Specific Uses of Errors in the Teaching Experiment

Stance of Learning	Level of Math Discourse		
	Performing a Specific Math Task	Understanding Some Technical Math Content	Understanding the Nature of Mathematics
Remediation	7	8	7
Discovery	11	7	2
Inquiry	5	9	9

TABLE 7.2. Occurrence of Specific Levels of Student Involvement When Capitalizing on Errors in the Teaching Experiment

Level of Student Involvement	Number of Error Activities Reflecting It
<i>I</i> ₁ Teacher modeling	3
<i>I</i> ₂ Teacher-led student inquiry	20
<i>I</i> ₃ Independent student inquiry	4

The distribution of the 24 error activities with respect to the three levels of student involvement previously identified (see Table 7.2) also confirms the possibility and value of all these variations of the proposed strategy.

Not surprisingly, especially in consideration of my original conception of the strategy prior to engaging in this instructional experience, the great majority of uses of errors in the teaching experiment occurred in activities where the students actively engaged in inquiries around some error(s) under the teacher's guidance (level of involvement *I*₂). However, the fact that in a few occasions the students were able to initiate and lead the study of an error on their own (independent student inquiry—level of involvement *I*₃) provides evidence that secondary school students could eventually internalize the proposed strategy so as to be able to use it on their own without the support of a teacher—something that I would like to consider as the ultimate goal of introducing the strategy of capitalizing on errors in mathematics instruction. A look at the categorization of each error activity with respect to uses of errors and levels of involvement also shows that Katya and Mary were able to engage actively (i.e., at levels *I*₂ or *I*₃) in each of the nine uses of errors as springboards for inquiry identified by the taxonomy. Also note that different levels of student involvement were observed occasionally within the same error activity (see Error Activities 3C, 4A, and 11A).

In Table 7.3 I report a first categorization of the types of errors used in the teaching experiment, along with the number of error activities in which each was capitalized on.

The categorization reported in Table 7.3 is obviously specific to this experience and, thus, by no means comprehensive of all the possible types of errors that could be taken into consideration when planning error activities. Yet, at

TABLE 7.3. Types of Mathematical Error Capitalized on in the Teaching Experiment

Type of Error Studied	Number of Error Activities Where It Was Used
Computational mistakes	2
Incorrect/erroneous results	1
Incorrect/erroneous definitions	8
Incorrect/erroneous hypotheses	2
Incorrect/erroneous procedures	1
Incorrect/erroneous proofs	1
Incorrect/insufficient explanations	2
Tentative steps	3
Contrasting solutions/results	5
Contradictions	1
Unjustified transfer in different context	2
Things that "do not make sense"	4
Ambiguous expressions	2
Ambiguous instructions	1
Nonmathematical errors	1

least it suggests several types of mathematical errors that can provide valuable starting points for error activities. The variety of errors identified in this list also supports a posteriori the decision to interpret the term *mathematical error* in the most comprehensive way possible when planning implementations of the proposed strategy in mathematics instruction. Indeed, several of the error activities described in this section would not even have been conceived as possible if a more traditional and restrictive definition of error had been assumed.

The error activities generated in the teaching experiment also presented a considerable variety in terms of the source of the error studied. Tables 7.4 and 7.5 report the distribution of error activities with respect to who made the error

TABLE 7.4. Origin of the Errors Used in the Teaching Experiment

Error Made By	Total Number of Error Activities Where This Kind of Error Was Used
Same person:	
(a) for Mary	14
(b) for Katya	17
Other classmate:	
(a) for Mary	12
(b) for Katya	8
Teacher	4
Outside peer	2
More "naïve" person	none
More "expert" person (other than teacher)	none
Error inherent to mathematics	3

TABLE 7.5. Teachers' Input in Selecting the Errors Used in the Teaching Experiment

Error Was	Total Number of Error Activities Where This Kind of Error Was Used
Planned (i.e., previously selected and introduced by teacher)	10
Expected (i.e., made spontaneously by a student but expected by teacher)	6
Unexpected (i.e., made unexpectedly and pursued impromptu)	12

studied and the level of teacher input in its choice, respectively (according to the categories identified earlier in Chapter 6).

I would like to briefly comment on the results summarized in Tables 7.4 and 7.5. First of all, I was initially surprised by the great number of error activities that were generated by pursuing the study of some unexpected error. However, the very open-ended nature of our instructional sessions, combined with the personal attention invited by the small number of participating students, may help to explain this result.

It is also interesting to note that the great majority of the errors studied ended up being those made by one of the participants in the course of our mathematical activities (i.e., one of the two students or myself as the instructor). Finally, a survey of the 24 error activities also revealed that in this experience the source of the error studied did not affect much the specific use made of such error in instruction.

In sum, the error activities developed in this teaching experiment provided further illustration of the whole range of variations identified in Chapter 6, thus confirming that the strategy of using errors as springboards for inquiry is much more articulated than it might have initially appeared. Furthermore, the analysis developed in this section confirms that many of these variations are complementary to each other, and that mathematics teachers should try to be aware of these variations so as to take full advantage of the proposed strategy in their teaching.

EVALUATION OF WHAT THE STUDENTS GAINED FROM THE EXPERIENCE AND FROM THE USES OF ERRORS MADE IN IT

In order to evaluate how Katya and Mary benefited from a consistent use of errors as springboards for inquiry in the teaching experiment, I would like first of all to examine the learning opportunities that the various error activities developed in this experience provided them with. Table 7.6 summarizes these results.

It is worth remarking on the result that each of the learning opportunities hypothesized earlier were actually offered by more than half of the error ac-

TABLE 7.6. Learning Opportunities Offered by Error Activities in the Teaching Experiment

Learning Opportunity	Total number of error activities offering it
Experience doubt and conflict	24
Pursue math explorations	15
Engage in problem solving	13
Monitor and justify math activity	19
Experience initiative	15
Reflect on nature of math	13
Recognize math as humanistic	16
Communicate mathematically	24

tivities developed in the teaching experiment. This is indicative of the power of the proposed strategy to effectively promote these valuable experiences for mathematics students. These data are especially significant when one considers that the same learning opportunities, although deemed very desirable by both an inquiry approach to mathematics instruction and the most recent calls for school mathematics reform, are rarely offered to students in a traditional mathematics classroom.

In order to fully appreciate how capitalizing on errors contributed to Katya and Mary's learning of mathematics, however, one also needs to examine what the students learned as a result of the teaching experiment as a whole. Although one cannot assume that what the students learned in the teaching experiment was due uniquely to the use of errors made in it, the fact that error activities occurred consistently and often more than once in each of the instructional sessions (as shown earlier in this chapter) makes it clear that capitalizing on errors was an integral part of instruction and, as such, it must have contributed substantially to the outcomes of the experience. To highlight the role played by errors in achieving these results, however, in the following analysis I also explicitly discuss how specific error activities (hereafter identified with the abbreviation EA followed by their code number) seem to have affected particular learning outcomes.⁶

By the end of the experience, the students had reached an understanding of the nature of mathematical definitions that went well beyond that of most high school students. As shown both in their projects and in their responses to some review worksheets, both Katya and Mary came to appreciate the requirements of isolation of the concept, precision in terminology, and essentially usually imposed on mathematical definitions and knew how to create a good mathematical definition for a familiar concept by refining tentative definitions against significant examples and counterexamples. EA 2A (as illustrated in part in the "Stu-

⁶ For a more articulated analysis of these learning outcomes and extensive evidence from the students' work, see Chapter 10 in Borasi (1992) or Borasi (1988c).

den's' definitions of circle" case study [I/6]) was especially instrumental in achieving these results. In addition, the students showed that they could appreciate the necessity and value of tentative definitions (as shown in their responses to a review worksheet given to them at the end of EA 7A, as reported earlier in the "Students' polygon theorem" case study [K/6]) and were aware of the problems that could be created when working with definitions in new contexts (as a result of encountering surprises and errors when interpreting the familiar definitions of circle in taxicab geometry—in EA 12A, 12B, and 12C—and of multiplication and exponentiation in number systems other than the whole numbers—in EA 9A, 13A, and 13B). As a result of specific error activities developed as part of their inquiry into the notion of mathematical definitions, the students also learned several mathematical results regarding circles (as shown in the "Students' definitions of circle" case study [I/6]—EA 2A), isosceles triangles (EA 4A), polygons (as reported in the "Students' polygon theorem" case study [K/6]—EA 7A), operations with negative and fractional numbers (EA 9A, 10A, 10B, 13A), and variables and constants (EA 2B, 8A, and 8B).

One of the most significant outcomes of the experience, however, can be identified in the students' new appreciation for the more humanistic aspects of mathematics. This is revealed quite explicitly in the following quotes:

M: I was thinking about, you know, when we come to an equation like that [O9], when it just cannot be figured out by me, or by the next person, and then it just reminds me that this was all invented by people. It's not something like we are born and there is a tree and it has been there forever. It's like we invented this, out of our minds.

K: I thought math . . . everything that was going to be discovered in math has already been discovered and being a mathematician would be a really stupid thing to do because everybody already knows everything that there is to know. . . . But even the smallest thing is questionable. The same definition could get you something completely different, like with taxicab geometry.

These statements represent a radical change from the view of mathematics as cut and dried and impersonal that the students had expressed prior to the beginning of the teaching experiment. To fully appreciate the influence of the use made of errors on such a change, it should also be mentioned that the students' revealing observations quoted here were generated in the final interviews when they were recalling the significance and implications of EA 13B (in the case of Mary) and EA 12B (in the case of Katya).

Some positive changes in the students' mathematical behavior could also be observed throughout the experience. It was evident, as the unit developed, that the students' critical attitude and independence increased, making them more active

and in control of their mathematical experience. First of all, they became more and more cautious with regard to both the results of their own mathematical activity, and the results suggested by authority (i.e., the instructor's input and/or what they had been taught in previous mathematics courses). The students also took increasing advantage of the opportunity to shape their mathematical explorations by generating new questions on their own rather than simply complying with the tasks imposed by the instructor. Whereas this was already somewhat evident in the report of EA 7A (see "Students' polygon theorem" case study [K/6]), it played an even greater role toward the end of the teaching experiment (especially in EA 12C, 12D, and 13B, the latter reported in the "High school students' 09" case study [I/6]) and culminated when the students spontaneously helped the instructor define their final project—the very task on which they would be evaluated!

The mathematical achievements experienced in the teaching experiment, along with a new view of mathematics making them perceive it as a more accessible and creative domain, all contributed to better attitudes toward school mathematics and an increased self-esteem as mathematics students. In the interviews conducted at the end of the experience, the students commented favorably on the different approach to learning mathematics they had experienced in the teaching experiment and were able to recognize the important role that a focus on errors had played in it. They said that they had found the experience more interesting, complex, and challenging than they had ever thought possible in a mathematics class and were surprised at their own achievements and ability in a subject where they had always felt at a disadvantage. As a result, they felt more confident in their own ability to do mathematics and declared themselves more willing to pursue other mathematics courses in the future. A comment by Mary is especially revealing of these changed attitudes:

M: I didn't think I could figure out math before. I used to think that the material was to blame, but I really think it is the way people were told to teach it. And I think that it is the way they were taught.

These learning outcomes are remarkable, especially if one considers the short duration of the teaching experiment (equivalent to little more than 2 weeks of regular instruction) and the weak mathematical background of the two students.

This evaluation of what Katya and Mary learned as a result of this instructional experience supports the hypothesis, articulated earlier in Chapter 6, that a consistent use of errors as springboards for inquiry could help teachers promote the fundamental goals called for in future school mathematics (e.g., Borasi, 1992; NCTM, 1989). More specifically, the data from the teaching experiment enabled us to conclude that, at least in that specific instructional experience, capitalizing on errors contributed to:

- *Enabling the students to better understand the nature of mathematics as a discipline.* Several error activities provided Kaya and Mary with a concrete and accessible means to address abstract metamathematical issues such as the nature of mathematical definitions, and in addition the students were made more aware of the role of uncertainty, limitations, and personal judgment in mathematics as a result of discussing some "borderline" cases of mathematical error.
- *Facilitating the learning of significant mathematical content.* This was often achieved as a result of the reflections and explorations developed in specific error activities by taking advantage of the potential of errors to create cognitive conflict (as exemplified in the cases of the concepts of circle, polygon, and exponentiation).
- *Making the students more proficient in "doing" mathematics.* Kaya and Mary's problem-posing, problem-solving, communication, and reasoning skills improved as a result of their meaningful practice in the context of specific error activities, and the two students became more independent mathematical learners as a result of learning to monitor and analyze their own errors.
- *Making the students more confident in their ability to learn and use mathematics.* This was achieved especially as a result of developing a more healthy attitude about making mistakes and of perceiving mathematics as more accessible to them, and showed concretely in the students' attitudes and behavior toward the end of the experience.

Chapter 8

Capitalizing on Errors in Mathematics Instruction: Examples From the Classroom

The teaching experiment on mathematical definitions discussed in the previous two chapters was instrumental to the systematic implementation and careful monitoring of the proposed strategy. Yet, the reader may still reasonably wonder whether the working hypotheses about using errors generated within such an unusual experience would actually apply to regular classroom instruction as well. In order to dispel this doubt and further enrich my discussion of how we can capitalize on errors in mathematics instruction, I now report on four other experiences where the proposed strategy was implemented, by myself as well as three other teachers, within mathematics courses at the secondary school and college levels. Each of these experiences is analyzed using the categories and working hypotheses previously developed in Chapter 6. In the second part of the chapter I then summarize some implications about implementing a use of errors as springboards for inquiry within regular classrooms.

ERROR CASE STUDIES REPORTING ON ERROR ACTIVITIES DEVELOPED IN SECONDARY AND COLLEGE MATHEMATICS CLASSES

The instructional episodes reported in this chapter were all developed as part of the already mentioned research project "Using errors as springboards for inquiry in mathematics instruction."¹ This project was designed with the goal of

¹ This 2-year project was supported by a grant from the National Science Foundation (Award No. MDR-8651582). A final report summarizing the results of this project (Borasi, 1991b), as well as detailed reports on the experiences from which the error case studies reported in this section have been derived, are available from the author.

developing and evaluating the instructional strategy of capitalizing on errors in the context of secondary school mathematics. Eleven instructional experiences where the teacher made a conscious effort to take advantage of errors as learning opportunities were developed and studied. Each of these experiences included at least a complete instructional unit and lasted from 2 weeks to a semester. One of these experiences took place at the college level and the other 10 were developed in three different secondary schools. The mathematical concepts covered in these experiences included topics within arithmetic, algebra, geometry, probability, logic, and calculus. Four teachers, all with quite different mathematical backgrounds and teaching experiences, participated in the planning and/or implementation of these experiences with the author. With the exception of the teaching experiment discussed in Chapter 7 and another teaching experiment I conducted in the same school, these experiences were developed in regular courses as an integral part of classroom instruction.

The four instructional episodes reported in this section represent only a very small selection of the more than 150 error activities developed overall within the 11 experiences comprising the project. However, they have been selected as representative not only of the variety of ways in which errors were capitalized on, but also of the different instructional contexts in which the proposed strategy was implemented.

The context of the first error case study ("Students' $P(A \text{ or } B)$ " case study [M/8]) was a probability unit taught in a suburban ninth-grade mathematics classroom. In this case study I report on a series of three lessons that were almost fully devoted to resolving the controversial issue of whether the probability of drawing "a jack OR a diamond" from a standard deck of cards is $\frac{1}{4}$ or $\frac{1}{2}$. This case study illustrates how a whole classroom can engage in worthwhile mathematical discussions and activities motivated by the curiosity that is triggered when several class members are convinced of the correctness of alternative results.

The "Students' geometric constructions" case study [N/8] developed within a unit on geometric constructions taught in a mathematics course offered to (mostly tenth-grade) students attending an alternative high school that is part of a large inner-city school district. This experience shows how the constructive use of errors that occurs within genuine problem-solving activities, illustrated earlier in the "Students' polygon theorem" case study [K/6], is also possible in the context of mathematical inquiries conducted by a whole class.

In the "Students' $\frac{1}{4} = \frac{1}{2}$ " case study [O/8] I describe the opening lesson of a precalculus course that I taught to (mainly eleventh-grade) students in the same alternative high school that provided the context for the previous case study. This lesson had been planned around a study of the "error" $\frac{1}{4} = \frac{1}{2}$ (discussed earlier in "My $\frac{1}{4} = \frac{1}{2}$ " case study [F/5]) and produced a number of surprising outcomes that made me aware of some elements that are necessary for a successful implementation of a use of errors as springboards for inquiry.

Finally, in the "College students' 00" case study [P/8], I report on a number of activities and discussions that were motivated by the discovery of a contradiction when attempting to evaluate 00 within a precalculus course designed for a small group of remedial college freshmen. It is interesting to compare this experience with the one reported earlier in the "High school students' 00" case study [L/6], where more naïve mathematics students approached the same error activity in a teaching experiment context.

Error Case Study M: Building on Errors to Construct the Formula for the Probability of "A OR B" in a Middle School Class ("Students' $P(A \text{ or } B)$ " Case Study [M/8])

The instructional episode reported here developed over three consecutive lessons in a ninth-grade mathematics course taught by Tracy Markham, a junior high school teacher with 4 years of teaching experience. This was a mathematics class of about 30 students within a suburban junior high school. The course curriculum was the one mandated by the New York Regents for students intending to take the statewide "Course I" exam (the first of an integrated series of three high school mathematics courses intended to substitute the traditional sequence of Algebra I, Geometry, and Algebra II). However, because the students enrolled in this class were considered "average" or "below average" and thus insufficiently prepared for Course I, the class met for two additional periods each week to provide them with further instruction and support.

I worked closely with Ms. Markham to help her redesign some of the units in this course and make them more accessible to her students. The strategy of capitalizing on errors was used both in the planning of these units and importantly as the units were taught. From the very beginning of the school year, Ms. Markham had also established a number of innovative learning practices—such as small group work and keeping a math journal—that were used in the episode reported here.

The probability unit, taught toward the end of the school year, was one of the units that Ms. Markham and I redesigned especially for this course. Because many probability results are rather unintuitive, we could often predict that some student would make certain errors and thus plan activities that would take advantage of them. This was the case with the series of lessons on constructing the formula for the probability of the disjunction of two events (i.e., $P(A \text{ OR } B)$) reported in this error case study.

The first lesson started with the following journal assignment written on the board:

1. Find the probability of picking a diamond or a jack when picking one card from a standard deck of cards. Explain your answer.

2. Find the probability of picking a spade or a face card² when picking one card from a standard deck of cards. Explain!

The previous day, the students had already solved problems involving the probability of the disjunction of two *independent events*. In that case, they had had no problem constructing and applying the formula:

$$P(A \text{ OR } B) = P(A) + P(B).$$

The specific exercises assigned on the board, however, were adding a new dimension to the problem because in both cases the two events considered were no more independent—in the first problem, for example, the same card could be *both* a jack *and* a diamond.

The students worked on their own on these problems for a while, writing their answers in their journals, while the teacher moved around looking at their work and answering questions. When most students seemed to be done, the teacher asked each student to share their answer to the first problem, as she reported and then tallied these results on the board:

$\frac{1}{52}$: 13 students

$\frac{1}{52}$: 1 student

$\frac{1}{52}$: 13 students.

The fact that the class was almost evenly split between supporters of $\frac{1}{52}$ and $\frac{1}{52}$ as a solution made the question of what the correct solution of the problem should be very real. Soon a lively discussion on this issue was generated, as people from both camps were sure of their solution and eager to convince the others about it. As the teacher asked students to provide an explanation for their answer, it became very clear that each group was able to provide a rationale for their proposed solution and ready to stand by it:³

T: *How did we get [these results]? Some did $\frac{1}{52}$. Where did it come from, Aaron?*

Aaron: *There are 13 diamonds and 4 jacks.*

T: *There are 13 diamonds and 4 jacks and? ... (She writes this on the board)*

Aaron: *52 cards, put it over ...*

T: *So you add in here ... out of 52. So what would be the probability of getting a diamond?*

St: $\frac{1}{52}$.

T: *So you said $\frac{1}{52}$, and the probability of getting a jack?*

St: $\frac{1}{52}$.

T: *$\frac{1}{52}$, and then you added those. And now we know where that came from, Mary?*

Mary: *I got 16.*

T: *Why?*

Mary: *Because you already have one of your jacks in diamonds.*

T: *Ok, so how did you figure that out?*

Mary: *I had 13 cards for diamonds and 3 others jacks.*

T: *Ok, so you said that there are 13 diamonds and three other jacks?*

Mary: *Yes, one of the jacks is in diamonds.*

T: *Ah ... So one jack is already here, Mary?*

As this dialogue was taking place, several other students began to talk among themselves, as Mary's observation raised the issue of whether the jack of diamonds was indeed counted twice or not. Several students got involved in the discussion at this point, although members of each camp seemed to just keep repeating their arguments without getting any closer to a resolution. The following excerpt of the dialogue is illustrative of this kind of discussion:

Aaron: *But Miss Markham, it doesn't ask for ... it just asks for all the jacks and all the diamonds.*

St: *You're counting it twice!*

Aaron: *It doesn't matter! It just says all the jacks and all the diamonds. It didn't say you had to separate them*

St: *... you can't do that!*

Adam: *You count the same thing twice.*

T: *Why not?*

Adam: *Because we've already ...*

St: *It says OR!!*

Aaron: *Just count all the diamonds. Forget about the jacks. Count all the diamonds. There are 13 diamonds.*

T: *So the probability of getting a diamond is? Is $\frac{1}{52}$ right?*

Aaron: *Right, and count all the jacks and there are 4. It is 4 jacks. There!*

Although more and more students got involved in the argument, and their tone and participation indicated that they were anxious to resolve this apparent dilemma, it became clear that this kind of discussion would not lead them very far, because every party was very sure that their procedure was the correct one

² The class was familiar with the use of the term *face cards* to indicate jacks, queens, and kings.
³ All the dialogues reported in this section are excerpts of the verbatim transcripts of the lessons; in order to identify the speaker, I used the abbreviations T (for the teacher) and St (for an unidentified student), as well as fictitious names for specific students.

but could not convince the other group. At some point, however, a student (Tyler) asked for a deck of cards so that he could prove his point. The teacher then suggested that the class divide into groups of four and gave each group a deck of cards, asking the students to take out all the diamonds and jacks. The students very quickly and eagerly got into their groups and started to sort out and count their cards. Then the teacher asked each group to give their answer and explain how they got it.

Contrary to the teacher's expectations, even this concrete activity of separating all the "favorable cases" did not succeed in convincing the supporters of the solution $\frac{1}{4}$ of their error. Neither their failure to produce 17 cards that satisfied the given conditions, nor further arguments from their classmates, did much to change these students' opinions. Indeed, when at the end of this activity the teacher asked everyone who believed the answer to be $\frac{1}{4}$ to stand up, about half of the class did so!

Despite this result, the discussion that developed around this controversy was mathematically quite valuable, because it provided students within each camp with an opportunity to make their reasoning explicit and to communicate it effectively to their peers:

John: You have got to add all the Jacks and all the diamonds!

Diane: The ones that are saying 17 is right, it is like saying there are two jack of diamonds in there

St: And there can't be two jack of diamonds, else it wouldn't be 52.

St: It wouldn't be a fair deck!

St: ... but there are only going to be 3 jacks.

St: These are two different things.

St: Yeah, but there are not two jack of diamonds. It would be different if there were ...

In a further attempt to clarify the issue, the teacher herself took a deck of cards, separated all those that were "a jack or a diamond," counted them aloud and wrote each of them on the board, so as to show that, unless the jack of diamonds is counted twice, indeed there are only 16 cards satisfying the condition—even if there are 13 diamonds and 4 jacks in each deck of cards. Because the students agreed with her procedure, she thought that the original issue had been resolved at this point. Thus, when at the beginning of the next lesson she tried to "summarize" the results the class had previously agreed on, she was surprised to find out that some students still held to their original belief:

T: OK, you are going to put these cards and you are going to go through the whole deck and you are going to, right in front of me,

separate them. We are looking for a diamond, a jack or both. So put down all the cards, go through them all. You can put a discard pile. How many cards did you separate that will satisfy that condition?

John: 16.

T: 16 cards, right? So if I picked out of that deck of 52 cards, if I just randomly picked a card, what are the chances of me picking a card that was a diamond, a jack or both.

John: 17.

T: Show me all 17 cards. You said out of 52 cards there are 17 cards, show me them.

John: Well, if you add them up there are 17.

As shown by this exchange, the real problem was that some students still did not see the connection between separating all the cards satisfying the condition "jack OR diamond," on one hand, and the question of finding the probability of drawing a jack or a diamond, on the other. This became even more evident as the students engaged in a similar problem (finding the probability of drawing "a spade OR a face card") and were unable to resolve the controversy between the proposed solutions of $\frac{3}{4}$ and $\frac{3}{2}$, even when asked to separate all the favorable cases using an actual deck of cards.

The teacher then tried yet another approach to solve this new problem. On the board, she wrote down all the spades in one column, all the face cards in another column, and then asked the students to identify all the common cards. The students had no difficulty in identifying the jack, queen, and king of spades, which the teacher then listed in a new column, labeled "both" (see Table 8.1).

This representation seemed to help most students see that indeed, by adding all the spades and face cards separately, they ended up counting some "common cards" twice, so they needed now to "subtract" them in order to obtain a correct count. Besides providing an explanation for the error that caused the original controversy, this approach had the added advantage of suggesting the standard algorithm to solve problems involving the probability of a disjunction, as shown in the following dialogue that developed in the case of finding the probability of drawing a heart or a queen from a deck of cards, after the three columns of "hearts," "queens," and "both" had been compiled on the board:

TABLE 8.1. Table Created By the Teacher on the Board to Compute P (spade OR face card).

Spades	Face Cards	Both
Ac, 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K	JH, QH, KH, JD, QD, KD, JS, QS, KS, JC, QC, KC	JS, QS, KS

- T: You have the probability of getting a heart, you have the probability of getting a queen and the probability of getting a heart and a queen. (She writes on the board: $P(\text{Heart}) = \frac{1}{2}$, $P(Q) = \frac{1}{4}$, $P(H \text{ and } Q) = \frac{1}{8}$, what is the probability of getting a heart?)
- Adam: 10, wait!
- T: How many hearts in a deck?
- Adam: 10 ... no 13 when you count face cards.
- T: 13—you have to count all of them. So the probability of getting a heart is $\frac{1}{2}$. What is the probability of getting a queen?
- Adam: 4.
- T: So $\frac{1}{2}$. What is the probability of getting a heart and a queen?
- Adam: 17.
- T: There are 17 ... ?
- Adam: Not really cause ... you add both of them, but you already have one of the queens on the hearts.
- T: Forget these two column a minute. I am going to ask you to pick one card from a deck of cards. What is the probability that that card is a heart and a queen?
- Adam: 1 out of 52.
- T: How come?
- Adam: Cause there is only one queen that is a heart.
- T: There is only one queen of hearts right? Notice I put this word "AND" in here. Hearts AND queens, there is only one queen of hearts. So now we have the probability of a heart, probability of a queen, and probability of both. Now we got our columns as you were describing them. Now what?
- Adam: It is not going to work, what I said. Cause, if you ... add 13 and 4 you get 17 ...
- T: So you would add these?
- Adam: And then subtract the both column.
- T: Why?
- Adam: Cause you already used one of them. I don't know, I got 17 but now ... there is four queens and 13 hearts, there is only 12 actually cause you used one in the hearts column.
- T: Does this make more sense today?
- Adam: Yeah.

Indeed, in the third lesson, after having applied successfully this procedure to a number of other problems (i.e., $P(\text{black OR club})$, $P(6 \text{ OR red})$, $P(\text{red OR face card})$), the students were ready to generalize and represent this algorithm in symbols as:

$$P(A \text{ OR } B) = P(A) + P(B) - P(A \text{ AND } B).$$

This formula was even tested and proven right (somewhat to the students' surprise) even in "borderline" cases such as finding the probability of obtaining an even number or a 3 when rolling a die and even finding the probability of obtaining 2 or 7 when rolling a die.

This whole-class instructional episode provides a good illustration for some of the variations within the strategy of capitalizing on errors identified in Chapter 6.⁴ First of all, if we look at the source of the errors discussed throughout the three lessons, we can notice that these errors were all made by students in the class as they engaged in mathematical activities, although the teacher had somewhat expected these errors to occur on the basis of her previous teaching experiences. Thus, she and I had been able to plan in advance a number of strategies and activities (such as "counting" the cards and "creating columns") that could be naturally proposed in class so as to capitalize on certain errors once they were actually made by students in the class—something that made the teacher herself feel more confident and "in control" throughout the activity, despite its open-ended nature. It is also important to note that, because the students did not know a priori which of the contrasting solutions proposed was the correct one (although the teacher did), the stance of learning assumed throughout the three lessons was one of discovery. Within such a stance, the students worked at two different levels of mathematical discourse as they engaged both in the solution of specific mathematical tasks (such as resolving the controversy generated by the proposal of contrasting answers for most of the problems approached) and in efforts directed toward a better understanding of specific content (i.e., some fundamental probability concepts and results such as the relationship between computing probabilities and counting favorable cases, and the general formula to evaluate the probability of the disjunction of two events). Thus, the specific uses of errors illustrated in this error case study can be identified by a combination of discovery stance and task level (i.e., discovery/task use) and a combination of discovery stance and content level (i.e., discovery/content use).

This instructional episode also provides compelling evidence of the potential of errors, especially when they seem plausible to several students and are considered in contrast to the correct solution, to generate constructive doubt and conflict and, consequently, provide some valuable learning opportunities. It was indeed this genuine conflict that motivated the students to provide justification for the results they supported (regardless of whether such a result turned out to be "right" or "wrong") and communicate them effectively to the teacher and the other students in the class. It is important to remark that the best discussions developed in these lessons occurred when the students were trying to convince each other of the plausibility of their solutions—something that rarely occurs in traditional mathematics classrooms, where student talk is almost uniquely di-

⁴ I refer also the reader to Appendix A for a glossary of the terms and abbreviations used here.

rected to the teacher. In the process, several students surprised the teacher with their ability to reason mathematically, an ability that they had not previously revealed in more traditional lessons.

The episode reported in this error case study shows how errors can do more than simply stimulate some worthwhile mathematical activity. It was in fact the very nature of the error studied (i.e., the fact that it derived from an unwarranted generalization of an algorithm previously developed for a more restrictive case) and its analysis that led the students to derive a more general algorithm (and respective formula) for calculating the probability of "A OR B." Furthermore, this error case study represents a good example of how a use of errors as springboards for inquiry should not be considered incompatible, but rather complementary, to a diagnosis and remediation approach. The error activity developed here, in fact, indirectly provided the teacher with a powerful means to elicit her students' probability conceptions and, thus, contributed to her understanding of some of her students' difficulties with this topic.

To conclude, I would like to reiterate how this experience provides anecdotal evidence that constructive uses of errors, student mathematical inquiry and communication are all possible and valuable in the context of regular classroom instruction. Indeed, even just the small selection of dialogue reported in this case study illustrates how several different students participated actively in the activities and discussions generated around errors in these teacher-led lessons (level of involvement /2). I would also like to remark on the productive use made of small-group activities in these lessons (i.e., in the "cards counting" activity organized toward the beginning of the episode) and of student journals (where each student was asked to write down the answer they supported and their reasons for doing so, at different points in time during the lesson) as a means to more directly engage individual students. On the negative side, some teachers may question the fact that the activities reported in this error case study took about three class periods when most teachers would devote only one to the same topic. Undoubtedly, when a teacher tries to engage and hear from most students in a class of over 20 students a lesson will take considerable time. However, given what the students "learned" about probability, favorable cases, and reasoning and communicating mathematically, I would argue that the extra time was very well spent.

Error Case Study N: Students Dealing Creatively With Errors When Doing Geometric Constructions ("Students' Geometric Constructions" Case Study [N/8])

The lesson reported here was developed within a high school mathematics course taught by Judith Fonzi at the School Without Walls, the same urban alternative high school where the teaching experiment on mathematical defini-

tions took place.⁵ Fonzi, a mathematics teacher with a master's degree in mathematics and a total of 8 years of teaching experience, had been a teacher in the school for 3 years. About 20 students, mostly tenth graders, were enrolled in this class.

The mathematical topics usually covered in the second course of the New York State Regents Sequence (Course II) were used loosely as a guideline for the curriculum of this course. As typical of all Fonzi's courses, and most of the courses taught at the School Without Walls, an inquiry approach informed the teaching of the course. The episode I have selected as representative of this approach and of the spontaneous use of errors as springboards for inquiry within it took most of a class period of about 1 hour and occurred within a unit on geometric constructions taught early in the school year.

In the previous few classes, the students had already successfully verified the three most commonly used criteria of congruence for triangles. In each case the class had first engaged in constructing a triangle given those specific elements (i.e., two sides and the angle between them [SAS], a side and the two angles adjacent to it [ASA], or three sides [SSS], respectively) and then proved that any triangle satisfying those conditions would be congruent to the first one. Rather than following the steps of constructions suggested by their textbook or by the teacher, the students themselves had been expected to come up with a way to construct the triangle and to show to the rest of the class how such a method would "work" (both by carrying their construction out successfully and by being able to justify it using geometrical and logical arguments). This kind of exercise provided the students with plenty of opportunities for genuine and challenging problem solving as well as mathematical reasoning.

In the episode reported in the following, the students engaged in verifying yet another criterion for the congruence of triangles: whether two angles and a side adjacent only to one of them are sufficient to determine a triangle uniquely (this criterion is abbreviated as AAS hereafter). The students had been asked to work on this construction for homework, although actually only one student reported having been able to accomplish the task. The teacher then decided to engage the whole class in verifying the construction proposed by this student, along the lines of what done in previous classes. This time, however, the proposed construction turned out to be more problematic and controversial, thus engaging the whole class in mathematical discussions and problem solving that ultimately led to two alternative procedures for constructing a triangle satisfying the given conditions.

The teacher went to the blackboard and asked the student who had done the homework, Sam,⁶ to give her specific directions so that she could reproduce his construction on the board. He started by asking the teacher to draw two angles

⁵ An abbreviated version of this episode was previously reported in Borasi (1995).

⁶ Once again, fictitious names have been used to identify specific students in this error case study.

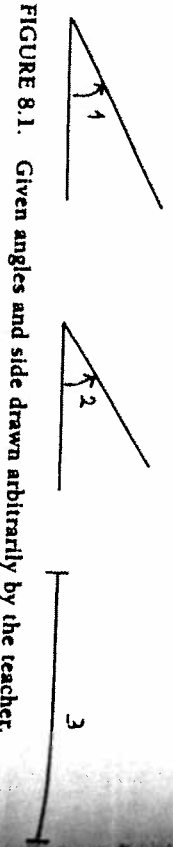


FIGURE 8.1. Given angles and side drawn arbitrarily by the teacher.

(specifying that they could not both be obtuse) and a side. In response to these directions, the teacher drew arbitrarily the two angles (labeled 1 and 2) and the side (labeled 3) shown in Figure 8.1.

To get other students directly involved in the activity, the teacher then explicitly asked for the participation of the rest of the class in these first, easy steps of the construction:⁷

- T: *Who knows the next step? I know Sam does. Who else knows the next step? ... We're just checking it [AAS] out to see if it could possibly be that the situation created a triangle. What's the next step?*
- Tracy: *Draw one of the angles on the side.*
- T: *Measure the first angle on this working line here? (pointing to line 3) Okay.*

Using a protractor and a rule, the teacher then proceeded to put into practice the step thus suggested, by accurately measuring and reporting on another part of the board the side and one of the angles, and continually checking with Sam and Tracy to make sure that she was following their instructions correctly. The figure reproduced in Figure 8.2 was thus obtained.

At this point, another student (Linda) suggested that the teacher replicate Angle 2 on the other extreme of Side 3, but then corrected herself once she realized that this would not correctly represent the condition given (because one of the angles should not be adjacent to the given side):

- T: *Okay, there's Angle 1, Tracy, what am I ... Who can tell me what I'm going to do next?*
- Linda: *Construct Angle 2 on the other ...*
- T: *Construct Angle 2 on this endpoint? (the teacher points to the other extreme of Side 3)*
- Linda: *Yes. (Some students begin to comment among themselves and cause Linda to correct herself) On the opposite ray.*
- T: *Why are you changing your mind?*



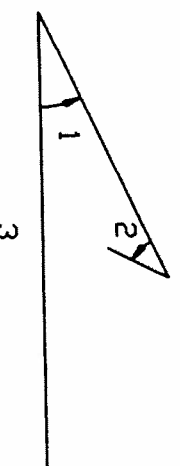
FIGURE 8.2. Figure produced by the teacher following the directions "Draw one of the angles on the [given] side."

- Linda: *Because for angle-angle-side you need the angle at the top.*
- T: *Does everybody see why she changed her mind? If I put Angle 2 over here (at the other extreme of the given side) it will be angle-side-angle (ASA) and we said that if we are writing them this way (AAS) the convention would be that they are supposed to go in this order. So if I stick it over here I'm doing it in the wrong order.*

Notice how in this case a false step is quickly identified, diagnosed, and re-mediated by the students themselves as a natural component of their problem-solving process. At the same time, it is also important to remark about the role assumed by the teacher in the exchange reported—because she both explicitly invited the student who made and corrected her error to make her reasoning public and, furthermore, repeated and rephrased her explanation so as to make it more accessible to other students and to involve the whole class in the process.

A brief pause occurred while the teacher tried to follow Linda's second direction literally, by producing Angle 2 at the end of the extension of the other side of Angle 1 (called Ray 1 hereafter), as shown in Figure 8.3.

The students were quick to realize that this result was not acceptable, and some of them started to look for some ways to fix it.



⁷ All the dialogues reported in this error case study are excerpts of the verbatim transcript of the lesson; in order to identify the speaker the abbreviations T (for the teacher) and SS (when several students spoke together), as well as the students' fictitious names, have been used.

Pat: It doesn't work.

Sam: Oh, oh no. You made the line too long or a ... Erase the line.
T: Erase Line 3?

Sam: Not the whole way. Okay.

Sam's suggestion was to "adjust" Line 3 so that the ray from Angle 2 would meet it in the "right" spot (implicitly suggesting that when he did his construction at home using this procedure he must have been extremely lucky in choosing the correct positioning of Angle 2 on Ray 1 or, alternatively, must have used some trial and error). Most students in the class, however, were not willing to accept a trial and error procedure of this kind. Jane, in particular, was able to articulate precisely the key problem with this procedure:

Jane: You've got to know the lengths of the sides before you add another angle to the [figure]. Do you know what I mean?

T: I think I might know what you mean, I am not sure if anybody else knows what you mean.

Jane: If you can't do it on that side (indicating Line 3), then how are you going to do it on that side (indicating Ray 1) in the right way? You don't know the length of that side.

This precise articulation of the problem may have contributed to its solution, because at this point other students were able to suggest a creative procedure to construct the desired triangle that made use of the first failed attempt:

Todd: Extend the line.

T: Extend which line, Todd?

Todd: The side ... the one that makes up the angle (indicating Ray 1).
T: This side? What, Shea?

Shea: If you copy the angle that's right above [Line] 3, that doesn't have a number ... (she points to angle $CC'A$ in Figure 8.4) If you copy that [angle] to the end of Line 3, then just make the top of Line 1 be the top of the other angle. Do you know what I mean?

The idea behind Shea's construction can be better understood by looking at her final product, reproduced in Figure 8.4 (the letters identifying specific points have been added by me so as to make the narrative that follows easier to understand).

Shea's procedure is indeed a very clever way to resolve the problem of knowing exactly how much Ray 1 needs to be extended, taking explicit advantage of the information provided by a prior incorrect result. What a beautiful example of capitalizing on errors! Most interestingly, Shea herself was able to realize and articulate it:

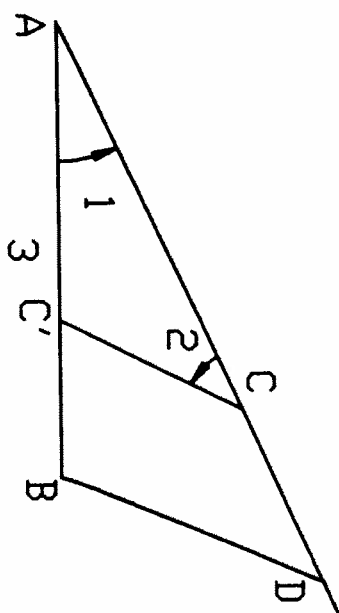


FIGURE 8.4. Diagram illustrating Shea's construction approach.

T: [This] is probably about what Todd wanted me to do. He wanted me to make this line longer ... Todd suggested: If this line were longer, it might work ...

Shea: Yeah, but you don't know how much longer to make it.

Shea: It works, but you can't do it unless you do it wrong first.

The rest of the class now seemed convinced that this construction would work and were able to individually repeat such a construction on another example. Some students were also able to explain why the procedure worked by observing that given that the angles ACC' and ADB (see Figure 8.4) are equal by construction, then the lines CC' and DB are parallel and, hence, the triangles ACC' and ADB are similar.

After this activity was accomplished, the teacher also tried to come back explicitly to Shea's comment, in order to point out that making an error first was a necessary part of the procedure, something that could not be avoided if we wanted to eventually reach the correct result. Not every student seemed ready to agree with this statement. While the teacher attempted to move on to the question of whether Shea's construction would yield only congruent triangles, some students continued to argue among themselves about the appropriateness of the procedure itself. Overhearing them, the teacher decided to bring their controversy to the attention of the whole class:

T: Jim ... [you said to] Jane "No, you don't have to do it wrong the first time to make it work." And Jane said "So, how do you do it?"

Jim was invited to the board to prove his point. After some unsuccessful attempts, he had to give up. At this point, however, another student (Todd) was ready to take his place at the board and show an alternative procedure to con-

construct a triangle given AAS that would not require "to do it wrong first." His construction relied on the known property that the sum of the angles in a triangle is 180° . He used this property to construct the third angle of the triangle, which could then be used to construct the triangle using the ASA procedure created in the previous class (as illustrated by Todd's final product, reproduced in Figure 8.5).

Todd's procedure took even the teacher by surprise, because she herself had never thought of this possibility before. This reaction is reflected in the following dialogue, which occurred after Todd had completed the construction of the third angle on the left side of the blackboard and was starting to construct the triangle next to it:

T: Are you starting over?

Todd: No, I'm starting the triangle now.

T: Oh, wait a minute, I think we need to hear what he's doing here. You are just starting your triangle now?

Todd: Yeah.

T: So what you were doing was preliminary work? And what was that? You see, the trouble was we couldn't get an angle at this end. And since all three angles of a triangle equal 180° (he points to the picture he has just completed on the left) Angle 1, Angle 2 and you just ... the rest is that angle.

Pair: Wow.

Shea: That's really neat.

Both the teacher and the other students were quite impressed by the ingenuity and the novelty of this procedure, although some students spontaneously questioned the validity of his procedure. This, in turn, initiated a discussion on why Todd's construction would work:

Student: It does look like hers (Shea's). Hers just leaned a little more.

Mary: Proof. Proof.

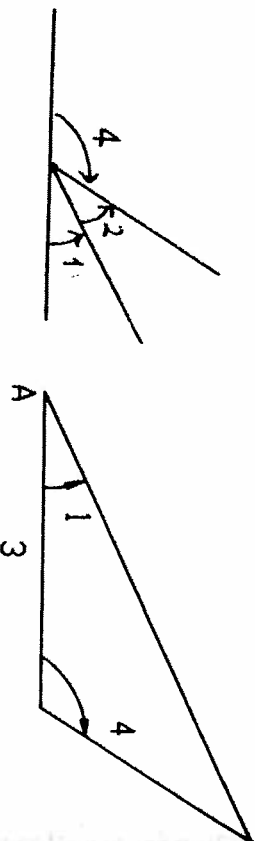


FIGURE 8.5. Todd's final triangle and preliminary construction.

T: You've got to prove what, Mary? You've got to prove that the Angle 2 that we were supposed to have really does end up there.

SS: (interrupting) He did [prove it].

(As this conversation is taking place, Todd finishes his construction of the triangle and the class applauds.)

T: Who is the other person in this room who obviously thought about this picture and can explain it?

Jim:

He constructed Angle 1 and then added Angle 2 to it. Also three angles ... all three angles equal 180° . The portion left from Angle 1 and Angle 2 is ... hum, Angle 4 ... So then you go Angle 1, Line segment 3, [and then Angle 4] goes from the end of Line segment 3.

T: Was Angle 4 part of the given?

SS: No.

This observation allowed the teacher to point out that often in mathematics one has to use information and results that are not immediately given in the text of the problem:

T: Before this one (Todd's procedure) could work, what had to happen?

Linda: You had to think about it.

T: (With emphasis) Yeah you had to think about it! That's true.

Jim: And you had to remember that the angles equaled up to be 180° .

T: So you had to bring in some other information? ... What [else] did you use to make it work? What did he make really and truly to construct that triangle?

Don: He used two angles and a side.

T: Look very carefully for a minute at the triangle that he used. I got there, but I got there in rather an odd-ball way. Look very carefully at the triangle he made, remember how he made it.

Mary: He used angle-side-angle first.

Jane: Yeah, he did.

T: Angle-side-angle. Which was something that we already proved. Right?

This observation led to a final discussion of the difference between AAS and ASA, as well as of the cumulative nature of mathematical results, based on the realization that previously proven results can be used in new proofs.

This error activity illustrates once again how a constructive use of errors can occur in a variety of ways within a short period of time (less than an hour of classroom instruction in this case) and even within the same error activity. First

of all, because the task of constructing a triangle using AAS was both new and challenging for the students, it engaged them in genuine problem-solving activities where they had to carefully monitor and justify their steps and, of course, make moves in the wrong as well as the right direction (discovery/task use errors). However, when Shea used the information provided by a first unsuccessful trial and error attempt to place Angle 2 on Ray 1 (see Figure 8.4), she was well aware that such a construction was incorrect and, yet, believed that it could provide a useful step toward reaching the correct solution (remediation/task use of errors). The dissatisfaction several students felt about her procedure (even after agreeing about its validity) further motivated some of them to find an alternative method to produce the same construction—a task that had not been set by the teacher and took even her by surprise (inquiry/task use). As a result of all these activities involving errors, the students dealt with such as properties of similar triangles and the fact that the sum of the interior angles of a triangle is 180° (thus illustrating a use of errors at the understanding content level of discourse within all three learning stances). At the same time, some questions dealing with more abstract issues such as “Do you sometimes need to do something *wrong* first in order to make it work in mathematics?”, “How do you use already established results in a new proof?”, and even “Is it possible in mathematics to have two different but equally acceptable procedures that yield the same correct solution?” were also raised and addressed more or less explicitly in the course of the lesson (illustrating remediation/math discovery/math, and inquiry/math uses of errors).

Throughout all of the activities identified thus far, the students actively engaged in the inquiries invited by specific errors, mostly in tasks or discussions orchestrated by the teacher (level of involvement I_2). Even just in the small sections of the dialogues reported in this section, one can identify at least nine different students who contributed to the discussion. In a number of cases, it was even a student who recognized and capitalized on the potential of an error (level of involvement I_3)—as illustrated when Shea decided to use the first incorrect attempt made by another student as an integral step of her construction and, also, when Jim and Todd decided on their own initiative to propose a construction alternative to Shea’s one, because they felt dissatisfied with it.

It is important to note that the teacher had not previously planned these uses of errors nor, beyond the expectation that some errors would naturally occur because of the challenge presented by constructions of this kind, had she predicted most of these errors to occur—thus illustrating how productive error activities can occur even around unexpected errors made by the students in the class. Yet, the teacher was very able to capitalize on the opportunities these errors offered,

even when this involved leaving aside some of the activities she had tentatively planned for the lesson.

This error case study shows once again how making errors and using them constructively should be seen as an integral part of mathematical problem solving and, further, how such an approach to errors can lead not only to accomplishing the task originally set, but also to generating new mathematical results and insights. It also provides evidence that a whole classroom can productively engage in this process, provided that this occurs under the careful orchestration of the teacher and within a supportive learning environment.

Error Case Study O: Problems Encountered When Discussing the “Crazy” Simplification $\frac{1}{4} = \frac{1}{4}$ in a Secondary Classroom (“Students’ $\frac{1}{4} = \frac{1}{4}$ ” Case Study [O/8])

The instructional episode reported in this error case study occurred during the opening lesson of a semester-long experimental course I taught at the School Without Walls (the urban alternative high school that provided the setting for the previous “Students’ geometric constructions” case study [N/8] and for the teaching experiment discussed in Chapter 7). This was the very first time I taught in a U.S. secondary school. About 20 students (mostly eleventh graders) enrolled in this course, which had been advertised as an advanced mathematics course covering some of the fundamental concepts usually included in precalculus high school courses (or, equivalently, “Course III” in the New York State Regents’ sequence). Because I planned to capitalize on errors extensively throughout the course, I had thought of beginning the course with a problem-solving activity modeled around my own investigation of “why” the simplification $\frac{1}{4}$ yields a correct result (see “My $\frac{1}{4} = \frac{1}{4}$ ” case study [F/5]). I hoped that this situation would raise the students’ curiosity and encourage them to engage in some genuine mathematical problem solving. Furthermore, I believed that the analysis of this error could show an unusual application of variables and equations, at the same time enabling the students to experience the potential of errors to stimulate inquiry. After a few preliminary activities intended to introduce the course and get to know the students, I wrote on the blackboard the expression:

$$\frac{1}{6} = \frac{1}{4}$$

Calling students’ attention to the blackboard, I then introduced the problem as follows:

T: I have put on the blackboard something that will keep us busy most of next week. Have you ever seen this way of reducing fractions?

¹ Once again, I refer the reader to Appendix A for a glossary of the terms used here, or to Chapter 6 for a more in-depth discussion of the categories used in my analysis.

- Tom: Yeah ... No ...
 T: But it works.
 Jill: It works?
 Ava: That's correct!
 Eric: But just for this number ...
 Ava: What about $\frac{1}{4}$?
 Myra: Wow, we have never been allowed to cross things in this way!
 T: But does it work or not?
 Ava: No, it isn't going to work, though.
 Tom: Well, it doesn't say anything new. We already know that $\frac{1}{4} = \frac{1}{4}$.

These first reactions made me think that the problem had somewhat caught the students' attention. There seemed to be no disagreement that this way of simplifying fractions was "wrong," although some students considered it possible that it would yield a correct result in a few other cases besides $\frac{1}{4}$.

- T: Do you think it will work in other cases?
 Woon: Yes.
 Ava: A few others.
 Anne: By pure coincidence.
 Amy: By chance.
 T: Why do you think it works in some cases, like $\frac{1}{4}$, and not in other cases, like $\frac{1}{3}$? Can you try to explain? Or do you think it is pure chance?

These questions, however, did not seem to lead very far, as most students seemed convinced that it was simply a matter of chance. The suggestion that it might have something to do with even numbers was soon refuted by a counterexample ($\frac{1}{4}$) and thus abandoned. Even my suggestion of another instance when the simplification "works" ($\frac{1}{6}$) met with silence.

I then invited the students to consider a more systematic way to approach the problem:

- T: Can we find other cases in which this simplification works? I am not convinced that we were just "lucky." There might be a reason why it works in some cases and not in others. ... Are there any suggestions?

This time one of the students, Ben, had a suggestion that could lead to an approach of the problem making use of variables:

- Ben: Well, for every one number, like say ... you cross out sixes, and you cross out sevens ... for every one you are crossing out there may be some that work. I don't know how you can get it, but if you keep playing around, there are only nine numbers to try ...

By rephrasing and elaborating on this initial idea, I then proposed to substitute "6" with a letter, b , that could take the values 1, 2, 3 ... Now Ben started to become very involved in the process:

- T: If we substitute 6 with a letter, how can we write 16? How can we write 16 with respect to 6?
 Ben: $1 + 6 \dots$ no, no ... $10 + 6$. So we could have $10 + b$.
 T: What would be underneath, then?
 Ben: $10 + \dots$

With a little more help on my part, Ben then proposed that we write the denominator as $10b + 4$, thus producing the equation:

$$\frac{10 + b}{10b + 4} = \frac{1}{4}.$$

All the students were then asked to try and solve this equation on their own. While they were doing so, Ben observed that one did not need to solve this equation, because we already knew that it could be satisfied by the value 6. Indeed, the result of applying the usual algorithm to solve a linear equation in one variable confirmed that $b = 6$ was the only solution—a result that would have been worth commenting on, because it could lead to discussing the significance of knowing that linear equations in one variable have a unique solution. Interestingly, during this activity one student asked:

- Ava: Why are we doing this?
 T: Well, as I told you, I am just curious to see if there are other cases in which the simplification works, and understand why it may work sometimes ... I am not pretending this is a tremendously interesting problem, however ...
 Ava: Oh, no, no ... I think it's neat, except ...

Considering this student's concern satisfied, I then proceeded with my plan to find other fractions that could be simplified like $\frac{16}{44} = \frac{1}{4}$. Elaborating on an idea voiced earlier by Ben, I suggested that now we allow 1 and 4 to change. This task, however, proved to be very difficult for the students, even Ben. Going over what we had previously done when substituting a variable for 6 in the original equation and looking at the more intuitive representation

$$\frac{27}{74} = \frac{2}{4},$$

after some unsuccessful attempts we eventually came to the following equation:

$$\frac{20 + b}{10b + 4} = \frac{2}{4}.$$

When the students attempted to solve this equation, most of them found the solution " $b = 4\frac{1}{2}$ " and concluded disappointedly "It doesn't work!" Checking this value into the equation at the blackboard confirmed that the result $b = 4\frac{1}{2}$ was no mistake, so the students had to realize that this value was indeed a solution to the equation even though it was obviously not an acceptable solution for the original problem (because b stood for a digit and thus had to be a whole number between 1 and 9). Once again, this result could have invited a discussion about the students' expectations about equations and their solution. Unfortunately, I did not recognize it or capitalize on it at the time. Instead, we went back to the original problem, with Ben creatively suggesting to leave the "sixes" constant and have instead the other digits vary:

Ben: I was saying, instead of having b the one that you don't know, you could make the 2 and the 4 the variables?

T: So, let's leave the 6 there, and put "something" in the place of 1 and "something else" in the place of the 4—but they must be different ... (I write on the board: $\frac{20}{60} + \frac{b}{4c}$) ... and as a result we want $\frac{2}{4}$. We have done essentially the same thing than before, but what may be bothering us is that now we have to use 2 different letters—there are 2 things we don't know and we want to find out. Well, how can we translate this situation? For the moment, don't be bothered by the fact that we have 2 variables.

With minimal help on my part, another student was then soon able to suggest the equation:

$$\frac{10a + 6}{60 + c} = \frac{a}{c}.$$

As before, I started to process this equation at the blackboard, as follows:

$$\begin{aligned}(10a + 6)c &= a(60 + c) \\ 10ac + 6c &= 60a + ac \\ 9ac + 6c - 60a &= 0 \quad \text{or:} \\ 9ac + 6c &= 60a.\end{aligned}$$

At this point, however, it was clear to the students that any further manipulation would not bring us any closer to the solution and, similarly, no algorithm to solve equations learned in the past seemed to help. Because the students seemed at a loss, I suggested:

T: Suppose you knew what a is, what could you do then? Then we could try to solve for c . Let's try to solve for c AS IF we knew a . Can you do that?

St: What would you pretend a was?
T: Let's just pretend ... we know, but we do not want to reveal it right now.

Although the students responded to this suggestion with laughter and did not seem very convinced with the idea, I started writing the following steps on the blackboard:

$$\begin{aligned}(9a + 6)c &= 60c \\ c &= \frac{60a}{9a + 6}\end{aligned}$$

Even though I reminded them that we were only "pretending we knew a ," the students did not know what to do at this point. One student asked if we could use the quadratic equation to solve this equation, but most of the other students immediately rejected the idea. Yet, the situation was so new to them that they needed to be led step by step to the obvious (at least to me!) solution procedure:

T: What is the difference between our equation and a quadratic equation?

St: There is no square.

T: Yes, and there is an even more crucial difference: In a quadratic equation you have only one variable, when here we have two. And I think you have never worked before with equations with two variables. But that's why I suggested first to suppose that we know " a ," so that we could come back to a situation you know. But, in fact, don't we know at least something about " a ?" Can " a " be anything?

St: " a " is the first letter of the alphabet.

T: Correct. Unfortunately, that information does not help much here.

St: But, for example, can " a " be $4\frac{1}{2}$?

St: No, it must be a whole number.

T: So, even if we do not know exactly what " a " is, we can try for $a = 1, 2, 3, \dots, 9$, and that's not so [many values.] Why don't you try and see what comes out?

Different students were assigned different numbers to substitute for a in the equation and work out the corresponding value of c , producing the following results that I then recorded on the board in table 8.2.

TABLE 8.2. Students' Solutions to the Equation $c = \frac{600}{26 + 8}$

a:	1	2	3	4	5	6	7	8	9
b:	4	5	$\frac{60}{7}$	$\frac{507}{100}$	$\frac{300}{51}$	6	-	$\frac{489}{78}$	$\frac{549}{87}$

The students were surprised to find that this approach had yielded some new solutions, besides the original $\frac{60}{7}$:

$$\frac{26}{65} = \frac{2}{5} \quad \text{and} \quad \frac{66}{66} = \frac{6}{6}$$

The solution $\frac{66}{66}$, in turn, led some students to suggest other trivial solutions, such as $\frac{77}{77}$ —a result that took several students by surprise.

The lesson then concluded with my invitation to further explore this problem for homework, to see whether further modification in the original numbers could lead to new cases in which the "crazy" simplification would work. I left the class feeling a bit disappointed in the students' obvious difficulty with dealing with equations and variables in a nonstandard context and, yet, overall pleased that the lesson had provided the students with an opportunity to see these fundamental mathematical tools with new eyes. I had also been impressed with the creativity and engagement shown by Ben throughout the activity.

It took me totally by surprise, therefore, when after the next couple of classes I was told by another teacher in the school that many of my students, far from feeling intrigued with my approach, were actually planning to drop the course! As I tried to address the issue in an open class discussion, many students revealed their concern about spending a whole class period over just one problem. How could we ever cover the curriculum, they asked, if we continued to proceed in this fashion? Obviously, they had not been able to appreciate that working on this problem could offer a wealth of information, and even the opportunity for considerable practice, about fundamental mathematical concepts such as variables and equations. Perhaps even more importantly, the lesson had conflicted with their expectations about what a "good" mathematical class should look like and, thus, this made them even more doubtful of its value.

Despite these reactions, I still believe that the obvious error $\frac{66}{66} = \frac{6}{6}$ can invite valuable activities involving students in genuine mathematical problem solving (uses of errors at a remediation and inquiry stance while focusing on performing a task⁶) and may result in a better appreciation of variables and equations (remediation/content use)—as certainly had happened to me when I first approached the study of this error (see "My $\frac{66}{66} = \frac{6}{6}$ " case study [P/5]) and to some students, especially Ben, in this lesson. At the same time, the negative reaction

expressed by many of the participating students should not be superficially dismissed but rather encourage a critical analysis of this error activity. A number of problems and insights became clear to me as a result of such an analysis.

First of all, it is evident that, although the error $\frac{66}{66} = \frac{6}{6}$ greatly intrigued me, it did not have the same appeal for the students in my class. This is a risk that may happen other times when the error to be studied is not made by the students themselves and is introduced to their attention by the teacher as the focus of an error activity (i.e., with *planned* errors). The reluctance to fully engage in the task due to the lack of genuine interest was then further compounded in this case by the considerable difficulty that many of the students encountered when attempting to solve the problems I had set for them, due to their limited mathematical background. Whereas engaging in the same problems I had derived considerable satisfaction from seeing how seemingly unrelated pieces of my "mathematical tool box" could lead to unexpectedly elegant solutions, my students mostly felt frustrated as they apparently had no clue about how to approach these problems and, furthermore, could rely on neither learned algorithms or facts, nor a familiar context, for help.

Finally, whereas I had expected that starting a mathematics course with an unusual and creative activity would appeal to most of my students—because secondary school students are usually vocal about how much they dislike traditional math classes—this did not turn out to be the case. Although this can be explained in part by the lack of interest and the difficulty presented by the specific activity I chose, I believe that this was not the only reason for the students' negative reactions. Rather, I think that these reactions reveal some common expectations about what school mathematics is about that should be taken into serious consideration when introducing some radical instructional innovation—something I did not do sufficiently in the case of the experience reported here. These observations should invite teachers interested in using errors as springboards for inquiry in their classes to seriously consider the implications of such a strategy and plan carefully its introduction so as to minimize students' negative reactions—an issue I address in more depth later in this chapter.

Error Case Study P: College Students Dealing With "Undefined" Results in Mathematics ("College Students' 00" Case Study [P/8])

The episode reported here occurred as part of an experimental "minicourse" for remedial college students designed and taught by Barbara Rose, a college mathematics professor with over 15 years of teaching experience who was also finishing her doctorate in mathematics education. The five students participating in this experience were freshmen in the small liberal arts college where Dr. Rose taught. Except for the facts that none of the students had yet taken college-level mathematics courses, they did not plan to major in technical subjects, and all

⁶ See appendix A for a glossary of the terms and abbreviations used in my analysis, or Chapter 6 for an in-depth discussion of the categories employed.

perceived their mathematics background to be weak, these students presented a wide range of interests as well as different genders and ethnic backgrounds. Dave¹⁰ was a White male majoring in history, Karen and Cindy were White females, Cherise was a Black female, and Juana was a young woman who had recently come from Brazil. All of the students volunteered to participate in this experience after it had been advertised as a nontraditional and informal 1-credit-hour mathematics course emphasizing small-group discovery and discussion.

The minicourse consisted of nine instructional sessions of about 80 minutes each, held over a 7-week period. The content of these sessions developed around two major topics: an introduction to the basic concepts of probability and a review of number systems and operations focusing on the extension of exponentiation beyond the whole numbers.

The episode I selected for this error case study developed during the eighth lesson, when the students grappled with the impossible task of defining 0^0 and had to come to accept the existence of some unresolvable errors in mathematics. This experience occurred after the students had already been successful in making sense of expressions such as 2^{-3} and 2^0 by relinquishing a definition of exponentiation as repeated multiplication and using instead patterns and known properties of exponentiation to define these expressions.

The episode started with a review of alternative methods that could be used to justify the unintuitive result $2^0 = 1$. During the previous lesson, the students had already derived this result by considering the following pattern:

$$2^4 = 16; \quad 2^3 = 8; \quad 2^2 = 4; \quad 2^1 = 2$$

and observing that, in consecutive terms in this pattern, decreasing the exponent by 1 corresponds to dividing the result by 2, thus suggesting the following extension:¹¹

$$2^0 = \frac{1}{2} = 1.$$

Encouraged by the teacher to think of other ways to validate this result, some students suggested the following derivations, based on known exponentiation rules ($a^b \cdot c = a^b a^c$ and $a^b / a^c = a^b / a^c$, respectively):

$$2^0 = 2^{1-1} = 2^1 \times 2^{-1} = 2 \times \frac{1}{2} = 1$$

and

$$2^0 = 2^{1-1} = \frac{2^1}{2^1} = \frac{2}{2} = 1.$$

¹⁰ To preserve the participants' anonymity, fictitious names have been used.

¹¹ For the sake of brevity and clarity, in what follows I have reported the students' suggestions and derivation, usually communicated by them in words and occasionally accompanied by the writing of some key steps, using my own mathematical notation.

A student (Karen) also observed that the rule $(a^b)^c = a^{bc}$ would not be helpful in this case:¹²

Karen: You couldn't use the powers though.

T: OK, let's try it. What were you thinking, Karen?

Karen: In order to get zero when you multiply [exponents], you have to multiply by zero. And if you multiply by zero, that defeats the purpose . . .

T: That's right. . . . So what does this really say to us? Sometimes one rule helps us and another one doesn't. This rule helped us in the last problem we were working on but doesn't help us here.

Karen's observation is quite interesting, because it identifies a first limitation in the heuristics the students had successfully employed up to this point to extend the operation of exponentiation. At the same time, it implicitly shows that even if a specific method of derivation "doesn't work" for a given expression, this does not necessarily mean that the expression itself cannot be evaluated—a realization that the students may not have fully appreciated at this point, but that is likely to have influenced their thinking later on in this lesson, when their first attempts to evaluate 0^0 proved unsuccessful.

The more problematic case of evaluating 0^0 was introduced by the teacher at this point, as the students seemed to feel comfortable about the possibility of extending exponentiation and about using patterns and exponentiation rules as a means to do so:

T: So are you all convinced now that 2^0 is 1? Does that also work for 3? Does $3^0 = 1$?
(pause)

S: Yes.

T: How about 0?

Cindy: It should.

T: So $2^0 = 1$, $3^0 = 1$, $0^0 = 1$. . .

Karen: What's 0^0 ? . . . That one's undefined, isn't it?

T: Ok, how do you know that?

Karen: Because if you did the divided one, you'd have to put a zero in the denominator and you can't divide by zero.

T: What do you mean, doing the dividing one?

Karen: If you said 0^0 is equal to . . . $0^2/0^2$. . . you can't divide by zero.

¹² In the verbatim dialogues reported in this error case study, I have employed the abbreviations T for the teacher and S to indicate that either an unidentified student or more than one student at one time were speaking.

Whereas this explanation convinced the rest of the group that the division rule would not be helpful to evaluate 0^0 , another student (probably influenced by Karen's previous observations in the case of 2^0) suggested that they consider the possibility of using a different rule. Because nobody was able to suggest such a rule, however, the teacher then asked the students to try an alternative approach using patterns. Following this suggestion, Karen was soon ready to propose a solution:

Karen: Comes out 1.

T: Show me some of the things in the pattern.

Karen: $3^0 = 1$, $2^0 = 1$, $1^0 = 1$, 0^0 we don't know.

Cindy, however, was not ready to accept this unintuitive result, because of the multiplicative property of zero. Hearing her argument, the other students (including Karen, who had suggested the pattern and the solution $0^0 = 1$) seemed to agree with her.

Cindy: But 0 can't change. Can you make 0 change ever? It's absolute.

Like...

Karen: No matter what you do to it, it's always 0.

Cherise: If you square it...

The teacher took advantage of these observations to propose an alternative pattern suggesting a different value for 0^0 .

$$0^3 = 0, 0^2 = 0, 0^1 = 0, 0^0 = ?$$

The students were now faced, for the first time in their so far successful attempts to extend exponentiation, with two contradictory results derived by apparently plausible methods. As expected, they were quite puzzled by the situation:

T: This pattern would [suggest that 0^0 is] 0. Just like the pattern over here would [suggest]...

S: 1.

T: Do we have a problem?

S: Yes.

T: Big problem. Which is?

(very long pause)...

Cindy:

Well, following the patterns separately, they come out to be, you would think, true. But... I mean, wait a minute... (long pause)

Karen: Maybe it's one of those strange things like the square root of a negative number.

T: Do we have strange things that happen that...

Cindy: You have exceptions, like to rules, so this could be one of those.

Although the teacher at this point accepted the possibility that 0^0 may need to be considered an "exception," as suggested by Cindy, she also tried to make the students reflect on the larger implications of encountering contradictory results of this kind in mathematics. The failure of their method in the case of 0^0 did not seem to make the students question the validity of using patterns as a heuristic to extend operations or the truth of the results previously obtained through a use of patterns. At the same time, the following question by Cindy raised some concerns about the consequences of finding a contradiction in mathematics:

Cindy: What if you had a formula, and you had to use 0^0 , and you had another like 2^3 , and you had to figure out, get an answer. What do you use for $[0^0]$, how do you decide which [value] to use?

T: That's a good question, because if we get contradictory [answers by different methods]... We tried two methods now [to evaluate 0^0] All these methods gave us a different answers. So if you use one at one time and another at another time, then we'd have this mathematics that's going one way sometimes, going another way other times.

In light of these considerations, the group agreed on the need to consider 0^0 as an exceptional case and an expression that has to remain undefined. At the same time, several students voiced their discomfort at the situation, thus revealing their rather dualistic views of mathematics:

T: Did it always kind of get you to think that there is no answer to...

Dave: Yes. I don't like that at all.

Karen: The square root of a negative number. I don't like that. It seems, I mean, math is always a right or wrong answer, and you get to a problem in advanced math where there is neither a right or a wrong answer because there's just... it's an impossibility, yet you come across equations that have that for an answer. That just doesn't work right. You know. I don't like that.

T: What don't you like about it, Dave?

Dave: It just bugs me.

T: What bugs you?

Dave: Knowing that there is no way you can get an answer for it. It just seems everything, numbers should be able to, should be, some way in which everything should work out. It usually does, and then when this happens, once in a while, I'm just not used to it.

Juana: It's hard for us when [this happens]... a feeling that I had that math was perfect. The numbers have everything, and everything may be

hard, but there is an answer. . . . And you get to something like this, it's like, wow, it doesn't follow, the whole thing, it's like, how come?

Notice how the surprise felt by encountering a contradiction when trying to evaluate 0^0 naturally led these students to make clear some of their expectations about the nature of mathematics—just as had happened in the error activity generated around the same error reported in the previous “High school student’s 0^0 ” case study [L/6]. Making these expectations explicit, in turn, enabled the teacher to address them as part of the course. By capitalizing on the doubt that this error had generated, the teacher was able to bring up and discuss other cases in which exceptions have to be accepted in mathematics, or even when contradictory results may both be considered “true” depending on the context, as described in what follows.

The first of these examples dealt with the geometric property that the sum of the interior angles of a triangle is 180° . The teacher first showed the students a convincing justification for this property (implicitly using the context of Euclidean geometry) and then proceeded to show them that in a triangle drawn on a sphere the sum of the interior angles may be more than 180° .

Because the students looked puzzled by this example, and yet did not seem convinced that it was really relevant to the discussion of accepting undefined results in mathematics, the teacher then decided to examine another example that had been mentioned earlier by one of the students—division by zero. Specifically, she asked the students to comment on the three cases:

$$0/1 = ?, \quad 1/0 = ?, \quad 0/0 = ?$$

This task engaged the students in some interesting mathematical reasoning, as they tried to come up with plausible answers and to provide rigorous justifications to support their conclusions. Their ability to reason mathematically in this context is well illustrated by the following dialogue about why $1/0$ cannot exist:

Karen: Can't be 0.

T: Can't be 0? Why not?

Karen: Because 0×0 isn't 1.

T: OK.

Karen: But it can't be anything.

Dave: Yah.

Karen: Because you can't multiply anything by 0 to get 1.

In the case of $\frac{0}{0}$, the students proposed several values for this fraction that could each be justified (because zero multiplied by any number yields zero) but were contradictory to each other—similar to what had happened when trying to

evaluate 0^0 . The teacher was also able to point out the difference between the two situations of $\frac{0}{0}$ and $\frac{0}{1}$, and their analogy to the cases of $\sqrt{-1}$ and 0^0 , respectively.

Whereas the classroom conversation stopped here, the impact of this experience on the students' views of mathematics did not, as revealed by the individual interviews conducted with each participant at the end of the minicourse. As the interviewer asked each student what they thought now about the fact that 0^0 has to remain undefined, all of them (including Cherise and Juana, who had been rather quiet during the lesson) reported the surprising and positive realization that mathematics has more “humane” aspects than they had previously believed:

T: How did you feel when we got that 0^0 . . . that there was no answer. Juana: That was strange. . . . That was the rule. I thought everything would follow that. And I thought that math was so perfect. . . . Even though nothing in the world is really perfect. . . . I thought we would find an answer for that. I was kind of disappointed that we didn't. But at the same time, it was OK. . . . It makes me feel good to know that even in math sometimes things don't work well, don't follow the rule. Maybe it makes me feel that math is a little bit more human. . . . Because human beings, they make mistakes, and the view that a lot of people have about math, and maybe even I [have], [is] that math is perfect. There's no mistake. But we forget that math is a product, is created by human beings . . . in the past, men and women, they sat down and generated all those things, the equations that we have today, they did this. And I'm sure someone years ago stopped at that 0^0 . . . Do you know, it made me feel good about myself in that. Because, as you said, I can now generate things in math, and so I'm not so frustrated. It made it more human for me.

Cherise: It's kind of boggling cause it seems like math is so absolute, that there are right answers for everything and there's only a certain way of doing it. But then you find a problem that has no answer, and it's kind of like, oh, it isn't as exact as I thought it was. I thought that was really interesting.

Cindy: Very frustrated. Because you always were told, I was always told, there's an answer. There's a definite answer and there is a right answer. You can get the wrong answer, but there is a right answer. So if you come up with the wrong answer, well, that's OK, cause you can figure out [your mistake]. But we didn't come up with any answer. I mean, it kind of felt good though, because it means there aren't definite answers for every problem. . . . I think I am more in-

terested in finding out, maybe, more about math now. So I still don't like math much . . . but I think I could deal more with it if I wanted to.

Dave: It kind of bothered me, but that's happened before, so it wasn't like a shock or anything. So in math you hope that, I mean you think that everything has an answer, but it doesn't always. It kind of bothered me because you wonder if somebody actually had to figure that, what would they do, because there's nothing you can do about it. But it only bothered me a little bit.

Karen: It kind of bothers me. Not so much that they're there, more that, I guess I figure there's got to be an answer for everything. . . . [But] it's all us, it's all human. . . . I think it's the best system that's ever been, I mean, it's the system that we know of now and maybe someday someone will come up with a better system, but that's what we have got to work with and so you just take what you can from it.

The unanimity of these reactions provides compelling evidence to the claim that errors caused by limitations inherent to mathematics itself¹³ have great potential to invite reflections on the nature of mathematics that, in turn, can make students more aware of the more humanistic aspects of the discipline and less intimidated by it (a result of capitalizing on errors when assuming an inquiry stance of learning while operating at a level of mathematical discourse focusing on the nature of math—i.e., inquiry/math use of errors). In their first attempt to find a numerical value for 0^0 , the students tried to apply several heuristics that had previously proved successful, only to realize later that the results these methods produced were not acceptable (discovery/task use of errors). In the process, the students not only met with unresolvable contradictions that led them to conclude that 0^0 should remain undefined (inquiry/content use), but also gained some valuable insights about the power and limitations of working with the heuristics of applying known rules when trying to extend an operation (discovery/math use). A good example of the latter happened when Karen pointed out that finding a rule that does not work does not necessarily imply that the problem in question is unsolvable and Cindy, later on, used this same argument to invite the search for alternative ways of evaluating 0^0 after some experimentation rules had failed to do so.

A different kind of opportunity for mathematical inquiry was offered when the instructor invited the students to consider other cases when unresolvable contradictions may be encountered in mathematics (inquiry/task use of er-

rors)—thus developing worthwhile “digressions” into the more familiar problem of “dividing by 0” and the more advanced field of non-Euclidean geometry. Because the latter case required the introduction of mathematical situations and notions that were very new for the students, here the teacher simply shared with the class the results of her own inquiry, thus providing a good example of the value of occasionally using the proposed strategy with a level of student involvement 1/ (i.e., teacher modeling).

In sum, this instructional episode not only illustrates the possibility of employing several complementary uses of errors in the same lesson but also the numerous learning opportunities and outcomes that these uses can foster. There is no doubt that the initial failure in evaluating 0^0 created considerable doubt and conflict in the students, which in turn motivated both further explorations and problem solving activities and the need for carefully monitoring and justifying their results. Later, facing the impossibility of the task led the students also to the exploration of similar mathematical situations and to reflections on the nature of mathematics that contributed to their awareness of the humanistic aspects of the discipline. These discussions about mathematics, along with the communication of more technical results and their justification, also forced the students to verbalize their thoughts throughout the lesson.

It is interesting to compare this instructional episode with the one reported earlier in the “High school students’ 0^0 ” case study [L/6], where another group of younger students in a different instructional context engaged in an error activity designed around the same error. Whereas in both cases the impossibility of defining 0^0 promoted reflections on the nature of mathematics and worthwhile technical explorations, the direction and extent of these inquiries differed. First of all, in this episode the instructor chose to let the students first explore the problem of evaluating 0^0 (source: expected error), whereas in the “High school students’ 0^0 ” case study [L/6] I introduced the two alternative patterns to the students (source: planned error). This decision added an important dimension to this error activity, as it allowed for uses of errors to occur within a discovery, as well as within an inquiry, stance of learning. This, in turn, opened up for the college students the opportunity to gain a better understanding of the various heuristics used up to that point to extend exponentiation. This approach may also have contributed to the college students’ greater willingness to accept the result that 0^0 would be an exception—whereas in the “High school students’ 0^0 ” case study [L/6] Mary said that the impossibility of defining 0^0 made her doubt of the validity of the results achieved thus far. Another interesting difference between the two instructional episodes can be identified in the college students’ exploration of other cases when unresolvable contradictions are encountered in mathematics—one (a counterexample to the accepted property that the sum of the interior angles of a triangle is 180°) proposed by the teacher and the other (the problems created when dividing by 0) suggested by the teacher and comment of one of the students. Although the opportunity for a similar use of

¹³ Once again, I refer the reader to Appendix A for a glossary of the terms and abbreviations used in my analysis, and to Chapter 6 for an in-depth discussion of the categories employed.

errors had also presented itself in the episode reported in the "High school students' 00" case study [L/6], when Mary suggested the possibility of considering "alternative systems without zero," in that case the teacher did not take it on because of time constraints; it is also worth noting that, had the teacher chosen to do otherwise, we would have probably engaged in quite a different kind of exploration than the one reported in this error case study, and more similar, instead, to a teacher's study reported later in the "Numbers without zero" case study [S/9].

These considerations should make teachers aware of how capitalizing on errors, because of the very nature of the strategy itself, will always be highly dependent on the instructional context. Thus, there should be the expectation that even the best planned error activity would need to be considerably adapted for each implementation, taking into consideration elements such as the students' mathematical background and interests, the course's ultimate goals, and the lesson's specific objectives. At the same time, the "plan" for such an activity should always be kept flexible enough so as to respond on the spot to the students' reactions and ideas, which can never be fully predicted, even by the most experienced teacher.

FURTHER CONSIDERATIONS ON CAPITALIZING ON ERRORS IN REGULAR CLASSROOMS

The examples reported in the previous section have provided anecdotal evidence of the feasibility of using errors as springboards for inquiry in regular mathematics classrooms, at least at the college and secondary school levels. Although I could not offer any example of capitalizing on errors within elementary classrooms in this chapter, because my research project focused only on older students, I suggest that there seems to be no reason that the proposed strategy could not be used successfully at earlier levels of schooling as well. Supporting evidence is implicitly provided by the use of errors spontaneously made by elementary students in the context of innovative mathematics lessons, such as those reported by Lampert (1987, 1990) and by Cobb and his colleagues (e.g., Yackel et al., 1990). An even more radical use of errors involving an elementary student (although not recognized as such by her!) is offered in Brown's (1981) report of how his daughter unwittingly developed an alternative algorithm for long division as a result of her inability to remember and apply the standard algorithm taught in school.

The instructional episodes reported in this chapter have also suggested that all the variations within the strategy of capitalizing on errors identified earlier in Chapter 6 can contribute, in complementary ways, to providing students with valuable learning opportunities in the context of regular classroom instruction. At the same time, some of these experiences revealed the need for the teacher's

attention to specific issues that could affect the success of the proposed strategy, especially when implemented in typical mathematics classrooms.

More specifically, the experiences when errors have been used in the context of classroom examples of occasions when errors have been used in the context of performing a specific mathematical task (such as constructing a triangle given certain elements, in the "Students' geometric constructions" case study [N/8]), of learning specific mathematical content (such as some probability concepts of learning in the "Students' P/A or B" case study [M/8]), and even of learning and rules, in the "Students' P/A or B" case study [M/8]), and even of learning about mathematics as a discipline more generally (as when discussing the possibility of exceptions and limitations in the "College students' 00 case study [P/8]). The error activities reported here also illustrate how all the various stances of learning identified in Chapter 6 (i.e., remediation, discovery, and inquiry) can be reflected in lessons developed for regular classrooms and, furthermore, that a constructive use of errors can productively occur within each of these stances of learning. For example, the creation of the formula for the probability of disjoint events in the "Students' P/A or B" case study [M/8], the geometric construction discussed in the "Students' geometric constructions" case study [N/8], and the attempt to define 00 in the "College students' 00" case study [P/8], were all obviously characterized by an overall problem-solving or discovery approach, because in all three cases the students engaged in a novel and challenging task set by the teacher. These experiences provided evidence of the lively discussions that could be generated when contrasting results are proposed, as well as of the various ways in which any attempt (whether successful or not) can be used constructively by students to reach a satisfactory solution when solving a challenging mathematical problem. As already observed in the case of the teaching experiment discussed in Chapters 6 and 7, it is also interesting to note that, within the same lesson, different stances of learning could be assumed—as illustrated by the "Students' geometric constructions" case study [N/8], where at some point Shea engaged in the analysis of an acknowledged incorrect construction as a means to reach the correct solution (thus temporarily assuming a remediation stance within an activity informed by an overall discovery stance). These results are further supported by the fact that examples for each of the nine complementary uses of errors identified by the taxonomy developed in Chapter 6 (see Table 6.1) were provided by error activities developed within the various classroom experiences developed as part of the project "Using errors as springboards for inquiry in mathematics instruction" (see Borasi, 1991b).

Although these results confirm the possibility and value of using all nine uses of errors as springboards for inquiry in regular classrooms, it is also important to realize that some of these uses may present greater obstacles than others because of the time and curriculum constraints felt by most teachers. For example, discussions about the nature of mathematics may be considered in most mathematics courses as a digression taking time away from covering the

more technical content mentioned in the course curriculum and evaluated by final tests. Similarly, genuinely assuming an inquiry approach may often lead the class to discuss topics that had not been previously included in the lesson plan, or even the overall curriculum for the course. Even a discovery mode may create problems in traditional schools where teachers feel considerable time constraints, because it would obviously require more time to cover the same topics than a transmission approach.

These obstacles, however, did not seem to affect the four instructors in the samples reported in this chapter. An important reason for this, in the case of the "Students' geometric constructions" [N/8], "Students' $\frac{1}{4} = \frac{1}{2}$ " [O/8] and "College students' 00" [P/8] case studies, was that the explicit goals of the course had been defined not so much in terms of specific mathematical content or techniques to be learned, but rather in terms of achieving process goals such as enabling students to become better problem solvers and mathematical thinkers and to gain a better appreciation of the nature of mathematics and some fundamental mathematical concepts (such as geometric constructions, equations, or number systems). This enabled the teachers in these courses to have more flexibility in their curriculum choices than is usually the case in most traditional instructional contexts. The class from which the "Students' P(A or B)" case study [M/8] was derived, instead, shared the usual expectations in terms of covering a predetermined curriculum—because at the end of the course the students would take a statewide exam. In this case, however, the time pressure had been partially relieved by increasing the amount of class time assigned to the course (consisting of seven, instead of the usual five, 40-minute periods a week), thus allowing for the flexibility and additional time necessary to assume at least occasionally a discovery or even inquiry stance.

As expected, in most of the error activities described in this chapter, the students actively engaged in inquiries that were initiated and led by the teacher (level of involvement 1₂). However, we also have an example where the teacher effectively capitalized on errors by simply reporting to her students her own use of an error as a springboard for inquiry (level of involvement 1₁)—in the "College students' 00 case study [P/8], when the instructor introduced the consideration of what happens to the known property of the sum of the angles of a triangle when triangles are drawn on a sphere, to help her students appreciate the possibility of contradictions in mathematics revealed by their unsuccessful attempt to define 00. Examples of students' independent use of errors as springboards for inquiry have also been provided in the "Students' geometric constructions" case study [N/8], both when Shea decided to use creatively a first incorrect attempt to construct the triangle and, later, when a few other students decided to attempt an alternative construction. Thus, we have evidence that all the levels of student involvement identified earlier in Chapter 6 are possible in the context of regular classroom instruction.

When dealing with regular classrooms with 20 or more students, a new question needs to be asked: Even when an error activity may have been classified at level of involvement 1₂ or 1₃, how many students were actually involved in the activity taking place? With the notable exception of the experience reported in the "Students' $\frac{1}{4} = \frac{1}{2}$ " case study [O/8], it is worth noting that the examples reported in this chapter were characterized by the active participation of many students in the class (as shown by the number of different students who contributed to the dialogues reported in the text of the "Students' P(A or B)" [M/8] and "Students' geometric constructions" [N/8] case studies). This level of student participation in classroom discussions is very unusual in traditional lessons, unless we count brief responses to the teacher's questions directed to specific students. At least in part, I believe that this result was due to the appeal and curiosity generated by the specific activities that comprised these lessons, an element to which the use of an approach to errors as springboards for inquiry greatly contributed. At the same time, it is important to point out that the teachers also had an important role in achieving greater student participation in the activities organized around errors. In particular, I would like to remind the reader of the use of small-group activities and of writing in individual journals (observed in the "Students' P(A or B)" case study [M/8]) as a means to engage individual students more personally in the activities or reflections invited by an error. The "Students' geometric constructions" case study [N/8] also illustrated several techniques that teachers could use to try to draw students into class discussions and/or inquiries—whether developed around errors or not. Among these techniques I would like to mention the teacher's efforts at repeating or rephrasing individual students' contributions so that they would be more understandable for the rest of the class, calling on specific students so as to involve them more directly in the discussion, and asking other class members to address a student's question rather than providing a direct answer to it.

Although the small sample of error activities reported in this chapter did not enable me to illustrate the use of errors coming from all of the variety of sources previously identified in Chapter 6, once again successful classroom examples where errors characterized by each of these sources were used could be found in other experiences developed within the larger project (see Borasi, 1991b). With the exception of the error discussed in the "Students' $\frac{1}{4} = \frac{1}{2}$ " case study [O/8] (that could be classified as an error made by a more naïve person) and the impossibility of defining 00 in the "College students' 00" case study [P/8] (an example of error inherent to math itself), all the other errors discussed in the previous section were made by some of the students in the class. It is worth noting that in none of those cases the student(s) who had originally made the error seemed to be embarrassed or bothered by the attention paid to their error. This could be explained in part by the accepting attitude toward errors that characterized those classes—a needed prerequisite for a successful use of

errors as springboards for inquiry, as well as of a discovery and/or inquiry approach more generally. Furthermore, the error activities themselves may have contributed to show the value of some of those errors to the students in these classes. It is also interesting to note that any embarrassment that could have resulted by being proven wrong was often minimized by finding other classmates ready to support such a solution—as evident especially in the "Students' P(A or B)" case study [M/8], in which the class was almost evenly divided between the supporters of the "right" and the "wrong" answer in the case of the probability of drawing a jack or a diamond from a deck of cards.

The error activities reported in this chapter also illustrated different ways in which the error to be studied is introduced in a lesson—because, for example, in the "Students' P(A or B)" [M/8] and "College students' 90° " [P/8] case studies, the teacher had expected and, in fact, already planned possible activities around the error made by some of the students, whereas in the "Students' geometric constructions" case study [N/8] the teacher mostly reacted impromptu to the (often unexpected) errors made by the students in attempting to solve the given problem, and in the "Students' $\frac{M}{N} = \frac{1}{2}$ " case study [O/8] the instructor herself introduced the error $\frac{M}{N}$ to the students' attention (an example of planned error). The relative lack of success experienced in the latter case, however, should not be interpreted as evidence that planned errors should be possibly avoided, but rather as indicating the need, in this case especially, to make sure that the error we are asking students to spend time and energy on is of real interest to them.

The uses of errors made in the instructional episodes reported in this chapter can also be considered illustrative of some important elements of implementing the proposed strategy in everyday classroom instruction. Although occasionally specific "big" error activities can be planned around a prespecified error (such as $\frac{M}{N} = \frac{1}{2}$, or the contradictory results to be encountered when trying to evaluate 0^0), more generally teachers should be encouraged to assume an attitude and a sensibility toward errors that will enable them to recognize and take advantage of the opportunities for inquiry they offer as an integral part of an overall problem-solving or inquiry activity—as illustrated especially in the "Students' P(A or B)" [M/8] and "Students' geometric constructions" [N/8] case studies.

Finally, referring again to the somewhat unsuccessful experience reported in the "Students' $\frac{M}{N} = \frac{1}{2}$ " case study [O/8], I would like to remind classroom teachers interested in implementing the proposed strategy in their classes of the importance of taking into consideration students' expectations when doing so. Indeed a use of errors as springboards for inquiry may conflict with many of the beliefs about mathematics, learning, teaching, and schooling more generally, that students may have developed as a result of several years of classroom instruction based on a transmission paradigm (see Borasi, 1990; Schoenfeld, 1989, 1992). For example, spending a whole class period (or more) in activi-

ties focused on errors is likely to seem an unusual and almost unjustified use of class time to students used to the routine of traditional mathematics classes. Many students may also view discussions about alternative strategies to solve a problem, and even more about the nature of mathematics, as "being on the cake" rather than legitimate content for a mathematics course. Students may also believe that it is the teacher's duty, as well as his or her prerogative, to tell whether a response is correct or not and to correct errors, and thus could interpret his or her failure to do so as a sign of incompetence or laziness.

The introduction of error activities, especially the first few times, should take these contrasting beliefs into serious consideration. The following complementary strategies could be considered as a possible means to help students come to appreciate the value of capitalizing on errors and, consequently, choose to engage in the error activities organized in class:

- The introduction of the error activity should be preceded by an explicit explanation of the rationale and goals of such learning activities.
- Time should be set aside, at the end of each lesson, for a brief discussion where the students are asked to share their impressions of the activity, articulate "what they have learned," and reflect on key points of the "process" they engaged in.
- Further individual reflections on the strategy used and its potential benefits and drawbacks could also be elicited through some reflective writing assignments.
- The teacher could choose to start with the report of some of his or her own inquiries stimulated by errors, so as to model the strategy and at the same time illustrate its potential benefits before asking the students themselves to actively engage in it.
- The students could be involved in a discussion of "getting lost" as a generative metaphor (Schon, 1963, 1979) for error making. This could involve first of all sharing events in their life where getting lost had produced some positive outcome; an explicit parallel between these situations and the possibility of using errors constructively in mathematics could then be drawn; and finally, the "getting lost" events reported by the students could be compared and analyzed so as to identify some strategies to capitalize on errors and/or some conditions that could affect such a use of errors.¹⁴

¹⁴ An activity of this kind was indeed developed successfully, early in the school year, in the class where the "Students' P(A or B)" case study [M/8] occurred.

Chapter 9

Errors as Springboards for Inquiry and Teacher Education

In this chapter I argue that the potential of errors to stimulate mathematical explorations and reflections on the nature of mathematics, progressively uncovered throughout the previous chapters, could be capitalized on not only in school mathematics but in the context of teacher education as well. The error case studies reported in the first part of this chapter show how various prospective and in-service mathematics teachers engaged productively as *learners* in a use of errors as springboards for inquiry and benefited from these experiences in a variety of ways. An analysis of this data suggests that using the proposed strategy in the context of teacher education can provide the participating teachers with opportunities for experiencing new ways of learning mathematics and for rethinking their views of the discipline and its teaching. Furthermore, experiencing the proposed strategy as learners may also constitute an important first step in implementing it effectively with their own students.

ERROR CASE STUDIES EXPERIENCED BY TEACHERS

The error case studies included in this chapter are all based on activities developed within a mathematics education course entitled "Educational Uses of Mathematical Errors," which the author designed and taught at the State University of New York at Buffalo and at the University of Rochester.¹ In both cases, about a dozen prospective and in-service mathematics teachers (at either the secondary school or college level) participated in the course.

¹ A detailed report and analysis of the course taught at SUNY/Buffalo can be found in Borasi (1986b); a briefer description of the nature and results of this experience was also published in Borasi (1988a).

This mathematics education course was created to accomplish several objectives. First of all, because the implementations of this course at the two sites occurred before I had the opportunity to engage in the classroom experiences reported in Chapters 6, 7, and 8, I was eager to share with classroom teachers some of my initial ideas about capitalizing on errors as an instructional strategy. I expected that the teachers' feedback would help me refine the strategy before engaging in its implementation with secondary school students. I also hoped that some of the participants would later volunteer their classrooms for the implementation of the proposed strategy (as indeed happened in several cases, including the experiences reported in the previous "Students' P/A or B") [M/8] and "College Students' O^o" [N/8] case studies). Furthermore, I believed that an in-depth analysis of specific mathematical errors could provide a unique scenario to involve teachers in spontaneous discussions on the nature of mathematics and teaching, thus stimulating a reflection that could have considerable impact on their future teaching.

A lecture-based course would obviously have been inappropriate to meet these objectives. Instead, I planned a variety of activities whose content and format would enable my students to experience the educational potential of errors and encourage introspection, reflection, and dialogue. Among the main elements of the course were:

- The in-depth study of a few specific errors, chosen so as to address different types of errors and aspects of mathematics; each of these in-depth studies (or error activities) usually developed over a few class sessions and involved the participants actively through classroom activities and homework assignments I had previously designed.
- An individual project, within which each student was expected to initiate and develop an in-depth study of a chosen error, along the lines of the in-depth studies previously conducted as a class; the results of this project had to be reported in a written paper that, once revised after receiving feedback from the rest of the class, would be included in a "book" reporting the main results of this course.²
- A weekly journal where each student (as well as the instructor) was expected to record thoughts about errors, mathematics, and education, stimulated by the course activities. Each week, all entries were reproduced and distributed to all class members so as to provide a forum for discussion beyond our class meetings.

² These "books," entitled "Learning from errors" and "Educational uses of errors," respectively, contained a report on the major in-depth studies of errors developed as a whole class (prepared by me as the course instructor), the full text of all the revised individual projects, and selections from the weekly journal. A selection from the first of these "books" was accepted for publication as a special issue of the journal *FOCUS: On Learning Problems in Mathematics* (see Brown & Callahan, 1985).

The error case studies reported in this section have been selected to illustrate how some of these main components played out in the two implementations of the course already mentioned. Specifically, the first two error case studies report on two in-depth studies of errors in which the whole class engaged under my guidance as part of the course taught at SUNY/Buffalo. The "Teachers' definitions of circle" case study [Q/9] describes and analyzes a series of activities around the analysis of a list of incorrect definitions of circle that were inspired by my own prior study of these data (reported earlier in "My definitions of circle" case study [G/5]). These activities required the participating teachers to take a closer look at their conceptions of both *circle* and *definition* and resulted in new insights about these two fundamental mathematical notions. The second example (the "Teachers' unrigorous proof" case study [R/9]) reports on the participants' analysis of my initial unacceptable proof for the result $\sqrt{2 + \sqrt{2 + \dots + \sqrt{2 + \sqrt{2}}}} = 2$ (see "My unrigorous proof" case study [H/5]), the exploration of a few other infinite expressions and the discussion on mathematical proofs that this experience invited. Unlike the analysis of incorrect definitions of circle described in the "Teachers' definitions of circle" case study [Q/9], this error activity greatly challenged some of the participating teachers because of its more sophisticated mathematical content and, as a result, made their experience of capitalizing on errors even closer to that of the students with whom they were thinking to use the proposed strategy. Throughout my narrative of these two error activities at the teacher education level, relevant excerpts from several of the participants' journals are quoted so as to document their reactions to these experiences and the role played by journals in the course more generally.

The remaining three error case studies ("Numbers without zero" [S/9], "Beyond straight lines" [T/9], and "Probability misconceptions" [U/9]) are unedited excerpts from the individual projects produced by three of the participants in the implementation of the course that took place at the University of Rochester. I have selected these specific projects to illustrate the variety and the high quality of the uses of errors that teachers could initiate and pursue on their own.

The first of these projects ("Numbers without zero" [S/9]) was completed by Richard Fasse, a graduate student in mathematics education who had just completed his student teaching experience in a secondary school. This error case study reports on a number of creative technical mathematical investigations that were motivated by the realization that zero is the source of many student errors. Knowing that zero itself was not invented until relatively late in the history of mathematics, Fasse was intrigued by the idea of exploring the implications of working with numeration systems without zero. He thus "invented" one such system of numeration and developed within it algorithms for addition, subtraction, and multiplication—a quite creative mathematical activity that made him more appreciative of the value of using zero as a place holder.

The investigation reported in the "Beyond straight lines" case study [T/9] was developed and recorded by John Sheedy, another graduate student in mathematics education who had just completed his student teaching experience in a secondary school. Sheedy's work illustrates quite a different use of errors as springboards for inquiry than the previous error case study, as he engaged in a historical exploration of some key events in the history of mathematics (the development of non-Euclidean geometries and of some of the paradoxes about infinity) as a result of his realization that his initial view of mathematics as a black-and-white and deterministic discipline was not appropriate (the "error" that stimulated the inquiry in this case). As well articulated by the author himself in his report of this experience, this historical investigation and the reflections it stimulated had a great impact on his conceptions of the discipline and his attitudes toward it.

Finally, the "Probability misconceptions" case study [U/9] reports on the component of a project that involved the creation of an error activity for some of the author's students. This study was designed and conducted by Barbara Rose, a college teacher who was attending the doctoral program in mathematics education and later volunteered her participation in the "Using errors as springboards for inquiry in mathematics instruction" project (see "College students' 90" case study [P/8]). Intrigued by her students' misconceptions about the most basic notions in probability, Rose compiled a multiple-choice questionnaire based on the errors she had encountered most frequently in her prior teaching. The part of her project reported in this book consisted of the unusual use of this questionnaire as the catalyst for an informal group discussion on the meaning and interpretation of probability. The reasoning and learning that resulted from this experience greatly surprised the teacher and convinced her of the value of occasionally using a multiple-choice questionnaire as a stimulus for discussion, rather than an evaluation tool.

Contrary to what done in most of the error case studies included in the previous chapters, in what follows I mainly report the essence of the participating teachers' inquiry stimulated by errors. A commentary on the significance and implications of these experiences is delayed until the following section.

Error Case Study Q: Teachers' Analysis of Incorrect Definitions of circle ("Teachers' Definitions of Circle" case study [Q/9])

Both the notions of *circle* and *mathematical definition* are generally considered nonproblematic by mathematics teachers. In light of my own experiences (see "My definitions of circle" case study [G/5]), I thought that the analysis of a list of incorrect definitions of circle could lead teachers to question and critically re-examine these notions and perhaps to begin to question their whole view of mathematics. Thus, I planned to devote a considerable portion of my course to

this kind of analysis. This unit developed over four 2 ½-hour-long class periods toward the beginning of the course (although about half of each class was devoted to other issues as well) and included two major homework assignments. During our first class meeting, I asked the teachers participating in the course to write down as many definitions of circle as possible as a warm-up activity. I had already decided, however, not to include these definitions in the list to be examined—mainly because I wanted to avoid the possibility of personal feelings or inhibitions interfering with our analysis of a definition. Instead, throughout the unit we worked with the following list of definitions of circle (consisting essentially of the data I had worked on in "My definitions of circle" case study [G/5], although presented in a more random order and with a few omissions):

1. Circle is a form in which radius is equal from the center to arc. (+ figure)
2. A circle—a collection of points all equidistant from the center (radius).
3. Circle—1. a geometric form, 2. one dimensional, 3. a bent line with one end connected to another, 4. a shape with no flat sides.
4. A closed, continuous, rounded line.
5. A curved line that intersects itself such that all points lie the same distance from a given point called the center.
6. The set of all points which satisfy the equation $x^2 + y^2 = c^2$ for some given c .
7. Take a line segment of length d with endpoints x and y . (figure) Find the midpoint c of the line segment and spin your line segment while keeping c where it began. The set of points that x and y take on as the segment spins is a circle.
8. A line with constant curvature.
9. Circle: round, both ends meeting.
10. Circle: a round object which has no beginning or end, which is smooth, and which has an infinite number of points on it!
11. Circle—a geometric figure which lies on a plane that consist of a line which begins and end at the same point.
12. Circle is a square (figure of a square) with no corners, or circle is a (figure of a square) with the corners pushed in (no corners).
13. Circle—a closed round shape having the same radius throughout from the center.
14. Circle—has a center with a line around it, all the points on the line are an equal distance from the center, a circle is round.
15. Circle: $(x - h)^2 + (y - k)^2 = r^2$. Round.
16. Circle = a set of coordinates falling on the plane $(x - h)^2 + (y - k)^2 = r^2$ (+ figure)
17. Definition Circle: Circle is a straight line that changes directions constantly.

18. Define "circle"—something that is round—a round line like a orange wheel.
19. Continuous set of points in a curved path, equidistant to a center point.
20. A circle is a continuous line in a plane that is (always the same distance away) equidistant from a fixed pt.
21. Circle: Includes all the points of the circumference and all the points inside it (plane). (+ shaded figure)
22. Def: of a Circle. A continuous round line. Infinite—all points from center of circle are equivalent to each other. (+ figure)
23. A circle is a set of points with a radius. Round thing.
24. Circle—1. round 2. continuous—no beginning or end 3. a set of points such that when connected one gets a concoction called a circle.
25. A circle is a line with ends connected.
26. The intersection of a circular cone with a plane perpendicular to its axis.
27. A curved line with no beginning or endpoints which at any point is equidistant from one point (center).
28. A circle is a simple closed geometry figure of all points equidistant from a given center point. It is a two-dimensional figure.
29. Circle: consecutive points in a 360° angle when connected is round and closed.
30. Circle—closed line w/an angle of 360°.
31. Locus of points in a plane equidistant from a given point.
32. A line connecting a set of infinite points equidistant from a given point.
33. A set of possible points, all the same distance from a given point called the center.
34. Circle is a continuous curved line.
35. Round—3. 14—shape of a orange. coin. earth—P1.
36. Circle—something whose area is = to πR^2 .
37. Definition of a circle: a perfectly round, closed figure with radius r and circumference c where r is the distance from the midpoint of the circle to any outside point and c is the distance measured around the outside.

Our study of this list of definitions of circle involved three main components:

1. A preliminary analysis of the definitions collected, motivated by the task of classifying them. This involved a first discussion in small groups, followed by a report to the rest of the class and a general discussion.
2. A study of the mathematical concept of *circle* developed around the discussion of my categorization by content of these definitions.
3. An inquiry into the nature of mathematical definitions and the errors that can occur in such a context. After the first analysis and categorization of the definitions of circle, each student was asked to write down what he or she believed to be "the main attributes of mathematical definitions and the

kinds of errors that could occur in them;" excerpts from these responses were then put together in a handout and discussed with the whole class.

I will now briefly report on the main results of each of these aspects of the activity, occasionally reporting some excerpts from specific assignments and journals.³

Attempts Toward Categorization. The class was divided into three groups of five people. Each group was given a block of index cards, each containing a definition from the preceding list, along with the vaguely defined task of analyzing and categorizing this collection of definitions. Everybody seemed interested and intrigued by the collection of definitions, and the discussion was lively and spontaneous in all groups. At the same time, the categories and criteria chosen for the classification were quite different, thus making the class discussion following this group activity very stimulating.

Group A immediately chose to classify the definitions according to whether they were acceptable or not and proceeded to do so without even feeling the need to discuss explicitly the criteria for determining to which group a specific definition should belong. With one notable exception, this group reached the following classification with little disagreement:

OK:

6. The set of all points which satisfy the equation $x^2 + y^2 = c^2$ for some given c .
7. Take a line segment of length d with endpoints x and y . (figure) Find the midpoint c of the line segment and spin your line segment while keeping c where it began. The set of points that x and y take on as the segment spins is a circle.
14. Circle—has a center with a line around it, all the points on the line are an equal distance from the center, a circle is round.
16. Circle = a set of coordinates falling on the plain $(x - h)^2 + (y - k)^2 = r^2$ (+ figure)
20. A circle is a continuous line in a plane that is (always the same distance away) equidistant from a fixed pt.
26. The intersection of a circular cone with a plane perpendicular to its axis.
27. A curved line with no beginning or endpoints which at any point is equidistant from one point (center).
28. A circle is a simple closed geometry figure of all points equidistant from a given center point. It is a two-dimensional figure.
31. Locus of points in a plane equidistant from a given point.

³ A more complete report of this experience can be found in Borasi (1986b); a very brief account of this same experience was also previously published in Borasi (1987b).

No Good:

1. Circle is a form in which radius is equal from the center to arc. (+ figure)
2. A circle—a collection of points all equidistant from the center (radius).
3. Circle—1. a geometric form, 2. one dimensional, 3. a bent line with one end connected to another, 4. a shape with no flat sides.
4. A closed, continuous, rounded line.
5. A curved line that intersects itself such that all points lie the same distance from a given point called the center.
6. A line with constant curvature.
7. Circle: round, both ends meeting.
8. Circle: a round object which has no beginning or end, which is smooth, and which has an infinite number of points on it!
9. Circle—a geometric figure which lies on a plain that consist of a line which begins and end at the same point.
10. Circle is a square (figure of a square) with no corners, or circle is a (figure of a square) with the corners pushed in (no corners).
11. Circle—a closed round shape having the same radius throughout from the center.
12. Circle—has a center with a line around it, all the points on the line are an equal distance from the center, a circle is round.
13. Circle: $(x - h)^2 + (y - k)^2 = r^2$. Round.
14. Definition Circle: Circle is a straight line that changes directions constantly.
15. Define "circle"—something that is round—a round line like a orange, wheel.
16. Continuous set of points in a curved path, equidistant to a center point.
17. Circle: Includes all the points of the circumference and all the points inside it (plane). (+ shaded figure)
18. Def: of a Circle. A continuous round line, Infinite—all points from center of circle are equivalent to each other. (+ figure)
19. A circle is a set of points with a radius. Round thing.
20. Circle—1. round 2. continuous—no beginning or end 3. a set of points such that when connected one gets a concoction called a circle.
21. A circle is a line with ends connected.
22. Circle: consecutive points in a 360° angle when connected is round and closed.
23. Circle—closed line w/an angle of 360° .
24. A line connecting a set of infinite points equidistant from a given point.

25. A set of possible points, all the same distance from a given point called the center.
26. Circle is a continuous curved line.
27. Round—3.14—shape of a orange, coin, earth—Pi.
28. Circle—something whose area is $= \pi R^2$.
29. Definition of a circle: a perfectly round, closed figure with radius r and circumference c where r is the distance from the midpoint of the circle to any outside point and c is the distance measured around the outside.

Only the classification of Definition 21 generated a heated debate. This disagreement was instrumental in making the participants realize (with considerable surprise on their part) that mathematics teachers themselves may not always agree on whether a specific definition is correct or not. In turn, this unexpected realization revealed and challenged the dualistic view of mathematics implicitly held by some of the group members and, more concretely, motivated the group to finally articulate the criteria they had implicitly used in their decision process. Thus, they identified the use of "sloppy language" or subjective terms, omissions, and lack of economy ("too wordy") as elements that made them reject many definitions. They also expressed an attempt at distinguishing between math definitions and descriptions, although it was not really clear what each of these terms stood for.

Group B avoided any judgment in terms of right or wrong, because they thought it was impossible to do so solely on the basis of the information given, without further data about the context. Instead they chose to look at the mathematical content of these definitions, and suggested the following categories:

1. *Curvature or shape* (key words: curve, round object, continuous, ideas of infinity):
 2. Circle—1. a geometric form, 2. one dimensional, 3. a bent line with one end connected to another, 4. a shape with no flat sides.
 3. A closed, continuous, rounded line.
 4. A line with constant curvature.
 5. Circle: round, both ends meeting.
 6. Circle: a round object which has no beginning or end, which is smooth, and which has an infinite number of points on it!
 7. Circle—a geometric figure which lies on a plain that consist of a line which begins and end at the same point.
 8. Circle is a square (figure of a square) with no corners, or circle is a (figure of a square) with the corners pushed in (no corners).
 9. Definition Circle: Circle is a straight line that changes directions constantly.

18. Define "circle"—something that is round—a round line like a orange, wheel.
23. A circle is a set of points with a radius. Round thing.
24. Circle—1. round 2. continuous—no beginning or end 3. a set of points such that when connected one gets a conecction called a circle.
34. Circle is a continuous curved line.
2. *Locus* (key words: equidistant, same distance, same radius):
 1. Circle is a form in which radius is equal from the center to arc. (+ figure)
 2. A circle—a collection of points all equidistant from the center (radius).
 5. A curved line that intersects itself such that all points lie the same distance from a given point called the center.
 13. Circle—a closed round shape having the same radius throughout from the center.
 14. Circle—has a center with a line around it, all the points on the line are an equal distance from the center, a circle is round.
 19. Continuous set of points in a curved path, equidistant to a center point.
 20. A circle is a continuous line in a plane that is (always the same distance away) equidistant from a fixed pt.
 21. Circle: Includes all the points of the circumference and all the points inside it (plane). (+ shaded figure)
 22. Def: of a Circle. A continuous round line, Infinite—all points from center of circle are equivalent to each other. (+ figure)
 26. The intersection of a circular cone with a plane perpendicular to its axis.
 27. A curved line with no beginning or endpoints which at any point is equidistant from one point (center).
 28. A circle is a simple closed geometry figure of all points equidistant from a given center point. It is a two-dimensional figure.
 31. Locus of points in a plane equidistant from a given point.
 32. A line connecting a set of infinite points equidistant from a given point.
 33. A set of possible points, all the same distance from a given point called the center.
 37. Definition of a circle: a perfectly round, closed figure with radius r and circumference c where r is the distance from the midpoint of the circle to any outside point and c is the distance measured around the outside.
3. *Formulas* (degree measure and equations):
 6. The set of all points which satisfy the equation $x^2 + y^2 = c^2$ for some given c .

15. Circle: $(x - h)^2 + (y - k)^2 = r^2$. Round.
16. Circle = a set of coordinates falling on the plain $(x - h)^2 + (y - k)^2 = r^2$ (+ figure)
29. Circle: consecutive points in a 360° angle when connected is round and closed.
30. Circle—closed line w/an angle of 360°.
35. Round—3.14—shape of a orange, coin, earth—Pi.
36. Circle—something whose area is = to πR^2 .
4. *Construction* (key words: spinning, connecting points):
 7. Take a line segment of length d with endpoints x and y . (figure) Find the midpoint c of the line segment and spin your line segment while keeping c where it began. The set of points that x and y take on as the segment spins is a circle.
 25. A circle is a line with ends connected.

Group C was the most troubled, as humorously documented by this journal entry of one of its members:

Trying to categorize the definitions presented a large problem to our group. We first began with about 3-4 categories (which we never really agreed upon), expanded that list to approximately 8-10 (followed by much disagreement again), and finally changed the terminology completely. It seems that the various backgrounds of the people and their individual views as to what a "good" definition was kept taking the categorization process in different directions. We never did finish the damn thing and ended up with more homework!! (I hear the echo from the classroom!) (Warren,⁴ journal entry 2)

Unlike the other groups, Group C had started by trying to establish explicit categories and criteria to be used in the classification. A first suggestion to classify the definitions by their mathematical content was abandoned because several people in the group felt their mathematical background was not strong enough to conduct such an analysis. They moved next to a classification based on the kind of error present in the definitions; they set up several categories (such as vague use of terminology, self-contradiction, nonexclusiveness, etc.) but ended up by often disagreeing about where to place a specific definition, and even more disturbingly by finding some definitions that would belong to more than one group. Finally, they decided to focus on whether the definitions were "usable" in a teaching context, and distinguished those "directly usable," "usable with some modification," and "nonusable."

⁴ In this report, the participants have all been identified using fictional names.

Although some student in group C came out of the experience with a sense of frustration, this group proved to be the most critical and reflective, and the discussion of their "troubles" constituted a valid stimulus for reflection by the rest of the class. A student reported in his journal:

The diversity exhibited by our class groups in the categorization of definitions of circle say much about how we view definitions and their usefulness. I had no problem making a decision as to which ones I felt were "appropriate" for my students. I found interesting that people could merely categorize them according to their content (shape, locus, formula, etc.) and yet not let their feelings enter into the decision as to correctness or usefulness. (Warren, journal entry 3)

This preliminary categorization activity turned out to be quite valuable, as the participating teachers began to realize that deciding what constituted a good definition of circle was not as straightforward as they had initially thought and, furthermore, the in-depth analysis of even incorrect definitions could be quite productive for them as teachers, in ways that went beyond diagnosing and re-mediating the "error" present in them. As such, this exercise was instrumental to setting the stage for the following two components of the activity.

Focusing on the Concept of Circle. In preparation for a class discussion on the concept of circle, I provided the participating teachers with the following hand-out, summarizing the key points of my own prior analysis of our data with respect to their mathematical content:

Even a first glance at these definitions for "circle" surprises for several reasons. First of all, there is the extreme variety. Different individuals have focused on different properties of circle, and chosen to consider them as the most important ones for describing the notion. The nonuniqueness of the definitions given does not necessarily point out errors. Alternative correct definitions are possible and it will be interesting to figure out as many as possible. In most cases, however, the definitions given are somehow inadequate to precisely pinpoint the circle among other geometric figures. A listing of several properties of circles, not necessarily all or only the essential ones, is also very common.

In this hand-out we will present a possible categorization of the definitions given based on the properties of circle on which they focus.

(A) METRIC DEFINITIONS (1-2, 31-33):

⁵ The numbers reported in parentheses identify specific definitions from the list reported at the beginning of this error case study, which had been made available to the students in the course at the beginning of this activity.

This is probably the most common definition of circle given in elementary math textbooks, although it does not point out the most intuitive and visual properties of circles. It is interesting to notice that most of the people who gave this definition forgot to mention that the points considered must lie on a plane, therefore including in their definition spheres as well as circles. It could be argued, however, that as we generally deal with two dimensions in school geometry, most people did not even feel the need of explicitly mentioning the plane, perhaps considering it an implicit assumption. If we want to be precise, in fact almost picky, we might also argue that no one explicitly mentioned what "distance" was being considered, since Euclidean distance is probably the only one generally encountered in school. We may also mention that not everyone was precise in establishing whether ALL the points that satisfy the condition should be considered in order to have a circle.

(b) TOPOLOGICAL-PROJECTIVE DEFINITIONS (3, 4, 8-10, 25, 34):

We include in this category all those definitions that point to properties of circles that are of a topological or projective nature. As the circle is a metric notion, none of these definitions is able to distinguish circles from figures equivalent to them in those other geometries (like ellipses, for example). However, these are probably the most obvious and visual properties of circles, as they include:

- continuity (topological)
 - closedness (topological)
 - the fact of being a line (topological)
 - the fact of being a curved rather than a straight line (projective)
- (c) TOPOLOGICAL AND METRIC DEFINITIONS (5, 13, 14, 19, 20, 27, 28):

Here the metric and characterizing property of having points equidistant from the center is joined with some more intuitive and visual properties such as those considered in the previous category. It is interesting to notice that again the condition of lying on a plane is rarely mentioned.

(d) ANALYTICAL GEOMETRY DEFINITIONS (6, 15, 16):

Although very precise, this kind of definition was not suggested by many. Again, 2 out of 3 forgot to mention the plane, while one person limited his consideration to circles with center at the origin.

(e) GENERATIVE DEFINITIONS (7, 26):

Legitimate and precise definitions of circle, based on rotation, which give also a constructive method to obtain such figure.

(f) DIFFERENTIAL GEOMETRY DEFINITION (8, 17):

Such definitions could be considered "almost" correct in differential geometry (if we define curvature in terms of derivatives, there will be no problem of circularity). It is interesting to notice that this kind of definition is what is practically used in LOGO to design circles.

(g) PURELY VISUAL DESCRIPTION (18):

This definition invites us to consider how the notion of "circle" could be abstracted from its embodiments in nature. Interestingly, the examples chosen are all three-dimensional!

(h) "QUEER" DEFINITIONS (12, 21–24, 29, 30, 36, 37): In this category, we collected definitions that seem strange for quite different reasons:

- they present peculiar ideas
- they are surprising for what they leave out
- they are noticeable for all the things they put together.

Though each of them deserve a special comment, they are less relevant to our present interest on characterizing properties of circles than the previous ones, and we therefore withhold discussion in this context.

The students were asked to reflect individually on these considerations and to pursue a first analysis of the concept of *circle* on their own by means of the following written assignment:

Assignment 1

- a. Read the hand-out reporting a categorization and analysis of the list of definitions of circle, prepared by the instructor (reproduced earlier).
- b. Add to the existing categories those among the definitions produced by this class that in your opinion belong to it. If you feel it necessary or useful, modify the categories or add new ones of your own devising.
- c. Choose any one of the categories referred to above, and analyze it more deeply. For example, you can decide to comment on specific definitions; modify some definitions so that they become acceptable; try to explore what objects some definitions are really describing. These are only suggestions, be as creative as possible in your analysis.

Although the open-ended nature of the task made it difficult for many students, some produced quite interesting results. For example, a student chose to discuss the group of "analytic geometry" definitions and explored how it could be further expanded by considering different systems of coordinates:

One can define a circle as the set of points (x, y) in a coordinate plane which satisfy the equation $(x - h)^2 + (y - k)^2 = r^2$.

However, it is customary to DERIVE this equation in the coordinate plane using the metric definition of a circle as a starting point. This makes more

sense to me. Furthermore, the analytic-geometry definition assumes the "standard" coordinate system. If you used instead a coordinate system in which the distance-measure along the y -axis was different from the distance-measure along the x -axis, then this definition would result in what we normally call an ellipse (I suppose you could call it a "circle" in the new coordinate system). . . . I could go on with this notion by mentioning coordinate systems in which the axes do not meet at right angles, with log-log coordinates systems, etc. . . . (Richard, Assignment 1)

The same student also suggested adding an interesting new category to the ones discussed in our hand-out:

I would also add a category "polygonal" to include those definitions that treat a circle as an infinite sided polygon. . . . It is a way that the basic concepts of "infinity" and "limit" can be introduced at a relative early age. . . . There is also the possibility to begin investigating how one might measure the circumference and area of a circle. (Richard, Assignment 1)

In the class discussion that followed this assignment we further analyzed definitions in each of the categories identified. This exercise made us all more aware of various properties of circles such as the equidistance of its points from the center, constant curvature, closedness, continuity, and a characteristic equation in a given system of coordinates. It also provided a deeper understanding of the relationship between circles and figures such as spheres, ellipses, eight-shaped figures, regular polygons, and even straight lines (because these are the only other lines with constant curvature, although equal to zero). A discussion generated around the common use of the word *round* also made the class wonder whether we should make the distinction between circle as a shape versus circle as a geometric figure more clear in school when presenting elementary geometry.

Examining the Nature of Mathematical Definitions. The previous activities had already spontaneously raised some questions about what constitutes a good mathematical definition. As a result, several people in the group started to doubt and challenge their own prior conception of mathematical definitions, as two teachers reported in their journals:

I am having a hard time with my definition of definition. Intellectually, I agree with the majority of my group . . . that a definition must be exact to be correct. . . . However, the part of me that sees the group as well as the black and whites, is much more lenient. That part believes in what I call the "personal definition." That part of me is willing to take into account a student ability when judging his definitions of mathematical

terms. If he is able to show a solid understanding of a term without a classic textbook definition, I find difficult to find him in error. I feel he has a firmer grasp of the subject matter than someone who is able to "parrot" a definition, but possesses no understanding of the concept. (Jill, journal entry 2)

I would think that students find it difficult to verbalize a definition of a picture, and will in fact describe it by its properties or what they heard in class.

But are verbatim theorems, postulates and definitions the best way to teach students about geometric shapes? Wasn't it interesting to observe the way the students interpreted the definitions? Shouldn't such observations make us, as teachers, be aware of our own techniques in presenting material? Perhaps we are the ones who aren't making ourselves clear enough to our students? (Sonia, journal entry 2)

With the goal of pursuing these questions in more depth, the students were then asked as a homework assignment to write down their opinions about what the main attributes of mathematical definitions are and what errors could occur with regard to them (Assignment 2). As often happens in these cases, many individuals raised interesting and insightful issues about the nature of mathematical definitions, although each generally tended to look only at some aspects of the whole picture. I decided to compile the most interesting contributions in a hand-out that was distributed in class as the basis for a discussion on the nature of mathematical definitions. In the class discussion that followed we ended up with the following categories and observations:

1. Use of terminology:
 - Vague terms should be avoided.
 - Only previously defined terms should be used.
2. Isolation of the concept:
 - The definition should not apply to other objects than the ones we want to define.
 - All instances of the concept should be included.
 - All exceptions should be included.
3. Essentiality:
 - No unnecessary information should be included.
 - Avoid unnecessary properties.
 - Do not mention or define unnecessary terms.
 - Avoid the use of long, big words.

* Given the length of this hand-out, I chose not to reproduce it here. Its text, however, can be found in Appendix B9 of Bornat (1986b).

4. Understandability:
 - The concepts and terms used should be appropriate to the level of the audience.
 - It should not be too complicated to decode.
 - It should provide an image for the concept.
5. Usefulness or usability:
 - It can be easily used or applied.
 - It can be easily remembered.
 - It is useful in other points of mathematics.
 - It is useful in applications.

In other words, we found ourselves sharing the view of mathematical definition passed to us by the community of mathematicians, even if the inclusion of categories such as understandability and usefulness allowed some educational concerns to surface. Once confronted with the task of evaluating specific definitions on the basis of the criteria just stated, however, the class often split and we had to realize that the notion of what constitutes a good mathematical definition was not as clear as it had first appeared.

For instance, how should the fact that many people forgot to mention the plane be interpreted? The class doubted that the authors of the first two definitions ("1. Circle is a form in which radius is equal from the center to arc." and "2. A circle—a collection of points all equidistant from the center (radius)") did not know the difference between circles and spheres. Rather, they assumed that the authors did not feel the need to specify that the points should all belong to the same plane, because it was obvious to them, especially if they thought of defining *circle* in the context of plane Euclidean geometry. Should such omissions then be considered as errors?

An additional problematic element came into the picture when the class considered the use and roles of definitions not only in mathematics, but in the context of mathematics instruction as well. For example, in evaluating a definition produced by a student, the class felt the need to take into account whether he or she was just "parroting" a memorized definition, or whether he or she understood the concept behind it. In the latter case, an imprecise but original definition might even be considered as a positive contribution—something teachers tend not to appreciate when striving for premature mathematical precision.

How should alternative definitions be evaluated for classroom teaching? The fact that the students must be able to understand them becomes a priority here. For instance, however essential and elegant, no one in this group would have chosen the differential geometry definition (i.e., "8. A line with constant curvature.") for a high school geometry course!

At this point, somebody even argued that the concern for essentiality and elegance, when presenting the definition of a new concept, does not seem to make much sense. On the contrary, as a teacher one would want to provide the

students with as much information and as many examples as possible about the concept in question. This remark made us reflect once more on the role of definitions in the teaching of mathematics. We were then able to identify the existence of some confusion between *definition* and a few other related but distinct notions. Such confusion had been at the basis of several of our problems in the previous evaluation of specific definitions. We reached partial clarification when we realized the difference between the *name*, *symbol*, *definition*, *description*, *model*, and *prototypical example* of a given mathematical concept. For a few people in class, the reflection on the nature of mathematical definitions did not end with this class discussion. Further observations appeared in journal entries for the following weeks. It is particularly interesting to follow the development of one student's (Elisa) conception of definition throughout our activities in this unit.

Elisa started with a very clear idea of what a mathematical definition should be, at least from the mathematical point of view. The following quote comes from her response to our initial written assignment, where she was asked to state the characteristics of good mathematical definitions:

A mathematical definition is a set of statements which completely describe the essential characteristics of a mathematical object, word, or concept. A good definition should use only previously defined terms or the generally accepted undefined terms with which one begins the investigation of any deductive system. It should encompass all possible objects in the class of objects defined. It should not contain any unnecessary information. It should not be redundant or circular. It certainly should not contain contradictions or vague descriptive phrases which are subject to different interpretations. A good definition is reversible. It can be expressed in prose or it can take the form of acceptable mathematical notation. It should use the most generally acceptable notation available. Certain definitions are good definitions but are not understood by people who do not have the necessary mathematical background. (Elisa, Assignment 2)

The analysis of the definitions of circle conducted in class, however, soon challenged her own conception of mathematical definition. The first evidence of this can be seen in the frustration and the doubts she experienced and reported in her journal, following the collective analysis of our list of definitions of circle:

I found the exercise of trying to categorize a series of "mostly incorrect" definitions of circle very frustrating. It is difficult for me to get beyond the fact that they are incorrect. (Elisa, journal entry 2)

I continue to be frustrated by the exercise of going into depth about categories of definitions of circle. . . . I found your comments and categorization of definitions of circle as well as those given by each group in class very interesting. The effect of class discussion and your handout was to expand my narrow conception of how to define a circle. It also raised many doubts in my mind concerning the whole question of adequate definitions. The task of making good definitions is complex. . . . Your categories and descriptions reminded me that the appropriateness of a definition is dependent on the mathematical sophistication of the person for whom the definition is given. (Elisa, journal entry 3)

The reflections on the nature of mathematical definitions stimulated by our class discussions and activities finally led Elisa to a finer and more relativistic conception of the notion:

I am still thinking about problems surrounding definitions. It seems to me that whenever we make mathematical definitions we are doing so, or should be doing so, in the context of certain undefined terms and previously defined terms which the "definer" and the "receiver of the definition" must previously agree. . . . Even when the context is clear, it is difficult to give fully precise definitions which could never be misinterpreted because our language and our symbols are not precise enough, and because words and symbols are often used to represent a variety of things. In order for a definition to be considered accurate and complete, the "definer" and the "receiver of the definition" must be on the same wavelength in terms of the word and symbols being used in the definition. If these conditions hold, then we can start giving more qualifications to the definition, i.e. whether it is reversible, whether it contains too much or just enough information, etc. (Elisa, journal entry 4)

I am still thinking about definitions. It seems that in our class discussions about definition we often use the word "definition" when we really mean "description." It is not necessarily true that a good mathematical definition must give a good image or understanding of a concept. It is most often the case that the understanding of the concept comes before the technical definition can be understood. Teachers need to get concepts across by using many different descriptions and examples which illustrate those concepts before they give a precise definition. As was pointed out in class, long before precise definitions of geometric shapes are given, children are taught to recognize these shapes and have a good, intuitive understanding of their properties. It is a common practice in teaching mathematics to try to give students some feeling of a concept before handing out the definition. For us as teachers, it is often much more important that our students understand and can use definitions correctly than that they can verbalize them precisely. Possibly, we would have more suc-

cess at stating the attributes of a good definition if we would differentiate carefully between what we mean by description and what we mean by definition. (Elisa, journal entry 5)

Although not everybody in class went to the same depth in their analysis of mathematical definitions, the experience at least started some doubt concerning the participants' original conceptions of this notion. As revealed in the responses to a questionnaire given at the end of the course, most people found the activity rewarding, although demanding and time consuming. The following comments have been selected to illustrate the participating teachers' explicit comments and evaluation of the experience:

The definitions of circle were an eye-opening experience for me. It clearly helped me to see my feelings and understanding about the nature of definitions in mathematics.

The definitions of circles was a very good activity, it gave all of us a chance to see just how it is each of us perceives such "obvious" concepts.

When we began this exercise, I merely saw most of the definitions presented as being wrong, period! As we continued to discuss definitions, my understanding of the difficulties they present was expanded. I now have a healthier respect for the difficulty of presenting adequate definitions. I still feel, though, that people have a tendency to confuse the concepts of "description" and "definition."

I became a little bored with the discussion of definitions of circle. However, it did generate a lot of thought. I still am trying to decide what a good definition consists of. If nothing else, it emphasized that an error to one is not necessarily an error to another.

In sum, this report shows how the analysis of mostly incorrect definitions of circle, structured by means of the various activities described throughout this error case study, provided an effective vehicle for the participating teachers to inquire into the concepts of circle and mathematical definition and acquire, as a result, a more articulated understanding of these fundamental notions.

Error Case Study R: Teachers' Reflections and Problem-Solving Activities Around an Unrigorous Proof ("Teachers' Unrigorous Proof" Case Study [R9])

With the goal of complementing the experience reported in the previous "Teachers' definitions of circle" case study [Q9], later in the same course I decided to engage the participating teachers in an articulated error activity that

would require them to deal with more challenging mathematical content and engage in more technical explorations. My intuitive proof for evaluating the infinite expression $\sqrt{2 + \sqrt{2 + \dots + \sqrt{2 + \sqrt{2}}}}$ (R[2] hereafter), described earlier in "My unrigorous proof" case study [H/5], seemed an appropriate starting point for such an activity.

Over the course of three weekly meetings (although not entirely devoted to this unit), the experience involved a first in-depth analysis of my unrigorous proof with the main goal of debugging it, its generalization for the evaluation of another infinite expression and, finally, an explicit discussion on the nature of mathematical proof. I report on the development of each of these activities separately.

First Analysis of My Proof for R[2] = 2. Our unit started with my presentation to the class of the following intuitive procedure to evaluate the infinite expression R[2]:

Let's call x the expression we want to evaluate:

$$\sqrt{2 + \sqrt{2 + \dots + \sqrt{2 + \sqrt{2}}}} = x$$

squaring both sides, we get:

$$2 + \sqrt{2 + \dots + \sqrt{2 + \sqrt{2}}} = x^2$$

The radical on the left side is again x because it also contains infinitely many terms. Therefore, the expression is equivalent to the second degree equation:

$$2 + x = x^2$$

Solving this equation for x yields the solutions $x = -1$ and $x = 2$. Because the radical is certainly positive, the root $x = -1$ has to be excluded. Therefore, it must be that:

$$\sqrt{2 + \sqrt{2 + \dots + \sqrt{2 + \sqrt{2}}}} = 2.$$

While presenting this derivation at the blackboard, I felt some of my students become uneasy. When I asked for reactions to this "proof," most students found it difficult to express why they felt the proof was "fishy," although only one person declared herself satisfied with it. I then suggested to the class to try and point out which specific steps of my proof they felt uncomfortable with.

The following observations were raised, in the same order as they are reported here:

1. In the second step, you do not have "exactly" the same radical. Can you still call it x^2 ?
2. Does the expression involve an infinite or finite number of nested radicals? The notation used does not seem appropriate if there are infinitely many terms.
3. As we had to discard -1 as a reasonable solution, how can we be sure that 2 is OK?
4. Why do we stop after squaring once, and do not instead go on isolating the radical and squaring it more times?
5. What is really $\sqrt{2 + \sqrt{2 + \dots + \sqrt{2 + \sqrt{2}}}}$?
6. Can we set it equal to x ?

Because we were close to the end of our meeting, we had only a few minutes left to start inquiring into these questions. The doubt expressed in Point 2 was easily solved by my stating that indeed the radical was intended to contain infinitely many terms, although I admitted that the notation chosen was not the one usually employed to indicate the presence of infinitely many terms. Point 3 was also quickly settled by one of the students who observed that even in "regular algebra," when we square both sides of an equation, we may occasionally end up with spurious solutions. Point 5 was then addressed. One of the participants (a mathematics instructor at a local college) immediately observed that the infinite expression should be interpreted as the limit of a sequence, namely:

$$\begin{aligned}x_1 &= \sqrt{2} \\x_2 &= \sqrt{2 + \sqrt{2}} \\&\dots \\x_3 &= \sqrt{2 + \sqrt{2 + \sqrt{2}}} \\&\dots \\x_n &= \sqrt{2 + \underbrace{\dots \sqrt{2 + \sqrt{2}}}_{n \text{ terms}}}\end{aligned}$$

The same person observed that only if such a sequence converges, we are allowed to set it equal to x , and operate with it as I did. Unfortunately, this early resolution caused the discussion of this important issue to end prematurely, leaving most of the other students uneasy and still not convinced.

No time was left for exploring the last question. I concluded this first activity (which lasted no more than half an hour) by assigning the reading of an ear-

lier (and less developed) version of the first part of "My unrigorous proof" case study [H/5], in which I discussed the evaluation of $\sqrt{2 + \sqrt{2 + \dots + \sqrt{2 + \sqrt{2}}}}$ and of the more general expression $\sqrt{a + \sqrt{a + \dots + \sqrt{a + \sqrt{a}}}}$.

When the class met the following week, I suggested that those who found the reading of the paper relatively easy and self-explanatory attempt on their own the evaluation of the somewhat similar expression:

$$E[\sqrt{2}] = \sqrt{2} \cdot \sqrt{2} \cdot \sqrt{2} \cdot \dots$$

Eight students chose this alternative. The other six students were instead invited to participate with me in an activity focused on analyzing in more detail my original proof for $R[2] = 2$. For about half an hour, the class then divided into two groups, each pursuing one of these activities.

Group 1: Report of a more detailed analysis of the proof. As the class split into two groups, I chose to join the one that had decided to discuss my proof in more detail. We started by reviewing all the questions raised in the previous session and, then, decided to focus on the key issue of what the infinite expression $R[2]$ really means and how we can approach its evaluation, which was still puzzling many of the students in this group.

One student suggested to start "getting a feeling for it" by approximating its value. Another student observed that there seemed to be two alternative possibilities to do so: "to start from the left, from the outside," and to "to start from the right, from the inside." He also said that he had problems seeing how to do the former in practice. The other students agreed, so we decided to choose the other alternative. Asked specifically what they were going to evaluate as approximations of the expression, the students spontaneously suggested $\sqrt{2}$, then $\sqrt{2 + \sqrt{2}}$, then $\sqrt{2 + \sqrt{2 + \sqrt{2}}}$, and so on. I formalized their suggestions by transcribing them in the form of a sequence:

$$\begin{aligned}x_1 &= \sqrt{2} \\x_2 &= \sqrt{2 + \sqrt{2}} \\x_3 &= \sqrt{2 + \sqrt{2 + \sqrt{2}}} \\&\dots \\x_n &= \sqrt{2 + \underbrace{\dots \sqrt{2 + \sqrt{2}}}_{n \text{ terms}}}\end{aligned}$$

I could then see that what had taken most of them aback in the previous class discussion was now starting to make more sense.

We then moved to the problem of determining whether this sequence approached some value. It was suggested that we first compute some values with a calculator to see whether some pattern would emerge or if the results would not change much after a while. This appeared to be a good approach, although somebody expressed the doubt "how can we be sure that nothing strange will happen later?" In particular, a student was concerned that the values would continue to increase "without limit." "Is infinitely real?" he asked, very concerned. I pointed out that the sequence seemed indeed increasing, so we could at least try to check whether it would admit some bound, that is, if all elements of the sequence were less than a certain value. Because of my intuitive proof, 2 seemed a good candidate. There was general agreement about this.

Before even attempting to prove this hypothesis, however, somebody raised the even more fundamental and challenging question "If the limit of the sequence is 2, does it mean that our expression is exactly 2, or that it just goes nearer to 2 without ever reaching it?" This question left everybody puzzled. They seemed to prefer the second alternative but were at a loss for providing a justification. I then suggested considering the similar case of the decimal expression of $\frac{1}{3}$. What is the value of .333333...? They all agreed that $\frac{1}{3} = \frac{1}{3}$, but they were again at a loss to explain why it is exactly $\frac{1}{3}$ and not "just a little less than $\frac{1}{3}$." Someone tried to justify it with the existence of a rule to translate this kind of decimals into fraction—but then he did not know how to justify the rule itself! Another student suggested as an explanation the fact the .3 was the result of a division algorithm, but neither solution seemed really satisfactory. I decided to conclude the discussion by observing (somewhat simplistically) that it was partly a matter of convention. Mathematicians decided to make sense out of infinite expressions like the ones we were considering by stating that their value was to be equal to the limit (if it exists) of a suitable sequence of finite numbers or expressions.

Despite the brief amount of time devoted to this discussion, I felt that the experience had contributed a considerable breakthrough in the participants' understanding of sequences, convergence, and limits. This result, to me, was even more important than the actual debugging of my original proof and the understanding of its more "rigorous" version and its generalization. These impressions were further confirmed by some of the students' journal entries referring to this activity, as illustrated by the following excerpt:

When we divided into two groups—one to discuss the proof of the evaluation of another infinite expression, and the other the actual proof and evaluation of $\sqrt{2 + \sqrt{2 + \dots + \sqrt{2 + \sqrt{2}} - 1}}$ —I decided to observe the latter. Much to my surprise, I was able to follow the advance, attack and retreat

taking place. I was not able to contribute much, but I observed some tactical plays that I will reserve for future use—especially the regrouping after each retreat. In the past, I have not enjoyed the process of proof. Probably, because I admit defeat too easily. It is important to step back, look at the original problem, regroup your thoughts and attack again. (Donna, journal entry 6)

Group 2: Report of their evaluation of a similar infinite expression. As my group engaged in the activity already reported, the other group proceeded to evaluate on their own the expression $E[\sqrt{2}]$. At first, they interpreted the expression as:

$$(((\sqrt{2} \cdot \sqrt{2}) \cdot \sqrt{2}) \cdot \sqrt{2}) \dots$$

and quickly reached the conclusion that the value of the expression, in this case, was not finite. Unhappy with this result, they then conceived the possibility of an alternative interpretation of the expression given, which led them to the defining sequence:

$$x_1 = \sqrt{2}$$

$$x_2 = \sqrt{2} \cdot \sqrt{2}$$

$$x_3 = \sqrt{2} \cdot \sqrt{2} \cdot \sqrt{2}$$

...

$$x_n = \sqrt{2} \cdot x_{n-1}$$

They then tried to compute the first few elements of this sequence with a hand calculator, obtaining the values:

$$x_1 = 1.4142$$

$$x_2 = 1.8477$$

$$x_3 = 1.9615$$

$$x_4 = 1.9903$$

$$x_5 = 1.9975$$

$$x_6 = 1.9993$$

$$x_7 = 1.9998$$

$$x_8 = 1.9999623$$

$$x_9 = 1.9999905$$

$$x_{10} = 1.9999976$$

These values suggested that the sequence converged to something near 2. A student then suggested applying my method to this new expression, obtaining the equation:

$$\sqrt{2}^x = x.$$

Solving this equation was not immediate, so they attempted some algebraic manipulations:

$$\sqrt{2}^x = x \Rightarrow \log \sqrt{2}^x = \log x \Rightarrow \frac{x}{2} = \frac{\log x}{\log 2}.$$

2 was an obvious solution to the equation once expressed in this form. But 4 was also discovered as a possible solution (because $\log 4 = \log 2^2 = 2 \log 2$). Were those the only solutions? In the attempt to answer this questions, they squared the original equation, obtaining:

$$x^2 = 2x.$$

By graphing this equation, they found 2, 4, and a negative value to be the only solutions. With a hunch that 2 was probably the correct solution (as a result of their earlier computations) they now tried to prove that 2 was a bound to the sequence $\{x_n\}$, that is, that for any n , $x_n < 2$, and they did that by induction. (This turned out to be easy to do, although many in the group felt uneasy about proofs by induction). They would have also liked to prove that the sequence was increasing (although they were sure that this was the case) and to find some kind of explanation for the extra solution $x = 4$. But they had to stop here as they ran out of the time allotted to the small-group activity.

Overall, I was quite impressed by the creative thinking and results this group had been able to produce without any intervention from the part of the instructor. As they reported to the rest of the class, it was also clear that all the group members felt quite proud, and even a little surprised, of these achievements.

Investigation of the Nature of Proof. Much as I had hoped, the activities described so far were instrumental in inviting some spontaneous reflections about the role of intuition and rigor in mathematics as well as on the nature of mathematical proof. These thoughts found first expression in several of the participants' journals. For example, after having read my own analysis and debugging of the original proof suggested for the result $R[2] = 2$, a student challenged the very fact of considering my original proof wrong:

I have done some thinking about the example $R[2] = 2$ and I cannot help but wonder why we are calling the original proof an error. While it is true that this proof applied finite operations to an infinite expression, I have been in more than one graduate mathematics class in which the professor would do this type of abbreviated mathematics. In his case, he knew full well that he was leaving out the details but he also knew that those details could be supplied and that they would agree with the given result. If a student were to raise a question about it, the reply from the professor would be that the student should check it out for himself. Once again we are face to face with the question of context.

It seems to me that if one were to accept the proof without being sure that it could be verified and then did not take the time to verify it, then that would constitute an error. It would be another example of getting the right result by the wrong method. Since in this case the mathematics was verifiable and verified then perhaps this is an example of just plain good mathematics. (Richard, journal entry 6)

Another student commented positively on the relationship between intuition and rigor in proving mathematical results:

Although I found parts of the proof difficult to follow (mainly because I'm a little rusty with Calculus), I found the article quite interesting. I believe very much in intuition. Really it is only after the light has been turned on that you can see the relationships, that a good rigorous proof can be written. Otherwise, it is being done by rote and I question its value . . . I agree that analyzing or debugging an intuitive proof can be interesting and lead to many new ideas. I'd like to think of the intuitive idea as the skeleton—the foundation. Rigor spruces it up and puts in more detail so that anyone looking at the proof sees the same thing. (Margaret, diary 5)

The activities of debugging the original proof and using it in a similar situation helped another student raise some questions about the criteria for a correct proof, at the same time re-evaluating the very process of proving:

After reading the "Intuition and Rigor" article I found myself wondering about proofs in general. Especially considering the point made about proofs "providing insight into some important heuristic in mathematics thinking." What is the criteria for a correct proof?

It would make a much more interesting discussion in classroom, to be able to generate discussions or generalizing proofs, making proofs for just special cases, and "spotting" errors in proofs.

This is an area that I have always DESPISED and I have now taken a new interest in its value as a teaching tool. (Donna, journal entry 5)

To prepare for a more general and explicit class discussion on the nature of mathematical proofs that would build on these spontaneous reflections, I gave a first writing assignment asking the students' opinion on the main characteristics of mathematical proofs and errors in them, followed by the reading of selected excerpts from Lakatos' *Proofs and Refutations* (1976). Somewhat to my surprise, the students' descriptions of mathematical proofs in the written assignment were rather stereotyped and uninteresting. They mostly characterized proofs by using vague adjectives (like *precise*, *rigorous*, *clear*, *concise*, *logic*) or they tried to describe what needed to be done in order to construct a good proof by mentioning elements such as "sequence of deductions from some premises to a conclusion," "based on previously established definitions and theorems," and "should examine all cases." With two notable exceptions, everyone seemed to share the feeling that what makes a proof acceptable is clearly established and known, at least among mathematicians, and that a good proof allows one to deduce without doubt the conclusions from the given premises.

In contrast, the identification of common errors in doing proofs turned out to be more varied and thought stimulating. I decided therefore to collect the most interesting excerpts in a hand-out⁷ and to distribute it in class as a starting point for the class discussion on the nature of mathematical proofs. The errors mentioned in the students' homework were mainly of the following four kinds:

- Errors involving missing steps.
- Errors due to a misuse of logic and inference rules.
- Errors due to misunderstanding of what has to be proved.
- Errors due to the fact that the proof is not general or comprehensive enough (special cases and exceptions have not been taken care of).

In the class discussion that followed we focused mostly on an analysis of the first two points and then moved to address a few important questions about the nature of mathematical proofs.

Contrary to what had happened in the case of the notion of definition (see "Teachers' definitions of circle" case study [Q91]), the discussion this time proceeded with some difficulty. Most students did not seem very confident in talking about mathematical proofs. At this stage, however, the participants were able to identify several common errors due to the misuse of logic, such as the belief that the converse of a theorem must also be true and, more generally, the

treatment of a conditional as if it were a biconditional. We also discussed a commonly shared uneasiness with the method of proof by induction, where it seems that the very thing one is expected to prove is used as an assumption (i.e., inductive hypothesis).

We passed then to consider errors as those identified in my intuitive proof for $R[2] = 2$. I challenged the class with the question of whether it is always possible to articulate all the necessary steps and assumptions in a proof, and how we can decide where to stop in such a process. Although not confidently, several students tried to answer the question by pointing out that even if we cannot "write out everything," what is left out should be either known to the reader or "obvious." Even when a student observed that there is some controversy even among mathematicians themselves about what really constitutes a proof, there was not much reaction from the rest of the class.

A real turning point in the discussion was reached when, partly in response to my suggestion of referring to their experience with proofs, the students started to talk of their own errors and difficulties with proofs. Now a quite different picture started to come out, as almost everybody declared that they had problems with regard to proofs. Several people shared the feeling of never knowing whether you have really proved something or not, a certain uneasiness in reading proofs given in a textbook because of the "obvious" steps that had been left out, the difficulty in figuring out what direction to take, and what things to use when trying to construct a proof. Most people admitted being quite poor at proving things and tried to identify causes of this handicap. Many teachers pointed out that their experience as mathematics students had always been more with understanding and memorizing proofs rather than making up their own proofs. Another participant came up with a very interesting insight in this regard: He compared the mathematical proofs we are presented with in school as the last draft of an essay. He observed that mathematics students never come to know of the false starts, retreats, corrections, intuitions, and fixing up that certainly occurred in creating the proof. How can they be expected to ever learn to come up with proofs of their own? How can students acquire the useful heuristics that mathematicians use in their research work? There was some discussion on these important issues and, although the class was not able to come up with solutions, at least the value of debugging intuitive proofs and of being presented sometimes with the genesis and history of some proofs were identified.

We also discussed, without reaching a satisfactory answer, why we need to prove a result and how we can motivate the activity of proving for our students. Everybody recognized, though, that the search for a proof could constitute a very creative activity in mathematics.

The discussion on the nature of mathematical proofs went on for more than an hour. More problems and questions were raised than we were able to solve, or even address as they deserved. Overall, this discussion and the activities that

⁷ Given the length of this hand-out, I have chosen not to reproduce it here. Its complete text, however, can be found in Appendix B.10 of Borasi (1986b).

preceded it presented a real challenge for the participants' conceptions of mathematics and errors and, for many of them, it also constituted an important turning point in their appreciation of the strategy of capitalizing on errors. This feeling was articulated by one of the participating teachers in his journal:

I think I understand the methodology of this course I am taking called "Analysis of Math Errors."

The instructor is broadening our appreciation of the potential value of math errors as a pedagogical tool by continually shifting the perspective we assume intuitively towards errors.

Here is what I mean.

First we are given an example of an error that involved a level of math ability that we all know so very well that we marvel that someone who "was told the rules" could possibly make. We then proceed to diagnose that the value of the error might be with the sometimes smug notion that we ourselves would never under similar circumstances make such an error.

Then we are given a problem at a level of math ability JUST WITHIN (in most cases) our own. Now it is not the same story . . . we "have been told the rules," perhaps in a calculus course or some other higher-level course, but we find ourselves misapplying them, not generalizing enough, or perhaps too much, in short, making the same types of errors that we find so mysterious when committed at a lower-level of math ability.

It is a rather humbling experience as well as a learning one as we flounder around with infinitely long nested radicals and watch the results, sometimes surprising and intriguing, emerge from the process. (Jeff, journal entry 5)

Error Case Study S: "Numbering Systems Without Zero"—A Teacher-Generated Exploration (by Richard Fasse) ("Numbers Without Zero" Case Study [S9])

Because zero is a source of so many errors, I wondered what it would be like if zero was eliminated. We know how cumbersome arithmetic with Roman numerals would be, so I invented a new number system to investigate.

Introduction. The first known use of zero was by the Mayans and Babylonians as a place holder around 3300 to 3500 B.C. The Hindus are credited with inventing our current numeration system with zero as a place holder around 300 to 400 A.D. It was not until much later that zero was accepted as a number.

Zero as a place holder allowed average citizens to perform calculations. Other numbering systems required complex mental reckoning to perform routine calculations, and persons capable of such reckoning were held in awe.

I thought investigating an alternate numbering system without a zero would provide insight to problems associated with zero in our current system.

Alternate Representation. Instead of using zero as a place holder, use a numerical subscript to represent the number of empty places. For example:

$$100 = 1_2 \quad 4060 = 4_16_1 \quad 40032 = 4_23_2$$

This system combines our current system with some aspects of logarithmic functions. Consider some simple examples:

$$\begin{aligned} 100 + 200 &= 300 & 1_2 &= 2_2 = 3_2 \\ 400 \times 400 &= 160000 & 4_2 \times 4_2 &= 16_4 \end{aligned}$$

I developed the addition, subtraction, and multiplication algorithms for this system as an exercise to discover how much more clumsy it would be than our current system. I did not document a division algorithm because it is a cumbersome combination of the multiplication and subtraction algorithms. The algorithms were developed for positive integers only. Also, it should be noted that I do not account for situations requiring more than nine consecutive place holders.

The most cumbersome aspects of these algorithms have to do with carrying and borrowing. I spent considerable time trying to devise a simpler approach, but was unsuccessful. In order to keep the algorithms short and readable, I have included examples to help illustrate the procedures.

Addition Algorithm.

1. Work from right column to left column of each number independently (after the algorithm has started, you may find yourself operating on Column 4 of one number and Column 2 of another). If at end of columns, go to 5.
2. If adding all subs, bring down the smallest sub, and subtract smallest sub from the largest sub.
Go to 1.
3. If adding all numbers, add and
 - If result < 10, bring down units.
 - If result = 10, carry 1 and bring down sub 1.
 - If result > 10, bring down units and carry 1.
 Go to 1.

4. If mixed, bring down number and reduce sub by 1. Go to 1.
5. Bring down excess.
6. Add any consecutive subs, for example, 111 = 12.

Examples are shown in Figure 9.1.

Subtraction Algorithm.

1. Work from right column to left column of each number independently. If at end of columns, go to 4.
2. If subtracting a sub from a
 - sub; bring down smaller sub and reduce other sub by same amount.
 - number; bring down number and reduce sub by 1.

Go to 1.

3. If subtracting a number from a:
 - sub; bring down complement of number, replace sub by as many 9s as (sub - 1) and reduce first number to left of sub by 1.
 - number of equal amount, bring down sub 1.
 - number, and borrowing not required; bring down result.
 - number, and borrowing required; add complement of larger number to smaller number and bring down result. If nothing to borrow from, result is negative. Otherwise, if borrowing from sub, replace sub by equal number of 9s and reduce first number to left of sub by 1. If borrowing from number, reduce number by 1, unless number is 1 in which case replace 1 by adding sub 1.

Go to 1.

4. Bring down excess.
5. Add any consecutive subs, for example, 111 = 12.

$$\begin{array}{r}
 106 \\
 +10023 \\
 \hline
 10129
 \end{array}
 \quad
 \begin{array}{r}
 1,6 \\
 +1,23 \\
 \hline
 9
 \end{array}
 \quad
 \begin{array}{r}
 1,6 \\
 +1,23 \\
 \hline
 29
 \end{array}
 \quad
 \begin{array}{r}
 1,6 \\
 +1,23 \\
 \hline
 1,129
 \end{array}$$

$$\begin{array}{r}
 10065 \\
 +5035 \\
 \hline
 15100
 \end{array}
 \quad
 \begin{array}{r}
 1,65 \\
 +5,35 \\
 \hline
 1
 \end{array}
 \quad
 \begin{array}{r}
 1,65 \\
 +5,35 \\
 \hline
 11
 \end{array}
 \quad
 \begin{array}{r}
 1,65 \\
 +5,35 \\
 \hline
 11
 \end{array}
 \quad
 \begin{array}{r}
 1,65 \\
 +5,35 \\
 \hline
 15111 \rightarrow
 \end{array}$$

FIGURE 9.1. Examples of additions performed using the usual and alternative notation.

$$\begin{array}{r}
 1006 \\
 -309 \\
 \hline
 697
 \end{array}
 \quad
 \begin{array}{r}
 1,6 \\
 -3,9 \\
 \hline
 7
 \end{array}
 \quad
 \begin{array}{r}
 99 \\
 -3,9 \\
 \hline
 996
 \end{array}
 \quad
 \begin{array}{r}
 99 \\
 -3,9 \\
 \hline
 996
 \end{array}$$

$$\begin{array}{r}
 4099 \\
 -399 \\
 \hline
 3700
 \end{array}
 \quad
 \begin{array}{r}
 4,99 \\
 -399 \\
 \hline
 1
 \end{array}
 \quad
 \begin{array}{r}
 4,99 \\
 -399 \\
 \hline
 11
 \end{array}
 \quad
 \begin{array}{r}
 4,99 \\
 -399 \\
 \hline
 11
 \end{array}
 \quad
 \begin{array}{r}
 34,99 \\
 -399 \\
 \hline
 3711 \rightarrow 372
 \end{array}$$

FIGURE 9.2. Examples of subtractions performed using the usual and alternative notation.

Examples are shown in Figure 9.2.

Multiplication Algorithm.

1. Work from right column to left column.
2. If multiplier is a sub, add sub to multiplicand. Go to 1.
3. If multiplier is a number, and multiplicand is a:
 - sub; and nothing carried, bring down sub.
 - sub; and something carried, bring down carried units and reduce sub by 1.
 - number; multiply.

If result < 10, bring down units.

If result multiple of 10, add 1 sub down and carry.

If result > 10, bring down units and carry excess.

Go to 4.

4. Add 1 sub to multiplicand and go to 1.

Examples are shown in Figure 9.3.

Conclusion. Zero as a place holder must have been one of the most important inventions ever. The ease of computation is amazing compared to any alternative. The approach presented here has some interesting attributes, but because columns cannot be "lined up" during the arithmetic operations it is quite cumbersome. However, if you consider the numbers being operated on in a push down/pop up stack or queue, an interesting computer metaphor results. Each number could be placed in a register and the algorithm applied to the contents of the registers. Some coding structure would be required to differentiate numbers from subs, but I am confident the algorithms could be programmed ef-

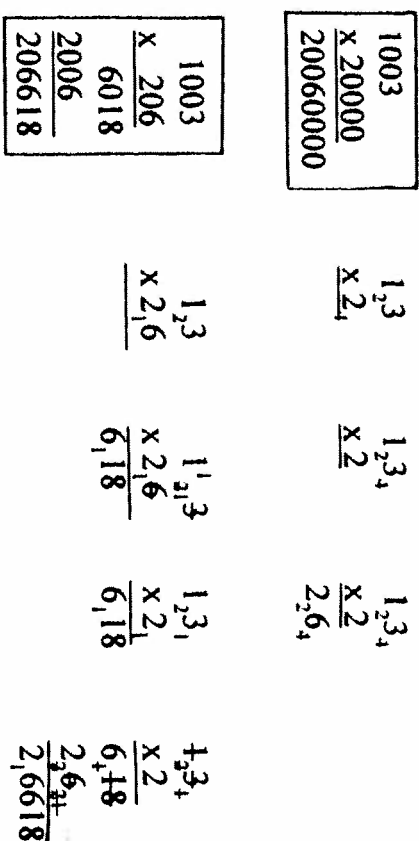


FIGURE 9.3. Examples of multiplications performed using the usual and alternative notation.

ficiently. However, except for purposes of illustration, I see no point in such a project.

Did you wonder what happens to these algorithms in base 2? Without zero, the numbers are represented by a string of ones either subscripted or normal; for example, 11111111. With only one level of subscripts, you lost the ability to represent consecutive zeros with any efficiency. This base 2 alternative model is isomorphic with zero notation base 2, and the zero is merely replaced with sub 1.

Error Case Study T: "Beyond Straight Lines"—A Teacher's Reflections and Explorations Into the History of Mathematics (by John R. Sheedy) ("Beyond Straight Lines" Case Study [7/9])

Parallel Lines. In the spring of 1969 I visited a friend of mine in Boston. One evening, having exhausted our other possibilities for amusement, he asked me if I played chess. "Of course," I replied. I have always enjoyed games, and I welcomed the challenge. Chess, I thought, was more interesting and far more elegant than checkers; played on the same board but with the added dimension of differently valued pieces that moved in special ways. The object was the same, however, that is, capture your opponents' pieces and protect your own.

We began to play. He moved his first pawn and I immediately set out to capture it. Each subsequent move on my part was followed by a usually direct and confrontational move on my part. I captured several of his pieces very early in the game and was delighted with my progress. Shortly, I perceived a threat to

my "important" pieces that I could not compensate for. At a loss for what to do, I made what even I considered to be a random distractionary move. My thinking was that he would go after my "sacrifice" piece and that the pause might allow me to fortify my position.

At that point he stopped me and asked why I had made such a foolish move—it did not make much sense. In fact, the way that I was playing the game did not make much sense to him. What was I trying to do? Capture his pieces, I told him. Why? What plan did I have? I had no plan. I merely reasoned that if I was clever and lucky I would be able to beat him. Luck, he told me, should have very little to do with it; there are certain rules that one adheres to, conventions that are employed especially at the beginning of play that reflect good play and development. He demonstrated the opening of a game: what was an acceptable move and what was an acceptable (logical) response. Further, he asserted that it was a game of balance between offense and defense; good play was structured play not merely random and opportunistic. I countered that I thought that being so predictable was boring. What sense did it make for me to make a move that he knew I would make? He explained that it was not a matter of my predictability but the predictability of a logical response; that actually considering the logic of various responses to any move and then counteracting with the best one, and further, that no move was insignificant, is precisely what makes the game interesting, challenging, and enjoyable. In the face of my reluctance to accept the "boundlessness" of games, he showed me that my simple play had no chance of ever succeeding. Any early gains that I could make by acting in my previous manner were merely illusory. The nature of the game is such that the error of a single move may not be immediately obvious. But, every move has a certain future implication that can be logically deduced; each move is inevitably linked to some future move, just as it is the reflection of those that preceded it.

We each had been playing a different game: I played an immediate, unreflective game, and he played a game of logical consequences that allowed him to engage his imagination at a more complex and satisfying level.

Reluctantly my eyes opened to the realm of new possibilities that the game represented. My restricted view had, in effect, limited the match to merely moving pieces about to pass the time. He showed me how it could be an expression of my ability to reason: to anticipate and visualize and demonstrate how I think, and to reflect on its implications. I had intuitively recognized my playing of the game as being an extension of my personality, which I certainly wanted to demonstrate at every opportunity, but in limiting my involvement I had diminished the scope of the challenge and my potential for satisfaction.

Since then I have enjoyed many games of chess, some immensely. I have, equally, anguished over errors that I have committed in the many games that I have lost. The game took on a whole new meaning once I accepted the responsibility for my thinking as demonstrated in my play on the board. Addi-

tionally, I learned that it is much easier to distinguish ourselves by superficial mannerisms than it is to engage each other on a level that recognizes what is essential about us all.

This Course. To each new situation we enter we bring a variety of notions, impressions, and expectations. I assumed that this course had to do with strategies that could be applied to remediation of errors. The implication for me was that errors in mathematics were easily recognized. This, to me, was one of the basic differences between mathematical thinking, mathematical truth, and the ambiguous nature of rightness and wrongness in any other area of my life or education. Does God exist? What does it mean to live a virtuous life? What is beauty? What is the meaning of *Don Quixote*? Did dropping an atomic bomb on Hiroshima "save" lives? What is the best car that I can buy? These questions can be variously debated. But, in the realm of science, and especially mathematics, there can be no debate or speculation as to the correctness of numbers and the rules that apply to them. There is no "what if" about these things. Mathematics represents the ideal of logical thinking. And it is, in a sense, outside of my experience.

I might not have even bothered to reflect on these things as they seemed so inherently and intuitively true. I accept "mathematical rules" as the given. To me is handed down accumulated knowledge that is represented each time I open a textbook. The challenge is for me to learn and apply the rules, as best I can. Then, having developed my own scheme of recognition and understanding of their workings, I may one day assist others who perhaps do not recognize and understand it as well. I am aware of the delightful nature of solving problems correctly, operating in a different symbolic language, and the usefulness of it all. This course has introduced me to various notions about mathematics that I would have thought were unnecessary to my education. Along the way it has generated some measure of confusion and discomfort. A footnote of a research paper that I read directed me to Kline's (1980) assessment of the current predicament of mathematics:

This book treats the fundamental changes that man has been forced to make in his understanding of the nature and role of mathematics. We know today that mathematics does not possess the qualities that in the past earned to it universal respect and admiration. Mathematics was regarded as the acme of exact reasoning, a body of truths in itself, and the truth about the design of nature. How man came to the realization that these values are false and just what our present understanding is constitute the major themes of this book . . . (Preface)

My immediate reaction was skepticism. What now, I thought. Are these the ravings of some "head-in-the clouds" type who is upset with the establishment

because it cannot accommodate his notion of the way things are? The aim of my efforts is generally to uncomplicate the complicated, to make sense of the diverse; to simplify my existence in a time when the barrage of information concerning just about anything and everything seems to have no responsibility or respect for a given standard. So many apparent contradictions are presented to us on such a regular basis that it seems one has all one can do just to maintain an even keel on the sea of our everyday lives.

What problems could there possibly be with math? I'm not aware of any problems. Math has always been straightforward and clear, and it works! The last thing that I want to hear, having spent all this time in a struggle to learn it, is that it has all been useless and wrong because $1 + 1$ no longer equals 2. I do not want to know that I have been relegated to membership in the Flat Earth Society because my perception of the way things are makes sense to me.

Such were my initial thoughts concerning a notion of mathematics as a fabric riddled with holes. What could be more certain than mathematics? Why, seemingly out of the blue, is there speculation and evidence that it has fallen into the realm of uncertainty, when, for what I know, it has worked well since the Greeks articulated it (and so many other things) such a long time ago?

Discovery. Until the early 19th century, the belief was that the laws of Euclid's geometry applied as well to the universe as to our terrestrial existence. The universe was a comfortable and known commodity. It was a profound occurrence, indeed, when other mathematically valid geometries emerged and discredited a way of thinking that had prevailed for over 2,000 years. That the universe could be described in some ways that were very dissimilar to the notions that men had grown accustomed to was unsettling because it meant that the universe reverted to an unknown quantity.

Euclid's geometry was born of practical knowledge about land surveying and architecture that the Egyptians had developed. The notions of points, lines, planes, and solids arose from everyday experiences with landmarks, footpaths, farmers' fields, and granite blocks. What he accomplished was to demonstrate that all of the geometrical theorems that had been accumulated up to that point could be derived from 10 basic postulates.

Two of these, the second and the fifth, apparently were viewed with some skepticism even in Euclid's day. It was not that mathematicians doubted that they were true (common sense seemed to indicate that they were) but there was some disagreement as to whether these two were self-evident truths. These postulates were, respectively, "A finite straight line can be extended indefinitely to make an infinitely long straight line," and "Given a straight line and any point off to the side of it, there is through that point, one and only one line that is parallel to the given line." The notion of infinity conspired against the former, whereas the compounded nature of the latter made it somehow conspicuous among the rest. The reluctance of some to accept these two as

universal truisms, however, did not impede the acceptance of Euclid's geometry as a standard for the next 2,000 years. It explained reality and made sense—things worked.

In 1824, Janos Bolyai made a discovery that appeared to finally resolve any question of uncertainty about the fifth postulate. He proved that the parallel line postulate was in fact a postulate, a self-evident truth. He went on, however, to assert that Euclid's was not the only geometry available to describe the universe. Bolyai's starting point was to replace Euclid's parallel lines postulate with another, namely, "Given a straight line and any point off to one side of it, there is, through that point, an infinite number of lines that are parallel to the given line." Then, from this and Euclid's nine other postulates, he derived theorems that were dramatically different from those of Euclidean geometry. What he described was a strange new universe that was quite unlike the earthbound projection that had been accepted for so long.

Actually, Karl Gauss, upon hearing of this new discovery, revealed that he had arrived at the same conclusions years earlier but had kept it to himself rather than incur the displeasure of his colleagues. Additionally, shortly thereafter, Nikolaus Lobachevski independently arrived at the same results. It was a way of thinking whose time had come.

In 1854, Bernhard Reimann revealed yet another geometry that differed from Euclid's. In Reimann's geometry, the second postulate states, "A finite line *cannot* be extended indefinitely to make an infinitely long line." Reimann's fifth postulate asserts that, "given a straight line . . . [there are] *not any lines* that are parallel to the given line."

For so long, the behaviors of points, lines, planes, and solids were described in correspondence with the perception of our senses. With the application of the new geometries—all logically founded—bizarre, nonsensical images of space were presented.

Mathematicians had considerable difficulty accommodating these new ideas. What made it so hard to accept was that Euclid's geometry had been delivered to science as a tool with which to define and measure space; it implied that the universe was an extension of our earthbound experience. It had to be so because it (the geometry) conformed to the rules and rigor of mathematical proof based on self-evident truth. When other interpretations appeared, derived and validated as they were in accordance with the rules that govern mathematical proof, their acceptance indicated a denial of what had preceded.

Additionally, as one could demonstrate that these other geometries "follow the rules," the question arose: Which do you accept as an accurate representation of spatial experience? Although one is obliged to grant the validity of the arguments for alternate geometries, they are resistible initially because they just do not seem to make sense. How is that possible? Is mathematics merely the invention of human imagination and not a body of universal truths based on common sense? Is reality merely the invention of the mind?

There were a variety of similar occurrences in mathematics, especially in the last 100 years. Georg Cantor, who revolutionized the concept of infinity late in the 19th century, discovered that his logic was not immune to paradoxical conclusions. In asserting that the whole could be equal to one of its parts is an inescapable and rational trait of the infinite realm, he was moved to remark, "I see it but I cannot believe it." This statement surely echoed the feelings of many who had to face the implications of the "new" geometries.

When Albert Einstein published his general theory of relativity in 1915, he revealed an explanation of how it is possible to infer the global geometry of something as immense as the universe by piecing together observations that could be made on the relatively minuscule human scale that we are restricted to. As an example, if you were asked to infer the shape of the earth by mapping out some small part of it, the procedure you might follow would be to pick a starting point; then, walk directly south 100 miles, then east 100 miles, then north 100 miles. Here, you would expect to be 100 miles west of your starting point. You proceed to measure the distance and discover that it is less than 100 miles. That is because you are moving on the surface of a sphere. Had all sides been exactly equal you could have deduced that the world was flat, but it is not. Reimann's geometry applies, and not Euclid's. As for "space beyond," the technology that we currently employ to map the far reaches of the universe implies that its shape seems to resemble Lobachevski's pseudosphere. (It should be noted that the truth of Euclid's geometry is context bound; it still works, but now we have recognized the scale of its application: The world looks different to the ant than it does to the bird.)

Euclidean geometry was effective for so many centuries that people mistook it for truth. So, the error that was exposed by the other geometries had to do with a certain way of thinking that has ultimately been undone.

History displays the incorporation of many ideas that were at one time considered distasteful and disorderly. That these ideas did not tamper with the basic foundational elements of mathematics—its logic—may have made them easier to accept and, so, conspired to maintain the impression of mathematical certainty. In *Mathematics, The Loss of Certainty*, Kline (1980) stated that the discovery of fundamental errors in basic arithmetic, geometry, and logic have left many in his field seemingly unaffected; they carry on as if the events of the last century had never happened; the certainty of their mathematics is undisturbed. He asserted that this denial is unhealthy. He felt that it is imperative for mathematics, never more fragmented than it is today, to face its crisis of uncertainty. And, he offered a more realistic perspective by means of an analogy:

Suppose a farmer takes over a tract of land in a wilderness with a view to farming it. He clears a piece of ground but notices wild beasts lurking in a wooded area surrounding the clearing who may attack him at any time. He decides therefore to clear that area. He does so but the beasts

move to another area. He therefore clears this one. And the beasts move to still another spot just outside the new clearing. The process goes on indefinitely. The farmer clears away more and more land but the beasts remain on the fringe. What has the farmer gained? As the cleared area gets larger the beasts are compelled to move farther back and the farmer becomes more and more secure at least as long as he works in the interior of his cleared area. The beasts are always there and one day they may surprise and destroy him but the farmer's relative security increases as he clears more land. So, too, the security with which we use the central body of mathematics increases as logic is applied to clear up one or another of the foundational problems. Proof, in other words, gives us relative assurance. (Kline, 1980, p. 318)

Reflection. My exposure to this historical view of mathematics makes the subject accessible to me. That mathematics was able to grow and maintain a certain perception of reality, for so long, was merely its good fortune. One can be deceived into thinking in a moment that things will always be as they are. Completeness and perfection are ideal, but change is the inevitable constant.

Life is difficult—this I reluctantly accept. Now, to see that mathematics is not immune to the uncertainty that afflicts my life is encouraging; it "humanizes" the subject. It, too, can be a slave, in its own way, to personal perspective, just as I am; just as we all are. I can relate to the feelings of those bewildered, amazed, and reluctant men (past and present) by reflection on any number of significant episodes in my life when my awareness was altered by a different perspective.

Our view of reality is like a map with which we negotiate the terrain of life. If our maps are to be accurate, we have to continually revise them because the world changes constantly and the vantage point from which we view the world changes as well. Uncomfortably, perhaps, we must admit that the map is never complete. We are directed, therefore, to keep an open mind and not so readily dismiss as erroneous some other perspective. We may even be encouraged to generate alternate perspectives in an attempt to further validate and continue to speculate about what we already hold (as we have done variously in this course). This attitude has potential: The map gets more complicated but it gets better, too. Just like playing chess.

Error Case Study U: Building on Probability Misconceptions—A Student Activity Created by a College Teacher (by Barbara Rose)
 ("Probability Misconceptions" Case Study [U/9])

Description of the Project. I have always been fascinated with probability and its numerous and varied applications, both theoretical and practical. As a

teacher, I am interested in students' concepts of probability, both as formal and working definitions. Hence my project is to have students respond to various probability situations and statements, most containing some degree of error. From the student's responses to errors and the errors they make in responding to them, I hope to better understand how students view probability and how errors can be used as a springboard for the learning and teaching of probability.

Procedures. I constructed a series of four questions on probability (reproduced later). Each of the first three consists of a situation followed by four statements for which students were asked to comment on truth value. The three situations were purposely selected to illustrate different contexts in which probability is used, from a more mathematical, cut-and-dried coin problem, to a weather prediction, to the gender of a baby. A fourth question asked for the definition of probability.

The following are not multiple-choice questions. Please comment on each part of the question.

1. Pretend I toss a fair coin 50 times. Comment on the truth of each of the following statements:
 - a) I could get 10 heads and 40 tails.
 - b) I would get 25 heads and 25 tails.
 - c) I would most likely get between 20 and 30 heads.
 - d) I couldn't get zero heads and 50 tails.
2. When we hear the weather forecaster predict the probability of rain today as 70%, comment on the truth of each of these interpretations:
 - a) You can't really count on the forecast since it is just a guess.
 - b) You can't really predict the weather since no one knows what the weather will be before it happens.
 - c) There is a better chance that it will rain than not rain, but 70% is an arbitrary number.
 - d) Under similar conditions in many cases in the past, it has rained about 70 times out of 100.
3. If a pregnant woman has had 8 children, all of whom are boys, comment on the truth of each of the following statements:
 - a) She will most likely have a boy because boys run in the family.
 - b) She has a high probability of having a girl because it is unlikely to have 9 boys in one family.
 - c) She is as likely to have a boy as a girl since the probability of having either is about $\frac{1}{2}$.
 - d) We have no idea what will happen.
4. What is the definition of probability?

The questions were administered in three different settings. Seventy-one earth science students (mostly ninth, but some tenth and eleventh graders) at Churchillville-Chili High School answered the questions with no context provided except that the teacher (who happened to be my husband) requested them to comply. In addition, 23 freshmen (mostly business students) at Roberts Wesleyan College responded to the written questions at my request. I did explain to them that I needed some input for a course I was taking that would hopefully eventuate in a better understanding of the teaching-learning process.

In contrast to their 94 written, individual responses, I solicited seven college students in the cafeteria who were willing to spend a half hour or so with me to help me with a project. None of these students had taken much math in college. For these students, we sat around a table and I asked them to verbally discuss the first three questions and comment on the truth of the statements. At the conclusion of the session, I requested that they individually answer Question 4—to write the definition of probability.

...

Results. Vast differences appear in response both to the content of the errors and in the method used to solicit them. Although the questionnaires reveal interesting responses to probability errors and fascinating errors in response to the errors, it is the small-group interaction that is rich in both the use of errors as diagnosis and exploration....

Small-group discussion. The discussion group consisted of seven students sitting around a table in a small, seminar room. I gave each of them the same questionnaire filled out by the other two groups, but asked them to read the question and respond verbally to the responses. My role, as teacher, was minimal; I occasionally asked them a question if I wanted a clarification on someone's comment or moved them to the next question when I saw time was becoming a factor. For most of the time, I simply observed the interaction and made notes of the conversation. All of the students participated in the discussion; in fact, there was often spirited discussion and heated debate. It was typical for one person to make an initial comment on one of the probability statements, followed by another's expression of agreement, challenge, or call for clarification. It was obvious that the answers "were in the making," for comments were subsequently retracted, modified, or expanded. For all three questions, the group came to consensus on an answer that showed good insight and understanding, except for one person who had an erroneous concept of probability.

On Question 1 concerning the coin, the students immediately discussed the words *would* and *could*—their meanings and impact on each response. Each contributed, contradictions arose, discussion ensued, minds were genuinely changed, new questions were raised (like would you bet the same on d as on

b?) and consensus was reached on each statement. Their answers revealed both an understanding of the importance of the language used and the probability concepts involved. For instance, they emphasized in (a) that although it is unlikely to get 10 heads and 40 tails, the word "could" made this a true but improbable statement. Likewise, they responded that the "would" in (b) was different than the "could" in (a), but still rendered the statement possible but not likely. The use of "couldn't" in (d) elicited an answer of "false" because this unlikely situation was still possible. Statement (c) was seen as clearly the most realistic because it gave a tolerance around the average response. There was one student who presented a problem for the rest of them. He thought that each combination (0-50, 1-49, 2-48, ..., 49-1, 50-) was equally likely and the others tried, unsuccessfully, to convince him of his error. They even got out coins, flipped them, and recorded the results. Due to the pressure of time, we had to go on before they convinced him, which left them all frustrated.

On the weather question, the students again examined the meaning of words like *arbitrary*. They discussed the difference between what it means to base prediction on facts, education, scientific research, and patterns, and one's ability to "count" on the forecast. One said, "So you can predict the weather even if you don't know what it will be because you are looking at patterns." They gave examples from their past of storms that did not happen and how that should be interpreted. They discussed what it meant to use a scientific method to reach an estimate and to what extent 70% was arbitrary. It appeared that they collectively, using these statements as a catalyst, came to a general understanding of what a weather prediction of 70% meant.

The gender question elicited the best example of errors as teachers. After they made smart comments like "Do they all have the same father?", "She's killing off the sperm," "She's not normal," "weaker chromosomes," and so on, they settled down to discuss the determining factors. When someone would comment that it all depended on the father, someone else would say, "But it's just like flipping coins, 50-50 chance." In response to (b), someone said "Under normal conditions, yes," to which someone responded "But that doesn't have anything to do with it. Don't take into account the other kids—it's 50-50." Someone else than said, "Yes, like a weighted coin." Another pointed out that it was similar to the 10 heads and 40 tails situation from Question 1. Although there had been the idea of independent events injected into the discussion of (a) and (b), there was not consensus on these statements until (c) was discussed. Then the contradictions between (a), (b), and (c) were apparent and minds were changed.

The combination of the errors themselves and the chance to interact with each other provided a wonderful format for learning. They saw the connections between the coin, weather, and gender questions, and they understood the distinction between the theoretical probability, short-term relative frequency, and the independence of events.

After the session was over, I explained the project and asked for their reactions to this kind of learning. They unanimously agreed that small-group interactive learning was preferred. They liked getting inside the heads of other people and thought it helped them get inside their own. They liked to make, find, and correct their own mistakes and felt it was a far more effective way to learn. The chance to argue about a concept, convince someone else, and be convinced was very appealing to them. In addition, they felt like a true participant in a small interactive group, rather than a passive listener. Their conclusion was that they would like to see more education structured in this manner.

All students, whether answering the questionnaire or participating in the small group, wrote out their definition to probability. Although there was great variety in style, completeness, and word choice, most students expressed the general notion of probability fairly adequately. The most common responses used words like *chance*, *likelihood*, *percentage*, and *frequency*.

Analysis/Interpretations. A small-group discussion presents an entirely different picture of the reaction to and use of errors. Whereas the questionnaire only gives a product of the responses to errors and a written record of other errors made in response to my devised errors, there is little evidence of process. So although I as teacher can diagnose their errors (to a limited extent), I cannot give immediate remediation or use these errors for exploration in that context. In contrast, for the small group there is diagnosis, remediation, and exploration—all without the intervention of the teacher. I am impressed that I did practically nothing during the entire session; every student took part and a healthy and vigorous conversation persisted. It was interesting to see the progression of thinking, the recycling of the ideas against the previous ones, the integration of the concept of probability from Question 1 to 2 to 3 and back again. On almost every question, someone would raise a question about the wording, so that language became an important consideration in the understanding of the question and its subsequent answer.

In addition, most students show little evidence that they learn from errors presented to them or from their own errors when they simply respond individually to a questionnaire, with no peer or teacher interaction. For the students in the small interactive group, however, the situation seems to be different, for they collectively force appropriate questions that address both the errors in the statements and their own errors as well. Although only one student appears to have changed answers on the questionnaire, there was considerable adjustment made in the small group.

There are other contrasts between the methods of questionnaire and small-group discussion. From the affective viewpoint, students responded differently. The students who completed the questionnaire did it obligingly, with little personal investment. After all, because there would be no feedback, what personal

or educational value would it serve except to possibly suggest some "food for thought?" In comparison, the students in the small group stayed well past the 20 minutes for which I solicited them. They were enthusiastic, for the feedback and stimulation appeared to satisfy both a cognitive and affective need. My reaction to the two methods, as teacher, is also interesting. Although there is plenty of "data" in the completed questionnaires, the group interaction is far richer than I had imagined. Much more emerged—about concepts, processes, and beliefs—than I had intended. Both students and teacher came away from the interaction as different because of the experience.

It would appear that most students do not know how to profit by errors without being taught how to do it. Maybe interaction and intervention is the first step. Although the desirable state would be for students to diagnose and remediate their own errors and even go farther by exploring where their errors could take them, few would appear to do this on their own. The first step may be for the teacher to assist a student in learning how to detect errors and what to do next. The preferred route, however, may be to use an interactive process whether the goal is diagnosis/remediation or exploration. For example, the students who completed the questionnaire are probably no more aware of their errors now than before they raised the questions, although the statements may have raised their awareness or curiosity about probability to pursue later. The teacher is the main one who benefits, for there is now a better understanding of what students believe about probability. For the group, however, each student gets immediate feedback on errors and a chance to remediate them on the spot. In addition, there is the opportunity to create doubt, play with the ideas, see where the errors may lead, and bounce new thoughts off others.

POTENTIAL BENEFITS OF ENGAGING MATHEMATICS TEACHERS IN A USE OF ERRORS AS SPRINGBOARDS FOR INQUIRY

The error case studies developed in the previous section have provided anecdotal evidence that mathematics teachers themselves can productively engage in error activities organized in the context of teacher education initiatives. Although the examples I have selected to report here all took place within a graduate mathematics education course, it is conceivable that similar activities could also be planned as part of methods courses, college mathematics courses attended by teachers, and even in-service workshops organized outside a university setting.

Although the error activities reported in the previous case studies could all be analyzed using the categories developed in Chapter 6, I have chosen not to do so here, but rather to focus my analysis on how these experiences could contribute to improving the preparation and continuing education of mathematics

teachers. Thus, in this section I discuss how the use of the proposed strategy in teacher education may provide the participating teachers with opportunities to:

1. Engage in genuine mathematical inquiry, problem posing, and problem solving.
2. Question and reflect on their views of mathematics.
3. Experience as learners a new strategy before they use it in their own classroom.

In my analysis, I also discuss why I believe that experiences of this kind could provide a valuable addition to current mathematics teachers' preparation as well as in-service programs.

Providing Teachers With Opportunities to Engage in Genuine Mathematical Inquiry, Problem Posing and Problem Solving

The successful completion of several college mathematics courses is a prerequisite for acquiring certification to teach mathematics in secondary school in most states and nations. Yet, because of the highly technical emphasis and the lecture mode of delivery characteristic of most college mathematics courses, even a secondary mathematics teacher may not have had much experience in solving novel and challenging mathematical problems and, even less, in initiating and pursuing more open-ended mathematical explorations. This is likely to be even more true for elementary school teachers, whose exposure to mathematics may have been minimal and often accompanied by feelings of inadequacy and fear.

In contrast, the activities reported in the previous "Teachers' definitions of circle" [Q/9], "Teachers' unrigorous proof" [R/9], and "Numbers without zero" [S/9] case studies created situations in which the participating teachers faced mathematical problems for which they had not previously learned an algorithm of solution—such as debugging an incorrect proof and evaluating a new infinite expression in the "Teachers' unrigorous proof" case study [R/9], categorizing a list of incorrect definitions of circle in the "Teachers' definitions of circle" case study [Q/9], or finding efficient ways to add, subtract, and multiply numbers written in a nonstandard notation system in the "Numbers without zero" case study [S/9]. Although some degree of problem posing was implicitly involved in all of these tasks, this was even more the case when the teachers engaged in the independent study of an error selected by them, as illustrated especially in the "Numbers without zero" case study [S/9].

It is interesting to observe that mathematics teachers can productively engage in genuine problem solving, problem posing, and inquiry both within mathematical topics that may be quite familiar and "easy" for them (such as circles in the "Teachers' definitions of circle" case study [Q/9] and arithmetic

operations in the "Numbers without zero" case study [S/9]) and in topics that are at the frontier of their own mathematical knowledge (such as calculus in the "Teachers' unrigorous proof" case study [R/9]). I believe that both kinds of experiences are worthwhile and can complement each other. Dealing with "simple" mathematical content may in fact help teachers who feel somewhat uncertain about their mathematical background and/or ability begin to take the risks involved in engaging in inquiry. The realization of the potential for problem solving and inquiry even in elementary areas of mathematics may also be important for some teachers to see how they could create similar experiences for their more "mathematically naïve" students. At the same time, I also think it is important for mathematics teachers to occasionally engage in activities involving more advanced mathematical topics, so as to experience an even greater challenge and, also, put themselves in a position that mirrors more closely that of their future students as they capitalize on errors.

Providing Teachers With Opportunities to Question and Reflect on Their Views of Mathematics

It seems reasonable to assume that a teacher's beliefs about the nature of mathematics will determine much of his or her decisions and behaviors in the mathematics classroom. For example, suppose a teacher believes that mathematics consists essentially of a set of predetermined rules and procedures to be mastered. Then it seems reasonable that she will focus her energies on preparing clear explanations and well-sequenced sets of application exercises, and will evaluate her students' performance mainly in terms of their ability to produce correct answers to the exercises and problems assigned to them. However, a teacher who perceives that the essence of mathematics lies in the employment of logical deduction to derive the consequences of a given set of assumptions is more likely to emphasize proofs and justifications versus correct results, both in her teaching and in the evaluation of her students' work. Although these two examples may seem rather extreme and obvious, research studies on teachers' beliefs have revealed that a teacher's conception of his or her discipline can influence his or her decision making in even more pervasive and subtle ways (see Thompson, 1992, for a review of the literature on this topic).

Although the results from studies assessing the belief system held by individual mathematics teachers are almost impossible to generalize, they have at least suggested that many teachers share with students and laymen a dualistic conception of mathematics—that is, they perceive mathematics as a rigid and impersonal discipline, where results are always uniquely determined and either absolutely right or wrong, and where there is no space for personal judgment and values (e.g., Brown et al., 1982; Cooney, 1985; Cooney & Brown, 1988; Copes, 1982; Meyerson, 1977).

Encouraging teachers to examine and reconsider their conceptions of mathematics, however, presents several difficulties. First of all, given the highly technical emphasis of their mathematical training, teachers have seldom (if ever) addressed issues regarding the nature of mathematics within previous mathematics courses and may not even recognize the possibility of controversy on such issues. Second, many teachers may feel uncomfortable in engaging in direct discussions of abstract and philosophical issues such as the role of rigor or the nature of truth in mathematics. Third, research on beliefs (e.g., Cooney & Brown, 1988) has made us aware that people's beliefs are not easy to access, nor is it simple to generate genuine dialogue on them, because many teachers may not even be aware of the beliefs about mathematics they hold. Therefore, these beliefs need to be "captured" in more indirect ways. To overcome these problems, Borasi and Brown (1989) suggested that teacher educators create "rich" situations that indirectly and spontaneously elicit the expression of teachers' beliefs and, consequently, stimulate dialogue amongst the participants, rather than initiate open discussions on the nature of mathematics. The error case studies reported in the previous section suggest that appropriate uses of errors (especially when the error is capitalized by assuming an inquiry stance of learning and focusing on the nature of math as level of discourse) may provide an effective means to create such rich situations in the context of teacher education initiatives.

Indeed, the previous "Teachers' definitions of circle" [Q/9], "Teachers' unrigorous proof" [R/9], and "Beyond straight lines" [T/9] case studies, in particular, have shown how mathematics teachers, as well as students at all levels of schooling, can capitalize on the potential of errors to raise questions with respect to some fundamental metamathematical notions (such as definition in the "Teachers' definitions of circle" case study [Q/9], and proof in the "Teachers' unrigorous proof" case study [R/9]) as well as more general issues regarding the nature of truth in mathematics (as illustrated both in the "Teachers' unrigorous proof" [R/9] and "Beyond straight lines" [T/9] case studies). Individual reflections as well as explicit discussions of these questions, combined with the personal experience of engaging in unusual mathematical activities (as discussed in the previous subsection), are in turn likely to challenge the dualistic views of mathematics held by many teachers and open them to the consideration of alternative ones.

Providing Teachers With Opportunities to Experience as Learners a New Strategy

Overall, whether they engaged in in-depth studies organized by the instructor (as in the case of the "Teachers' definitions of circle" [Q/9] and "Teachers' unrigorous proof" [R/9] case studies) or in independent projects where they took

full initiative of the inquiry stimulated by an error (as in the case of the "Numbers without zero" [S/9], "Beyond straight lines" [T/9], and "Probability misconceptions" [U/9] case studies) the teachers participating in my course ended up engaging as learners in several variations of the proposed strategy. In this way, they were able to experience in first person the excitement and the frustration of pursuing open-ended inquiry motivated by errors, the obstacles presented by their own preconceived notions of mathematics and errors, and the valuable outcomes that capitalizing on errors could produce.

I believe that this kind of experience is very important whenever a teacher is interested in implementing a novel instructional strategy that he or she never had occasion to witness as a student in prior mathematics courses. Such an experience can in fact provide a unique means to gain insights about the potential contributions of the strategy to one's instructional goals, at the same time enabling teachers to better predict their students' reactions when first introduced to it. Mindful of the well-known fact that most teachers tend to teach as they were taught, and that the traditional schooling most mathematics teachers went through is unlikely to have offered them much opportunity to capitalize on errors, experiences of this kind seem indeed a necessary prerequisite to understanding, appreciating, and effectively implementing the proposed strategy on the part of most mathematics teachers.

Chapter 10

Creating a Learning Environment Supportive of Inquiry

In my initial generative analysis of getting lost as a metaphor for error making (see Chapter 1) I raised the concern that a successful implementation of a use of errors as springboards for inquiry would require a conducive and supportive learning environment. In Chapter 2, I posited that an inquiry approach to mathematics instruction would satisfy such a condition. Indeed, the instructional episodes reported in Chapters 6 through 9 were informed by such an approach and were illustrative of its implementation in various instructional contexts. In this chapter, I would like to address what characterizes a learning environment supportive of student inquiry and how mathematics teachers could be prepared and supported in their efforts to establish such an environment. I hope that this discussion will be helpful to teachers interested in implementing the proposed approach to errors and, more generally, will provide a contribution to the current attempts to reform mathematics instruction.

In what follows, referring to the examples provided by the various error case studies developed throughout the book, I start by revisiting the claims made earlier in Chapter 2 that the proposed strategy implies a paradigmatic change in the way not only errors, but also mathematics, learning and teaching are conceived. Drawing from the instructional experiences reported in Chapters 6 to 8, I then try to highlight the major changes in terms of curriculum choices, evaluation, classroom dynamics, and social norms that seem necessary to create a learning environment supportive of student inquiry and, consequently, of a use of errors as springboards for inquiry. The chapter concludes with some suggestions for creating professional development initiatives that could support mathematics teachers as they engage in the challenging task of implementing the proposed approach to errors and the necessary supportive learning environment in their classes.

THE ASSUMPTIONS INFORMING A USE OF ERRORS AS SPRINGBOARDS FOR INQUIRY REVISITED

Earlier in the book, I argued that the inquiry approach to mathematics education informing the proposed strategy of capitalizing on errors relies on a number of fundamental assumptions about the nature of mathematics, learning, and teaching that are in sharp contrast to those that inform much of current mathematics teaching. The error case studies developed throughout the book have provided illustrations and supporting evidence that can now enable me to further articulate and discuss these assumptions and some of their implications for school mathematics.¹

Rethinking the Nature of Mathematics

Most people perceive mathematics as the "discipline of certainty" and, consequently, associate the ideals of objectivity, absolute truth, and rigor with mathematics. An inquiry approach, on the contrary, is informed by the belief that mathematics, like other products of human activity, is a humanistic discipline. Because mathematical results are not totally predetermined but rather socially constructed, their "truth" will depend on a number of factors including, besides logical coherence, the context of application, the criteria established by the mathematical community, and even to a certain extent personal values and judgments. Once it is accepted that mathematicians strive to reduce uncertainty without the expectation of ever totally eliminating it, ambiguity and limitations become integral and dynamic components of mathematical activity.

Let me expand on some of these points in light of the illustrations provided by specific error case studies:

- *Mathematical results are not predetermined but rather constructed.* The "historical" error case studies developed in Chapter 4 have provided compelling anecdotal evidence that mathematical results are not simply "discovered" in a straightforward manner, as the neat and organized way they are now reported in most textbooks or lectures may lead many students to believe. On the contrary, the troubled histories of the calculus and of the concept of infinity (as sketched in the "Calculus" [B/4] and "Infinity" [D/4] historical case studies, respectively) have revealed the centuries of intellectual struggle that were sometimes needed to produce even fundamental mathematical results. The debates and controversies that characterized the development of topics such as infinity ("Infinity" historical case study [D/4]) and non-Euclidean geometries ("Non-

Euclidean geometry" historical case study [C/4]) have also shown that mathematicians have occasionally proposed alternative yet legitimate solutions to mathematical problems. Thus, one has to realize that there is not just *one* unique way in which mathematics could develop. Rather, the mathematics community occasionally has to make choices among possible alternatives. These choices, although not predetermined, are far from being random either, as they are always guided by the consideration of potential coherence with the existing system and of the potential benefits that could be derived from each alternative. For example, the desirability of using numeration systems with zero is rather obvious once one realizes how computation becomes much more cumbersome in numeration systems without zero, as illustrated in the "Numbers without zero" case study [S/9]. More implicitly, these choices may also reflect the cultural values and political agendas of the time (as illustrated by both the case of infinity and non-Euclidean geometries).

- *Mathematical truth is not absolute, and may change in time.* Once one realizes that mathematical results are the product of human construction rather than of the gradual discovery of a predetermined system, one also has to accept that they are fallible like other products of human activity. As shown by Lakatos' historical analysis in the case of the development of one of the fundamental theorems of topology (see "Euler theorem" historical case study [E/4]), this construction of mathematical results often consists of an iterative process producing increasingly refined results, each of which can be assumed as "true for the time being," that is, until it is disproved and revised. If one abandons the hope of determining mathematical truth absolutely, it necessarily follows that mathematical results can only be sanctioned by a community of practice (the mathematical community of the time) on the basis of agreed-on criteria and of the existing mathematical knowledge. As illustrated especially by the history of calculus (see "Calculus" historical case study [B/4]) and of non-Euclidean geometries (see "Non-Euclidean geometry" historical case study [C/4]), both of these factors may change with time. The unrigorous way most results were derived at the beginning of the development of analysis, for instance, would not be considered acceptable today even from students in a beginning calculus course. Conversely, the absolute faith in the truth and uniqueness of Euclidean geometry, held for about two millennia by the mathematics community, had to be relinquished in light of the new evidence provided by the creation of the first non-Euclidean geometries, which revealed the possibility of a number of different axiomatic systems to represent spatial relationships that were all not only logically sound but also acceptable models of the physical space.

- *Mathematical truth depends on the context of application.* Whenever alternative systems or solutions are logically possible and plausible in mathematics, their use in a specific situation will require the consideration of the context of application. Besides the historical examples already mentioned in the previous

¹ For a more thorough articulation and discussion of the mathematical and pedagogical assumptions of an inquiry approach, see Chapter 11 in Borasi (1992).

points, I would like to remind the reader here of the evidence provided even by the more elementary examples discussed in the "Ratios" [A/1], "My definitions of circle" [G/5], and "Students' polygon theorem" [K/6] case studies. Adding numerators and denominators separately, for instance, is an obvious mistake if one is working with fractions, yet it is acceptable if one is trying to add ratios (as in the case of computing baseball batting averages—see "Ratios" case study [A/1]). Similarly, as discussed in "My definitions of circle" [G/5] and the "Students' polygon theorem" [K/6] case studies, respectively, the correctness of a given definition of circle or polygon cannot be fully determined *a priori* but rather will depend on the mathematical context in which it is interpreted as well as what we want to consider as instances of the concept in question.

• *Ambiguity and limitations are an integral part of mathematics.* Most of the examples developed in the preceding points also provide supporting evidence for the existence of ambiguity and limitations within mathematics. For example, the case of non-Euclidean geometries has revealed that the same real-life situation can be described by different mathematical models, each yielding some unique and different implications ("Non-Euclidean geometry" historical case study [C/4]). The discussion of specific definitions of circle conducted in "My definitions of circle" case study [G/5] showed how a given definition can determine a different set of objects and, consequently, imply a different set of properties, depending on the mathematical context in which it is interpreted. We have even seen that some of the properties associated with a familiar arithmetic operation may cease to hold when this operation is extended to new number systems (see the "Ratios" [A/1], "High school students' 0°" [L/6], and "College students' 0°" [P/8] case studies).

To conclude, the more relativistic, contextualized, and socially constructed view of mathematics that results from these considerations has the potential to affect not only the work of a small elite of professional mathematicians and philosophers, but also the everyday mathematical experience of nonspecialists. As shown by the examples reported in the error case studies developed throughout the book, the humanistic elements of mathematics discussed in this section affect all areas of mathematics, including elementary ones such as arithmetic and geometry. Appreciating these humanistic aspects of mathematics, in turn, could affect mathematics students at all levels in a number of complementary ways.

First of all, the recognition of limitations and "human" elements in the discipline could make it more attractive to people who have so far been intimidated by the absolute and authoritarian image of mathematics currently presented in schools—as expressed by the students participating in the error activities reported in the "High school students' 0°" [L/6] and "College students' 0°" [P/8] case studies, and by the teacher who authored the "Beyond straight lines" case study [T/9]. Furthermore, such a recognition could challenge some

common yet dysfunctional expectations about learning mathematics that have proven unsuccessful and invite students, instead, to realize that doing mathematics requires not only good technical knowledge, but also the ability to take into account the context in which one is operating, the purpose of the activity, the possibility of alternative solutions, and also personal values and opinions that can affect one's decisions.

An appreciation that mathematics as a discipline is not totally objective and predetermined, but rather is influenced by economic, cultural, and even political agendas—like other human domains—should also make mathematics educators question the choices made so far about what mathematics should be covered in the precollege curriculum. This, in turn, could lead mathematics teachers to realize the possibility and legitimacy of curriculum choices alternative to the pre-established course syllabi.

Most importantly, once we accept that mathematics is a social construct, the view of "knowing as inquiry" proposed by philosophers such as Dewey and Peirce can help us appreciate that the uncertainty that permeates the discipline should be perceived as a positive element rather than a limitation. The presence of ambiguity, limitations, and unavoidable errors in mathematics, revealed by the previous analysis, should be thus recognized as a major force for inquiry, and consequently for the production of mathematical knowledge, on the part not only of mathematics researchers, but users and students as well. Indeed, the evidence provided by all the instructional episodes reported in Chapters 6 to 9 supports such a claim.

Rethinking the Nature of Learning Mathematics

Another fundamental assumption of an inquiry approach to mathematics education, stated earlier in Chapter 2 on the basis of theoretical considerations as well as empirical evidence from psychological research, is that learning should be seen as a generative process of meaning making from the part of each student. Such a process is often stimulated by some perceived disequilibrium and involves making sense of situations and problems in light of the available data and one's previous knowledge and by building on social interactions. I believe that the instructional episodes reported in Chapters 6 to 8 have provided data that further support such a view of learning in the specific context of school mathematics. At the same time, they also contribute illustrations of what student mathematical inquiry actually looks like in practice.

Let me first comment more specifically on the some of the aspects of learning mathematics already articulated:

- *Learning mathematics as making sense.* A characterizing element of all the instructional episodes reported in this book is that at one point or another of the

activity the students were engaged in trying to make sense of some mathematical phenomenon. This was certainly the case as the participating students tried to critically examine specific incorrect definitions in the "Students' definitions of circle" case study [L/6], to understand the problems as well as the potential implications of some incorrect procedures in the "Students' homework" [J/6] and "Students' $\frac{1}{2} = \frac{1}{4}$ " [O/8] case studies, to decide whether a certain proposition was true in the "Students' polygon theorem" case study [K/6], to resolve an apparent contradiction in the "High school students' 0°" [L/6], "Students' P/A or B)" [M/8], and "College Students' 0°" [P/8] case studies, or to verify whether a triangle could be constructed given certain information in the "Students' geometric constructions" case study [N/8]. In all of these cases, the students had to make use of what they already knew about mathematics, even in areas that may not have seemed immediately related to the issue in question, along with information specific to the situation under study, to come to some satisfactory resolution. It is worth pointing out that, in several cases, making sense involved generating and pursuing new questions. Consider, for example, in the "High school students' 0°" [L/6] and "College students' 0°" [P/8] case studies, how realizing that different patterns led to contradictory values for 0° was considered by the participants as somewhat insufficient to understand why 0° should be undefined. In one case, a student wondered whether this unsatisfactory result could be avoided by creating an alternative system without zero (see "High school students' 0°" case study [L/6]), whereas in the other, the class tried to better appreciate the rationale and potential consequences of this decision by looking at other cases of undefined expressions in mathematics, such as those occurring when dividing by zero (see "College students' 0°" case study [P/8]).

- *Learning mathematics as stimulated by anomalies.* As it is to be expected given the focus of this study on capitalizing on errors, in most of the examples identified here, the catalyst and/or focus point for the learning activity was created by something puzzling—in other words, an *anomaly*. Although such an anomaly does not necessarily need to be associated with some sort of error, it seems crucial that inquiry-based mathematics lessons involve situations that are sufficiently problematic and open-ended to stimulate doubt and, consequently, curiosity so as engage students in genuine meaning making and learning.

- *Learning mathematics as a social activity.* Although each student engaged in learning mathematics has to make sense and construct his or her own understanding of the mathematical concepts, problems, or situations studied, this activity should not be conceived as occurring in isolation. Rather, the instructional episodes reported in Chapters 6 and 8 have shown how social interaction is a crucial component of this process. The "Students' polygon theorem" [K/6] and "Students' geometric constructions" [N/8] case studies were especially illustrative in this regard, in the context of a small-group activity and whole-class instruction, respectively. In both of these situations, all students

seemed to benefit considerably from their peers as they were forced to articulate their solutions and procedures, listen to other people's results and feedback on their own work, provide justifications for and/or revise their results when challenged by a peer, compare and evaluate alternative solutions proposed by different students, put together and/or elaborate partial individual contributions, and even reflect on the process as well as the product of such an activity and its significance.

Besides providing insights into the meaning and implications of assuming a constructivist approach to learning, as discussed earlier, several of the instructional experiences reported in this book have also provided rich illustrations of mathematics students engaged in genuine inquiry in the context of school mathematics. An analysis of these illustrations may now help to identify some characteristic elements of such an activity and thus further distinguish it from other learning experiences such as problem solving or discovery learning.

First of all, I think it is important to point out that student mathematical inquiry should not be identified only with the systematic and long-term activity taking place in articulated thematic units such as the one on the nature of mathematical definitions discussed in Chapter 7. Rather, one should also recognize as genuine inquiry the activities that occurred in shorter instructional episodes such as resolving the controversy created by two contradictory solutions to the problem of finding the probability of drawing "a jack or a diamond" from a deck of cards (in the "Students' P/A or B)" case study [M/8]) or constructing a triangle given certain conditions (in the "Students' geometric constructions" case study [N/8]). Despite their narrower scope, these experiences in fact still show the students engaged in an effort to make sense of the situation, considering and evaluating alternative solutions on their own without relying on the teacher's authority, debating among each other, and eventually even generating and pursuing new questions on their own.

Whether in the context of a long-term inquiry or within more isolated learning events, the students' mathematical activity in all of these cases is characterized by the facts that:

- The issue(s) addressed were sufficiently open-ended and controversial to allow for the generation of plausible alternative solutions and of genuine debate around them.
- The problems or issues discussed were not always set by the teacher and, at the very least, the students had a role in determining how the inquiry should develop and when it could be considered satisfactorily concluded.
- Digressions from the original planned activity were welcomed and encouraged.
- The students were expected to monitor and justify their mathematical activity and results.

- The students were expected to communicate their results convincingly to an interested audience (consisting at the very least of the other members of their class).

Rethinking the Nature of Teaching Mathematics. As it is to be expected, living by the assumptions about mathematics and learning articulated thus far has also affected the nature of the teacher's role and behavior in all the instructional experiences reported in Chapters 6 to 8. When looked at from the point of view of the teacher, these experiences indeed provide some insights into the meaning of reconceiving teaching mathematics as stimulating and supporting the students' own inquiries within a conducive learning environment. More specifically, in what follows I try to identify some changes in teacher's role that most distinguish these experiences from traditional mathematics instruction:

- *The teacher's role can be described better as "facilitator" rather than "instructor."* In all the instructional experiences reported in error case studies I through R, the instructor very rarely provided her students directly with information, either in the form of explanations or demonstrations. Rather, in all of these cases the instructor's main task has been to design mathematically rich and thought-provoking activities that would raise questions and engage the students actively in inquiry and meaning making. As these activities took place, the instructor still played an important albeit nontraditional role, whether in the context of whole-class instruction or small-group work. As well illustrated by the "Students' geometric constructions" case study [N/8] in the context of regular classroom instruction at the secondary school level, in inquiry-based lessons the instructor still holds important responsibilities such as monitoring the development of the activity and making decisions about how to best proceed after each stage, providing support to individual students as needed, and orchestrating sharing and discussion of results. It is important to appreciate that assuming such a facilitator role imposes much greater demands on the teacher than the traditional one.

- *Planning is not relinquished, although it takes on a very different form.* With the exception of the event reported in the "Students' homework" case study [J/6], all the instructional experiences discussed in the book were the result of careful planning on the part of the teacher (even if in some cases the lesson might have deviated somewhat from the original plan). Thoughtful planning is indeed a necessary prerequisite for success within an inquiry-based classroom, even if here the instructor is much less in control of the class agenda than in traditional classroom instruction, because he or she must always be ready to respond to her or his students' results and decisions. As illustrated especially in the case of the teaching experiment on mathematical definitions reported in Chapter 7, within an inquiry approach the teacher is first of all responsible for coming up with an initial question, issue, problem, or situation that is sufficiently rich and interesting to stimulate student inquiry. Materials and activities

that can help structure and stimulate such inquiry also need to be generated and tentatively structured in advance, so as to be available as options when needed to support specific students' explorations. Although the development of genuine student inquiry can never be fully predicted, an experienced teacher can in fact do much to foresee possible directions in which inquiry on a certain topic is likely to develop with her students and prepare supporting materials accordingly. At the same time, the instructor must always be ready to relinquish some of the activities planned if the students' inquiry moves in different and potentially more productive directions and, more generally, expect to use only a fraction of the ideas and materials developed in advance and to heed to develop new ones in response to the results of the students' work and decisions.

- *The teach needs to establish compatible beliefs and social norms in the classroom.* As already mentioned on various occasions, and dramatically demonstrated in the case of the "Students' $\frac{M}{N} = \frac{1}{4}$ " case study [O/8], a successful implementation of an inquiry approach requires students to develop and live by a different set of expectations about school mathematics than the one governing traditional mathematics classes. Teachers should not expect that such a switch will occur spontaneously as inquiry-based activities are first introduced in a mathematics classroom. Rather, explicit attention should be spent, especially at the beginning of the school year, to establish together with the students a new set of social norms that would be compatible and supportive of an inquiry approach. This could involve, first of all, some initial activities especially geared to elicit and discuss the participating students' beliefs about school mathematics. These experiences, however, should also be accompanied by ongoing reflections and discussions about the process followed in any nontraditional learning activity undertaken, so as to help the students better appreciate their rationale as well as potential benefits.

MAJOR IMPLICATIONS OF ADOPTING AN INQUIRY APPROACH TO SCHOOL MATHEMATICS

The instructional episodes reported in the book clearly illustrate how mathematics classrooms informed by an inquiry approach would look quite different from traditional ones. In what follows, I try to identify at least some of these differences with respect to curriculum goals and choices, classroom discourse and dynamics, evaluation, and social norms and expectations, respectively.

Curriculum Goals and Choices

The emphasis of "process over product" characteristic of an inquiry approach, along with the open-ended nature of the process of inquiry itself, requires first

of all a very flexible mathematics curriculum. At the risk of undermining the validity of the whole approach and sending students some mixed messages, within inquiry-based units students and teachers should feel free to pursue questions until a satisfactory resolution is achieved and to make digressions whenever promising new avenues of explorations open up. Thus, no curriculum that has been previously established in terms of a rigid list of mathematical content to be covered, however well constructed, would fully meet the needs of an instructional approach emphasizing student mathematical inquiry.

Yet, this does not mean that the mathematical content that students encounter as they engage in inquiry is irrelevant. On the contrary, I believe that the educational value of the experiences reported in Chapters 6 to 8 is due to the fact that the students engaged with some important mathematical concepts such as definitions ("Students' definitions of circle" [I/6] and "Students' polygon theorem" [K/6] case studies), circles ("Students' definitions of circle" [I/6] and "Students' homework" [J/6] case studies), analytic geometry ("Students' definitions of circle" [I/6] and "Students' case studies", probability of independent and dependent events ("Students' P(A or B)" case study [M/8]), geometric constructions ("Students' geometric constructions" case study [N/8]), arithmetic operations and their extension ("High school students' 0° " [L/6] and "College students' 0° " [P/8] case studies), or variables ("Students' $\frac{1}{4} = \frac{1}{4}$ " case study [O/8]). In fact, one could argue that because an inquiry approach requires more time, teachers must be even more conscious of selecting situations and topics for inquiry that are mathematically sound and valuable as the context of the inquiry. The recommendations provided by the *NCTM Evaluation and Curriculum Standards for School Mathematics* (NCTM, 1989) could provide some valuable guidelines to evaluate such choices, although I would like to warn against the danger of considering this document as another mandated curriculum to be covered. In sum, I suggest that it would pay for teachers to articulate their instructional objectives in terms of both some fundamental mathematical content (always keeping in mind that in this case more is less) and some processes that one would like the students to have mastered by the end of the course.

Reconceiving the goals of mathematics courses in this way will affect not only the content and organization of the curriculum for the course as a whole, but also how time is distributed among various activities and routines in everyday instruction. Students are very quick to realize which activities the teacher really values based on the time that is devoted to them in class. Thus, it would be crucial to devote considerable class time to small-group as well as whole-class explorations, to the sharing and discussion of the results of these activities, and to written and oral reflections on the process and its outcomes. Once again, because class time is a precious and finite commodity, this is likely to imply that the time devoted to other activities such as teacher explanations, review of homework, quizzes, and written exams may necessarily be reduced—as illustrated, for example, by the unusual routine of the mathematics lessons

reported in the "Students' P(A or B)" [M/8] and "Students' geometric constructions" [N/8] case studies.

Classroom Discourse and Dynamics

The considerations in terms of curriculum and goals articulated in the previous point have obvious consequences on classroom organization, as well as teachers' and students' behavior. Whereas traditional classrooms are centered mostly on the teacher (who explains, demonstrates, assigns worksheets, questions students, and evaluates their work), in inquiry-based classrooms the focus is on the students' own activities. As mentioned earlier in this chapter, whereas the teacher maintains an important role in planning and monitoring classroom activities, students also have an increased responsibility and say on the nature and direction of their work, as well as on its evaluation. Furthermore, whether the students work as a whole class, in small groups, or even individually on specific tasks, interaction among themselves is always an integral component of their learning experiences, as they act as a community of practice in constructing and critically examining their learning (as well illustrated especially in the dynamics of the instructional events reported in the "Students' polygon theorem" [K/6], "Students' P(A or B)" [M/8], and "Students' geometric constructions" [N/8] case studies).

This emphasis on interaction and collaboration also highlights the crucial role played by communication in an inquiry-based mathematics classroom. This was well illustrated by the error activities reported in Chapters 6 and 8, where students were continuously expected to talk and listen to each other, as well as the teacher, so as to share results, provide and receive feedback, resolve disagreements, verify the validity of procedures and conclusions, or reflect on the process they had engaged in. Thus, both the content and the participants' role in classroom discourse within inquiry-based classrooms are very different from those characteristic of the traditional classroom, where conversations are essentially dominated by and centered on the teacher.

It is important to appreciate, once again, that the classroom dynamics and discourses just described imply that teachers assuming an inquiry approach are going to lose the control that is powerfully, although implicitly, provided by a predetermined lesson plan combined with an emphasis on student individual seatwork. Thus, mathematics teachers will need to develop new approaches to classroom management to respond to the changed relationships and routines established in classrooms informed by a spirit of inquiry.

Evaluation

The mathematics education community has become increasingly aware that reform in curriculum and teaching practices needs to go hand in hand with a com-

patible revision of evaluation criteria and tools in order to be really effective (e.g., Marshall & Thompson, 1994; NCTM, 1989, 1995). This is especially true in the case of an inquiry approach to mathematics instruction, because currently students' measures of mathematical achievement rely heavily on multiple-choice and/or standardized tests, which by their very nature value the production of exact answers to rather mechanical tasks and do not even attempt to measure the kind of learning that students may derive from engaging in genuine inquiries.

In order to support the goals and spirit of an inquiry approach, evaluation should first of all be reconceived to address the acquisition not only of procedural knowledge, but also of conceptual knowledge, problem solving and posing heuristics, learning and metacognitive skills, creativity, independence, and attitudes toward the discipline. Because most error case studies have focused on isolated instructional episodes, this book has not provided many illustrations about what formal evaluation informed by these principles would look like in mathematics classrooms. Although some ideas in this regard can be gathered by the evaluation of the teaching experiment on mathematical definitions reported in Chapter 7, for this important issue I would like to refer the reader to the growing literature on mathematics assessment that is based on both research and innovative practices (e.g., Kulm, 1990; Leder, 1992; Lesh & Lamon, 1992; Niss, 1993a, 1993b; Romberg, 1992; Webb, 1992; Webb & Coxford, 1993).

Thinking of implementing the strategy of capitalizing on errors more specifically, it is also important to consider that students' negative attitudes toward errors have a quite justified root in the common use of errors as a main measure to determine grades in test situations (where the grade decreases as a function almost uniquely of the errors the student has made). Such a use of errors needs to be limited and complemented with other means of evaluation that would reward (rather than punish) risk taking and thus help remove students' justified concern for avoiding errors at all costs.

Social Norms and Expectations

The theoretical assumptions about the nature of mathematics, learning, and teaching discussed earlier, along with the instructional changes identified in the previous points, all have practical implications that can challenge considerably the expectations about school mathematics that most students have developed after years of traditional schooling. Consider the following list of typical expectations identified by Schoenfeld (1992) in his summary of the results of research on students' beliefs:

- Mathematics problems have one and only one right answer.
- There is only one correct way to solve any mathematical problem—usually the rule that the teacher has most recently demonstrated to the class.

- Ordinary students cannot expect to understand mathematics; they expect simply to memorize it and apply what they have learned mechanically and without understanding.
- Mathematics is a solitary activity, done by individuals in isolation.
- Students who have understood the mathematics they have studied will be able to solve any assigned problem in five minutes or less.
- The mathematics learned in school has nothing to do with the real world.
- Formal proof is irrelevant to processes of discovery or invention. (p. 359)

It is worth contrasting explicitly these beliefs with a few of the practices and expectations that characterize most of the instructional episodes reported in Chapters 6 to 8:

- Students are expected to make sense of mathematical problems and situations, to understand their own as well as their peers' solutions and their justification (see especially the "Students' polygon theorem" [K/6], "High school students' 0°" [L/6], "Students' P(A or B)" [M/8], "Students' geometric constructions" [N/8], and "College students' 0°" [P/8] case studies).
- Students can find the solution to novel and challenging problems (see especially the "Students' polygon theorem" [K/6] and "Students' geometric constructions" [N/8] case studies).
- It is worth listening to how other students have approached a problem, even when you have already reached a solution and you are convinced it is the correct one (see especially the "Students' P(A or B)" [M/8] and "Students' geometric constructions" [N/8] case studies).
- A mathematical problem may take a full lesson or even more in order to be satisfactorily addressed (see especially the "Students' polygon theorem" [K/6], "Students' P(A or B)" [M/8], "Students' geometric constructions" [N/8], and "Students' $\frac{1}{4} = \frac{1}{4}$ " [O/8] case studies);
- Justifying your results is an integral part of mathematical activity, and the responsibility of the student rather than the teacher (see especially the "Students' polygon theorem" [K/6], "Students' P(A or B)" [M/8], and "Students' geometric construction" [N/8] case studies).
- Discussions about mathematics as a discipline and about the process of learning mathematics are legitimate school mathematics activities (see especially the "High school students' 0°" [L/6], "Students' geometric constructions" [N/8], and "College students' 0°" [P/8] case studies).
- It is worth paying explicit attention to recognized errors (see especially the "Students' definitions of circle" [L/6], "Students' homework" [J/6], "Students' geometric constructions" [N/8], and "Students' $\frac{1}{4} = \frac{1}{4}$ " [O/8] case studies).

- Confusion is a necessary and valuable component of learning mathematics (see especially the "High school students' 00" [L/6], "Students' geometric constructions" [N/8], and "College students' 00" [P/8] case studies).

Although far from complete, this list further supports the claim, made several times throughout the book, that an inquiry approach requires establishing a new set of social norms in the mathematics classroom. As argued earlier in this chapter, the success of introducing any activity or strategy informed by an inquiry approach will be doomed to fail if the teacher does not devote explicit attention and class time to this issue.

SUPPORTING TEACHERS IN THE IMPLEMENTATION OF AN INQUIRY APPROACH IN THEIR CLASSROOMS

It is to be expected that implementing the radical instructional changes outlined in the previous sections along with the suggested uses of errors will be no easy task for mathematics teachers. Formal as well as informal professional development initiatives are needed to support interested teachers in such a challenging enterprise and to help them make it a success.

In the past few years, as an integral part of projects involving the implementation of some instructional innovation informed by an inquiry approach and in my role as teacher educator, I have actively participated in a number of such professional development initiatives. These firsthand experiences, together with the results of the growing literature on educational reform and teacher change (e.g., Schifter & Fosnot, 1993; Simon, 1994), have helped me identify some principles and specific kinds of experiences that I hope will be helpful for creating teacher preparation and enhancement programs aimed at promoting the strategy of capitalizing on errors and other reform initiatives informed by an inquiry approach.

More specifically, I suggest that teachers interested in implementing the strategy developed in this book be supported through long-term professional development initiatives that would enable them to: (a) experience as learners the kind of learning experiences that they would like to provide to their students, so as to better appreciate what these innovative learning approaches may in-

² These experiences comprise several professional development initiatives for mathematics teachers that I have organized within various projects supported by grants from the National Science Foundation—the already mentioned "Using errors as springboards for inquiry in mathematics instruction" (MDR-8651582) project, another research project focusing on developing a specific instructional strategy to support student mathematical inquiries, entitled "Reading to learn mathematics for critical thinking" (MDR-8850548), and a teacher enhancement project entitled "Supporting middle school learning disabled students in the mainstream mathematics classroom" (TPE-9153812).

volve and how students may react to them; (b) reflect on these experiences so as to become aware of the potential benefits and drawbacks of the proposed instructional innovation, and of its main instructional implications; (c) develop rich images of how the proposed instructional innovation could play out in specific instructional contexts, to be used as a concrete reference and model as they try to create similar experiences for their students; (d) find support, feedback, and inspiration as they actually begin to implement the proposed instructional innovation in their classes. In what follows I try to better articulate each of these components in the specific case of promoting the implementation of the proposed strategy of using errors as springboards for inquiry.

- *Enabling teachers to experience the proposed strategy as learners.* The experiences reported in Chapter 9 have illustrated how mathematics teachers themselves can fruitfully engage in challenging mathematical activities that involve a constructive use of errors. First of all, the whole group of teachers participating in a preservice or in-service course intended to promote the strategy of capitalizing on errors could engage as "students" in error activities previously organized by the instructor or facilitator in charge of the course (as illustrated by the "Teachers' definitions of circle" [Q/9] and "Teachers' unrigorous proof" [R/9] case studies). These error activities, in turn, may involve either mathematical content that is challenging for the teachers themselves, so as to put them in a situation very similar to that of their future students (as illustrated by the "Teachers' unrigorous proof" case study [R/9]) or mathematical topics from the K-12 curriculum, so as to show the potential for inquiry and mathematical challenge existing even within content the teachers are likely to address in their classes (as illustrated in the "Teachers' definitions of circle" case study [Q/9]). In both cases, the data reported in these error case studies suggest that as a result of these experiences the participating teachers could come to better realize the potential of errors to stimulate valuable mathematical activities and learning, as well as to recognize and overcome some of their own initial prejudices against errors. Furthermore, these experiences may be important for most teachers insofar as they might represent the first time they engaged in mathematical inquiries themselves, given the traditional lecture approach characterizing most formal mathematical training. As a complement to these whole-class error activities led by the instructor, it is important to remember that the teachers participating in my courses also engaged in a more independent use of errors as springboards for inquiry on their own, in the context of conducting the final project required by the course (see examples reported in the "Numbers without zero" [S/9], "Beyond straight lines" [T/9], and "Probability misconceptions" [U/9] case studies). I think this kind of experience is quite important for teachers, because implementing the proposed strategy in their classes would require them to be able to identify and use the possibility of capitalizing on errors on their own, so as to create valuable error activities for their students.

• *Encouraging teachers' reflections on the educational potential and implications of the proposed strategy.* As already argued earlier in this chapter in the case of students, even teachers participating as learners in error activities may not fully recognize the potential values and implications of these experiences, unless they explicitly reflect on and discuss the process they went through, what they learned from it, and their reactions to the activity. In order to achieve this goal, teacher educators should encourage such reflections through written assignments and class discussions. The "Teachers' definitions of circle" (Q/9) and "Teachers' unrigorous proof" (R/9) case studies have provided some concrete illustration of how journals can be effectively used as a place where the participating teachers articulate and share these reflections on an ongoing basis. A similar role, however, could also be played by more isolated and focused writing assignments, as well as by devoting some "reflection time" to discuss in class the participants' thoughts and reactions to the activity just concluded. As a result of these experiences, one would hope that the participating teachers would gain a better understanding of important elements such as the various and complementary ways in which errors could be used constructively, the kind of learning goals that each variation is most likely to support, the characteristics of a learning environment supportive of the strategy, potential students' reactions, and potential obstacles to the implementation of the strategy and ways to overcome them.

• *Providing images of how the proposed strategy can play out in practice.* Although experiencing as learners some error activities may already be an important step to visualizing what capitalizing on errors could actually mean in the context of mathematics instruction, it may not be sufficient for many teachers. As a prerequisite to implementing the strategy in their own classes, many mathematics teachers may like to "see" how specific error activities have been designed for students of age and ability similar to those of their own students, how these students may have reacted to such experiences, how other classroom teachers actually dealt with elements such as curriculum choices and classroom discourse in these situations, and so on. Ideally, this information would be best acquired by means of direct observation, for a sustained period of time, in classrooms where such experiences are taking place. This option, however, may not be available to interested teachers in most cases. As a valuable substitute, I believe that mathematics teachers may benefit from reading rich and detailed stories of classroom experiences where errors were capitalized on in a variety of instructional contexts dealing with different mathematical topics. It is my hope that the error case studies reported in Chapters 6 and 8 can in part perform this role, although I realize that they are still too few to reflect the reality and interests of all the mathematics teachers potentially interested in using errors as springboards for inquiry in their classes. Furthermore, it is important to recognize the limitations inherent in isolated instructional vignettes, because they can rarely provide sufficient information about what happened prior to the specific

episode reported (in terms of establishing classroom routines, social norms, and expectations) that actually made it possible or how the proposed innovation could inform some long-term instructional planning. This would call, in turn, for more extended and comprehensive "stories" (e.g., the one reported in Borasi, 1992) involving the implementation of a use of errors as springboards for inquiry as part of regular classroom instruction throughout an entire course.

• *Supporting teachers as they try to implement the proposed strategy in their classes.* The failure of many past attempts at educational reform (such as the "New Math" of the 1960s in the United States) have made teacher educators and reformers aware that in-service courses and workshops, however well designed and carried out, are rarely sufficient to enable the participating teachers to go back to their classrooms and implement the principles and techniques learned there on their own. Teachers especially need feedback and assistance at this stage, because it is just as they start incorporating new ideas into their teaching practice that they may fully realize the meaning and implications of these ideas and, at the same time, begin to seriously question some of their assumptions and prerequisites. Thus, to be really effective, a professional development initiative aiming at promoting a use of errors as springboards for inquiry in school mathematics should include, as an essential and integral component, some structured and supported field experiences. As such, I suggest first of all the creation of a support group that would meet on a regular basis to both provide ideas and constructive criticisms on the teachers' initial plans for innovative classroom experiences and, later, share and discuss the results of implementing such plans in the teachers' own classrooms. Such a support group would consist of a small number of participating teachers and at least a teacher educator or researcher playing the role of facilitator. The participating teachers could be further sustained in their efforts at improving their teaching practice by the suggestion of a sequence of activities that could help them introduce the innovation in question gradually in their classes—for example, in the specific case of the strategy of capitalizing on errors, one could suggest that the teachers first try to implement a modified version of an error activity they have previously experienced or read about, then design and implement an isolated error activity of their own, and later attempt to design a more comprehensive thematic unit that capitalizes on errors whenever appropriate. Help in systematically evaluating the results of these experiences and learning from them should also be provided to the participating teachers, and could include suggesting strategies to collect relevant data from their students and document the experience, providing a list of key questions to keep in mind as they reflect on what went on in class, or even making available some alternative models to evaluate instructional experiences.

Chapter 11

Conclusions

Throughout this book, the strategy of capitalizing on errors as springboards for inquiry has been progressively developed through the analysis of its implementation in different contexts within specific error case studies. In this final chapter, I summarize the key elements of this approach to errors and its potential implications for improving mathematics instruction and supporting education reform more generally. More specifically, in the first part of this chapter I review and synthesize variations in the proposed strategy and potential benefits that have been identified in previous chapters, as well as some considerations about capitalizing on errors in mathematics instruction at different levels of schooling. I then conclude the book by suggesting further research aimed at refining and generalizing the strategy of capitalizing on errors in a number of complementary directions, within mathematics instruction and other academic domains.

USING ERRORS AS SPRINGBOARDS FOR INQUIRY IN MATHEMATICS INSTRUCTION: A SUMMARY

The strategy developed in this book was originally motivated by the belief that errors could provide valuable learning opportunities for students if appropriately used in mathematics instruction. This instructional strategy calls for mathematics learners at all levels to engage in the study of specific mathematical errors in ways that should be guided not only by the desire to correct such errors, but also by the willingness to identify and pursue the more open-ended explorations that these errors can motivate. As such, the proposed strategy differs considerably from the approaches to errors that are currently most popular in mathematics education—namely, punishing or ignoring the errors made by students in class, or at best using the information these errors can provide for diagnosis and remediation directed by the teacher.

The error case studies developed throughout this book have provided compelling anecdotal evidence that professional mathematicians, mathematics teachers, and college and secondary school students, can all productively engage in these kinds of activities and benefit from them in a variety of ways. These results are further supported by the data collected in a number of other experiences conducted at the secondary school level and in teacher education contexts (see Borasi, 1986b, 1991b). The possibility of employing this strategy even with elementary school students is also suggested by the experience reported in Brown (1981), Lampert (1987, 1990) and Yackel et al. (1990). In sum, the approach to errors advocated in this book seems both valuable and feasible for mathematics learners at all levels, although both the content and the extent of the inquiry motivated by such a use of errors may vary considerably depending on the subject's mathematical ability, expertise, and interests.

The analysis of various implementations of this strategy in different contexts, as developed especially in Chapters 4, 5, and 6, has also suggested that using errors as springboards for inquiry should not be viewed as a monolithic strategy, but rather one that can vary considerably with respect to the nature and source of the error studied, the level of student involvement in the activity, and both the level of mathematical discourse and the stance of learning assumed in the lesson. In what follows, I briefly review the major variations within each of these variables and some of their implications for implementing the proposed strategy in mathematics instruction.

First of all, with respect to the level of mathematical discourse, it may be important to appreciate that the analysis of an error could engage students in mathematical activities operating at a different level of abstraction, such as:

- Performing a specific mathematical task.
- Learning about some technical mathematical content.
- Learning about mathematics as a discipline.

Although each of these mathematical activities are valuable and important for mathematics instruction, they may serve different and complementary goals such as increasing the students' ability in doing mathematics, contributing to the students' conceptual understanding of specific mathematical concepts and topics, and enabling students to appreciate the nature of mathematics and mathematical thinking, respectively.

Within each of these levels of mathematical discourse, the stance of learning informing a specific instructional activity can also influence the use of errors made in it, as each of the following complementary approaches could be assumed:

- Remediation—that is, acknowledging that errors could be analyzed with the main goal of determining what went wrong and thus correct it.

- Discovery—that is, approaching errors as steps in the wrong direction that can be used constructively when learning or solving something new.
- Inquiry—that is, recognizing and pursuing the potential of error to stimulate new questions and explorations.

Once again, although valuable error activities can be developed by assuming each of these stances of learning, it is important to appreciate that both the nature and the learning outcomes of such activities may differ. Finally, a combination of all these categories reveals the possibility of at least nine complementary uses of errors as springboards for inquiry, summarized in the 3×3 matrix reported in Table 11.1.

TABLE 11.1. A Taxonomy of Uses of Errors as Springboards for Inquiry

	Level of Math Discourse		
	Performing a Specific Math Task	Understanding Some Technical Math Content	Understanding the Nature of Mathematics
Stance of Learning			
Remediation	Analysis of recognized errors to understand what went wrong and correct it, so as to perform the set task successfully. (Remediation/task)	Analysis of recognized errors to clarify misunderstandings of technical mathematical content. (Remediation/content)	Analysis of recognized errors to clarify misunderstandings regarding the nature of mathematics or general mathematical issues. (Remediation/math)
Discovery	Errors and uncertain results are used constructively in the process of solving a novel problem or task; monitoring one's work to identify potential mistakes. (Discovery/task)	Errors and uncertain results are used constructively as one learns about a new concept, rule, topic, etc. (Discovery/content)	Errors and uncertain results are used constructively as one learns about the nature of mathematics or some general mathematical issues. (Discovery/math)
Inquiry	Errors and puzzling results motivate questions that may generate inquiry in new directions and new mathematical tasks to be performed. (Inquiry/task)	Errors and puzzling results motivate questions that may lead to new perspectives and insights on a concept, rule, topic, etc., not addressed in the original lesson plan. (Inquiry/content)	Errors and puzzling results motivate questions that may lead to unexpected perspectives and insights on the nature of mathematics or some general mathematics issues. (Inquiry/math)

The nature of an error activity will also depend somewhat on the type of error considered, because different kinds of errors—such as incorrect definitions, correct results reached through incorrect procedures, wrong results, conjectures refuted by a counterexample, or contradictions, just to name a few significant categories—are likely to invite different kinds of questions for exploration and reflection (see Chapter 5 for an explicit identification of some of these questions for each of the types of errors just listed). This fact should be taken into consideration when planning error activities intended to meet specific instructional goals. These considerations also suggest that, in order to take full advantage of the proposed strategy in school mathematics, curriculum developers and teachers should be invited to interpret the term *mathematical error* in the most comprehensive way possible.

As discussed in Chapters 5 and 6, variations within an instructional use of errors as springboards for inquiry could also occur with respect to the source of the errors considered along two different dimensions:

1. Who made the error studied, that is, whether the error was inherent to mathematics itself, or the person who made the error was the same person now engaging in the error activity, another member of the group engaged in the error activity, the classroom teacher, a peer outside such group, a more mathematically naïve person, or a more mathematically expert person.
2. How the error to be pursued was chosen, that is, whether the error studied was previously selected and introduced by the teacher within a planned error activity, was made spontaneously in class but expected by the teacher, or was totally unexpected and pursued impromptu.

An appreciation of all the possibilities identified by this categorization may be valuable for mathematics teachers interested in capitalizing on errors in their classes in two complementary ways. First of all, it may open up the consideration of a wider pool of possible starting points for error activities. Second, when planning such error activities, teachers need to be aware that their students, as well as themselves, may show different affective reactions depending on the source of the error studied.

Finally, it is obvious that planning error activities as part of classroom instruction is not sufficient per se to guarantee that the students will actually take advantage of the potential for inquiry and learning that these activities offer. In evaluating the effect of implementations of the strategy in specific instructional contexts, therefore, it will always be important to determine whether and to what extent each student actually engaged in an error activity. At the same time, it is also important to take into consideration that, depending both on the nature of the activity and on the individual student's participation in it, student involvement in an error activity can productively occur at different levels, as

students are likely to most often engage actively in an error activity organized and led by the instructor, but may also occasionally initiate and develop an error activity on their own, and at other times they may benefit simply from the teacher modeling the strategy, that is, reporting on an inquiry he or she has developed around an error.

These considerations suggest that all the variations within using errors as springboards for inquiry identified here, although complementary to each other, also present important differences in terms of the types of mathematical results that they can help to achieve, the educational objectives they may facilitate, their compatibility with current instructional goals and practices, their demands in terms of creativity and improvisation from both teacher and students, and their potential impact on students' conceptions of mathematics. Mathematics teachers interested in implementing an approach to errors as springboards for inquiry should be aware of these differences, if they want to best take advantage of the educational potential of this strategy in relation to their own instructional goals and teaching styles.

Keeping this caveat in mind, it is important to recognize that many of the error activities reported throughout the book provided the participating learners with several of the following learning opportunities, rarely offered within traditional mathematics lessons:

- Experiencing constructive doubt and conflict regarding mathematical issues.
- Pursuing mathematical explorations.
- Engaging in challenging mathematical problem solving.
- Experiencing the need for monitoring and justifying their mathematical work.
- Experiencing initiative and ownership in their learning of mathematics.
- Reflecting on the nature of mathematics.
- Recognizing the more humanistic aspects of mathematics.
- Verbalizing their mathematical ideas and communicating them.

On the basis of both theoretical arguments and anecdotal evidence, throughout the book I have also argued that active participation in error activities can enable students to better understand specific mathematical content and/or aspects of the nature of mathematics, acquire valuable problem-solving and problem-posing heuristics and metacognitive or communication skills, and/or increase their mathematical confidence and self-esteem. As these are all learning outcomes deemed especially important by most mathematics educators today, capitalizing on errors could provide a valuable contribution to achieving the goals articulated by the most recent calls for school mathematics reform.

Furthermore, as discussed in Chapter 9, I believe that teachers who take on the challenge of capitalizing on errors in their teaching may gain more than the

addition of a valuable instructional strategy to their "bag of tricks." Implementing such a strategy in their classrooms can in fact become the catalyst for valuable reflections about their pedagogical beliefs and teaching practices more generally and, consequently, invite radical changes in classroom routines, discourse, and social norms. In other words, I suggest that a use of errors as springboards for inquiry could provide interested teachers with a focus and a heuristic for their more comprehensive efforts of creating mathematics classrooms where students actively engage in mathematical inquiry and sense making as a community of learners.

To conclude this section, I would like to comment on the fact that my own initial conception of "using errors as springboards for inquiry" evolved considerably as a result of the progressive development and analysis of the proposed strategy on the basis of its implementation in different contexts, as reconstructed in this book. Initially, as reflected in my first publications on this subject (Borasi, 1985c, 1986b, 1987a), I thought of error activities mainly in terms of presenting students with a specific error to be examined and used in a number of different directions previously planned by the teacher, assuming a stance of inquiry. The various ways errors can be capitalized on, as summarized in this section, have revealed that these are just some of several ways of employing the potential of errors to invite students to engage in mathematical inquiries. I also came to realize that I did not initially appreciate the full possibility and implications of instructional situations where errors are spontaneously used in constructive ways as an integral part of a problem-solving and/or inquiry activity—revealed instead by examples such as the "Students' polygon theorem" [K/6] and "Students' geometric constructions" [N/8] case studies. In these cases, the idea itself of capitalizing on errors within a clearly defined and somewhat isolated error activity even starts to lose meaning. In other words, as errors become an integral part of the original activity, their use as springboards for inquiry may appear more invisible as well as more essential.

LOOKING AHEAD

As expected with every research study, the articulation of the strategy of using errors as springboards for inquiry and its instructional implications, developed throughout this book, has opened new questions and possibilities that call for further research. I would like to conclude with the identification of at least some of these potential avenues of inquiry and an invitation to pursue them.

First of all, the mathematics education community could benefit from the systematic analysis of further implementations of the proposed strategy in a variety of instructional contexts. This could both provide further illustrations and, thus, concrete "images" to support the creation of error activities on the part of interested mathematics teachers, and also contribute to support and refine the

working hypotheses about possible variations of the strategy and their potential benefits generated in this book. Results from experiences conducted with elementary and college students would be especially interesting, as they could shed light on the possibility and conditions necessary to capitalize on errors productively even with these student populations. More generally, valuable information about the educational potential of the proposed strategy, as well as on the dynamics of implementing instructional innovation informed by a spirit of inquiry more generally, could be gained from long-term studies of how specific teachers introduce a constructive use of errors in their regular classes and continue to use it in support of their teaching throughout the year (rather than within isolated "units"). Such research could also help examine the crucial issue of when and under what conditions mathematics teachers should decide to pursue the opportunities for inquiry offered by an error, given their specific instructional goals, curriculum, and time constraints.

It would also seem worth trying to generalize the proposed strategy regarding errors to propose new strategies supportive of an inquiry approach to mathematics instruction. In particular, it may be worth questioning what other elements, besides errors, could provide the catalyst for student inquiry. Earlier in the book, I argued that errors could be considered as the prototypical example of an anomaly—that is, something that does not make sense and thus could generate questions for reflections and explorations. I would like to now suggest the value of identifying other elements besides errors that could play the role of anomalies in school mathematics and, thus, stimulate mathematical inquiries.

The role attributed to anomalies in the inquiry process in Dewey and Peirce's philosophical theories of knowing also suggests that a use of errors as springboards for inquiry should not be seen as appropriate only to mathematics instruction. The metaphor of error making as "getting lost in a city," developed at the very beginning of the book, already illustrated that such an approach to error is implicitly assumed by some people in certain real-life situations. Thus, it would seem worthwhile to explore what capitalizing on errors might look like in instructional contexts involving the teaching of subject matter other than mathematics.

For example, it would be interesting to see how a conceptual change approach to science instruction could be further enriched by inviting students to derive the consequences of some of their misconceptions so as to develop alternative systems and, then, discuss the historical development and significance of explanatory theories different from those currently accepted by the scientific community. A use of errors as springboards for inquiry in the context of learning a second language also seems possible, as suggested by some personal experiences when I came to realize some peculiarities of the language I was learning as well as my own as a result of reflecting on some systematic error I was making. Keeping in mind that some research on language acquisition has suggested that novice learners should *not* pay too much attention to their errors

while attempting to speak a new language or their fluency would suffer, I think it would be worth considering the possibility of occasionally inviting students to examine their errors with the goal of raising questions about the differences present in diverse languages with respect to grammar, vocabulary meaning, and even cultural influences. Similarly, errors made in writing could be used as a vehicle to invite the exploration of grammatical and spelling rules, as well as meanings, within the student's own language. Furthermore, the changes made in several successive drafts could occasionally be explicitly examined to discuss the possibility for alternative organizations or even "messages" they uncover, as well as comparing different styles and addressing other aesthetic issues.

I hope that the experiences and considerations developed in this book have invited other educators to consider the value of capitalizing on errors in their practices and to pursue instructional research along some of the directions sketched here. Active participation in these enterprises could contribute to improving our understanding not only of the educational potential of using errors as springboards for inquiry in a wider variety of instructional contexts, but also, more generally, of the processes of learning, teaching, and instructional reform.

Appendix A:

Summary of Categories, Codes, and Abbreviations Employed in the Book

Throughout the book, for the sake of brevity I have used some abbreviations for concepts and categories that were often referred to. Although each term has been explained in the text itself when first introduced, I thought it would be helpful for the reader to report in this appendix a collection of all these abbreviations along with a brief explanation. When appropriate, I have also included the reference to the chapter(s) in the book where the reader can find a more in-depth discussion of the concept in question.

Dualistic view/conception of mathematics: A conception of mathematics characterized by the belief that mathematical results are predetermined and absolutely right or wrong.

Capitalizing on/using errors as springboards for inquiry: An approach to errors that interprets them as opportunities to generate doubt and questions that, in turn, can lead to valuable explorations and learning.

Capitalizing on errors: Abbreviation for *capitalizing on errors as springboards for inquiry*.

Error activity: An instructional episode where some error(s) has been *capitalized on as springboards for inquiry*.

Error case study: My report and discussion of either the in-depth study of a specific error or an *error activity*.

Level of (mathematical) discourse: The specific level of abstraction at which a learning activity is taking place—that is, whether one is focusing on performing a specific mathematical task, on understanding some technical mathematical content, or on understanding about the nature of *math* (see Chapter 6 for a discussion of this category and its possible attributes).

Task level: When the overall mathematical activity within which an error activity originates focuses on performing a specific mathematical task, such as solving a problem, performing a computation, attempting to prove a result, producing an acceptable definition for a given concept, and so on. (This term is used to indicate one of the categories within possible levels of mathematical discourse at which an error can be used or analyzed).

Content level: When the overall mathematical activity within which an error activity originates focuses on understanding some technical mathematical content—be it a mathematical concept, rule, or topic. (This term is used to indicate one of the categories within possible levels of mathematical discourse at which an error can be used or analyzed).

Math level: When the overall mathematical activity within which an error activity originates focuses on understanding about the nature of mathematics. This could involve understanding metamathematical notions (such as definition, proof, or algorithm), becoming aware of helpful heuristics as well as of their domain of application and limitations, appreciating what characterizes mathematical thinking and mathematics as a discipline, and so on. (This term is used to indicate one of the categories within possible levels of mathematical discourse at which an error can be used or analyzed).

Stance of learning: The overall approach, and particularly the degree of open-endedness, characterizing a learning activity and determining the possible interpretations and uses of errors such an activity (see Chapter 6 for a discussion of this category and its possible attributes).

Remediation stance (of learning): A stance of learning occurring when both the question and the answer informing the student activity are predetermined and known by authority, and furthermore the student is aware that his or her result is not correct (although he or she may or may not know what the correct result is). The expectation is that by analyzing an error one could identify what went wrong and correct it. (This term is used to indicate one of the categories within possible stances of learning that can be assumed when using an error.)

Discovery stance (of learning): A stance of learning occurring when the student is learning something new or trying to solve a genuine problem, although both the question and the answer informing the activity are perceived as predetermined and known to authority. Because the student is not expected to already know the answer, steps in the wrong direction are seen as a natural occurrence (although they may not always be immediately recognized as errors by the student) and there is the expectation that any result needs to be critically examined so as to determine whether it is correct or not. (This term is used to indicate one of the categories within possible stances of learning that can be assumed when using an error.)

Inquiry stance (of learning): A stance of learning occurring when neither the answers, nor the questions directing the student's mathematical activity, are perceived as necessarily predetermined, and detours as well as redefinitions of the original task are encouraged; questions raised by errors may thus initiate exploration and reflection in totally new directions, and even invite students to challenge the status quo. (This term is used to indicate one of the categories within possible stances of learning that can be assumed when using an error.)

Specific use of errors as springboard for inquiry: One of the nine possible ways of capitalizing on errors identified by the 3×3 matrix (see Table 6.1) obtained by combining possible levels of mathematical discourse and stances of learning that can be assumed when capitalizing on errors.

Remediation/task use (of errors): A way to capitalize on errors consisting essentially of the analysis of recognized errors so as to understand what went wrong, correct it, and thus eventually accomplish the task one had been set to perform in the learning activity in which the error itself occurred. (This is a use of errors as springboards for inquiry resulting from the combination of assuming a remediation stance of learning together with a level of mathematical discourse focusing on the performance of a set task).

Remediation/content use (of errors): A way to capitalize on errors consisting essentially of the analysis of recognized errors so as to clarify the misunderstanding of some technical mathematical content. (This is a use of errors as springboards for inquiry resulting from the combination of assuming a remediation stance of learning together with a level of mathematical discourse focusing on the understanding of some specific mathematical content).

Remediation/math use (of errors): A way to capitalize on errors consisting essentially of the analysis of recognized errors so as to clarify misunderstandings regarding the nature of mathematics as a discipline and/or some general mathematical issues (such as the nature of mathematical definitions or proofs, some problem-solving heuristics, etc.). (This is a use of errors as springboards for inquiry resulting from the combination of assuming a remediation stance of learning together with a level of mathematical discourse focusing on learning about the nature of math.)

Discovery/task use (of errors): A way to capitalize on errors consisting essentially of using errors and/or uncertain results constructively when solving a novel problem or task, at the same time continuously monitoring one's work so as to identify and correct potential mistakes. (This is a use of errors as springboards for inquiry resulting from the combination of assuming a discovery stance of learning together with a level of mathematical discourse focusing on the performance of a set task.)

Discovery/content use (of errors): A way to capitalize on errors consisting essentially of using errors, misunderstandings, and/or uncertain results constructively as one learns about a new concept, rule, topic, and so on. (This is a use of errors as springboards for inquiry resulting from the combination of assuming a discovery stance of learning together with a level of mathematical discourse focusing on the understanding of some specific mathematical content.)

Discovery/math use (of errors): A way to capitalize on errors consisting essentially of using errors, misunderstandings, and/or uncertain results constructively as one learns about some aspects of the nature of mathematics and/or some general mathematical issues (such as the nature of mathematical definitions or proofs, some problem-solving heuristics, etc.). (This is a use of errors as springboards for inquiry resulting from the combination of assuming a discovery stance of learning together with a level of mathematical discourse focusing on learning about the nature of math.)

Inquiry/task use (of errors): A way to capitalize on errors consisting essentially of using errors and/or puzzling results to motivate questions that may generate inquiry in new directions and suggest new mathematical tasks to be performed.

(This is a use of errors as springboards for inquiry resulting from the combination of assuming an *inquiry stance of learning* while operating at a *level of mathematical discourse* focusing on the performance of a *task*).

Inquiry/content use (of errors): A way to capitalize on errors consisting essentially of using errors and/or puzzling results to motivate questions that may lead to the exploration of and/or to new perspectives and insights on specific mathematical concepts, rules, or topics. (This is a use of errors as springboards for inquiry resulting from the combination of assuming an *inquiry stance of learning* while operating at a *level of mathematical discourse* focusing on the understanding of technical mathematical content).

Inquiry/math use (of errors): A way to capitalize on errors consisting essentially of using errors and/or puzzling results to motivate questions that may lead to the exploration of and/or to new perspectives and insights on issues regarding the nature of mathematics as a discipline and/or some general mathematical topics (such as the nature of mathematical definitions or proofs, some problem-solving heuristics, etc.). (This is a use of errors as springboards for inquiry resulting from the combination of assuming an *inquiry stance of learning* while operating at a *level of mathematical discourse* focusing on learning about the nature of *math*.)

Level of student involvement: The extent to which a student actually participates in the inquiry and activities generated within an *error activity* (see Chapter 6 for a discussion of this category and its possible attributes).

(*Level of involvement*) 1: *Level of student involvement* in an *error activity* occurring when the inquiry stimulated by the error is mostly conducted by the instructor and shared with the students (also referred to as *teacher modeling level of involvement*).

(*Level of involvement*) 1₂: *Level of student involvement* in an *error activity* occurring when the student actively engages in an error activity organized and led by the instructor (also referred to as *teacher-led student inquiry level of involvement*).

(*Level of involvement*) 1₃: *Level of student involvement* occurring when the student initiates and leads the inquiry around an error, with minimal input from the teacher (also referred to as *independent student inquiry level of involvement*).

Teacher modeling (level of involvement): *Level of student involvement* in an *error activity* occurring when the inquiry stimulated by the error is mostly conducted by the instructor and shared with the students.

Teacher-led student inquiry (level of involvement): *Level of student involvement* in an *error activity* occurring when the student actively engages in an error activity organized and led by the instructor.

Independent student inquiry (level of involvement): *Level of student involvement* occurring when the student initiates and leads the inquiry around an error, with minimal input from the teacher.

(*Error*) *source*: Term used to indicate where the error used in an error activity came from; it involves looking at who made the error and/or the level of control/input that the teacher had in selecting it (see Chapter 6 for a discussion of this category and its possible attributes).

Planned error: An error previously selected by the teacher and presented to the student within a planned error activity. (This term is used to indicate one of the categories within possible *error sources*).

Expected error: Error made by a student, but somewhat expected (or even prompted) by the teacher, and then promptly used in an error activity. (This term is used to indicate one of the categories within possible *error sources*).

Unexpected error: Error made *unexpectedly* by a student or the classroom teacher and then used for an impromptu error activity. (This term is used to indicate one of the categories within possible *error sources*).

Same student error: Error made by the same person now engaging in the error activity. (This term is used to indicate one of the categories within possible *error sources*).

Other classmate error: Error made by another member of the class (or group) now engaging in the error activity. (This term is used to indicate one of the categories within possible *error sources*).

Teacher error: Error made by the classroom teacher. (This term is used to indicate one of the categories within possible *error sources*).

Outside peer error: Error made by a peer not belonging to the class (or group) now engaging in the error activity. (This term is used to indicate one of the categories within possible *error sources*).

Naïve person error: Error made by a person more mathematically naïve than the one now engaging in the error activity. (This term is used to indicate one of the categories within possible *error sources*).

Expert error: Error made by a person more "expert in mathematics" than the one now engaging in the error activity. (This term is used to indicate one of the categories within possible *error sources*).

Math-inherent error: Error due to the limitation of mathematics itself rather than a person's mistake. (This term is used to indicate one of the categories within possible *error sources*).

Appendix B:

Title and Abstract of Error Case Studies

Note: After each error case study full title, in parentheses I have also indicated the abbreviated title I have used throughout the book to refer to that case study; the code number (in square brackets) indicates the consecutive letter used to identify each error case study, followed by the chapter in which the case study can be located.

Error case study A: $\frac{6}{7} + \frac{2}{7} = \frac{8+2}{7}$. ("Ratios" case study [A/1])

In order to give a flavor of the kinds of reflections and explorations that mathematical errors could invite, in this first error case study I examine the common error of adding two fractions by simply adding their respective numerators and denominators. This is done in a spirit of inquiry, that is, instead of trying to discover the causes of this common error with the goal of eradicating it, my analysis of this error is guided by the goal of identifying questions worth investigating that are invited by this error. This approach leads to uncovering the possibility for valuable mathematical inquiry in a variety of different directions. In particular, challenging this "error" by questioning "what if this result were true?" leads to recognizing some fundamental differences between *fractions* and *ratios*—despite the fact that in mathematics we use the same symbolism for both, and thus often tend to identify the two notions. These differences, in turn, invite the exploration of the nonstandard mathematical system of ratios. This case study suggests that errors, by presenting a possible alternative to established results, may suggest the exploration of simple nonstandard mathematical situations and, thus, provide even naïve mathematicians with the opportunity to engage in original inquiries similar to those that characterize the activity of mathematical researchers.

Error case study B: *Lack of rigor in the early development of calculus and its positive outcomes.* ("Calculus" historical case study [B/4])

The early development of calculus, one of the areas of mathematics that has found greater application within and outside the field, was marred by a number of errors and shaky proofs. This initial lack of concern for rigor, however, had the benefit of not stifling the creativity of productive mathematicians of the time (such as Euler) who were thus able to produce results that were ingenious and useful, even if they did not find a rigorous justification until much later. Furthermore, the uneasiness created by some of these results eventually led to a re-examination of the basis of earlier works, which in turn resulted in both a refinement of the methodology employed to derive and justify results within calculus and, even more importantly, in a revision of the foundations of mathematics more generally. This "historical" error case study shows how the constructive analysis of specific errors has not only helped mathematicians eventually eliminate those errors, but also occasionally provided the stimulus to review and radically refine the methodology and/or assumptions used.

Error case study C: *The surprising consequences of failing to prove the parallel postulate.* ("Non-Euclidean geometry" historical case study [C/4])

One of the most traumatic events in the history of mathematics—the discovery that Euclidean geometry did not represent the only way to describe spatial relationships—came about as the unexpected result of failing to prove Euclid's parallel postulate. Many historians of mathematics have identified Saccheri's faulty "proof by contradiction" of the parallel postulate—one of the many attempts to prove the erroneous expectation that the parallel postulate could be derived from the other Euclidean axioms—as one of the first seeds for the development of a geometry based on a different set of axioms than the one assumed by Euclid. This historical case study presents an extreme example of how errors have occasionally led to the development of entirely new areas of research in mathematics and even brought mathematicians to reconceive the nature of their discipline.

Error case study D: *Dealing with unavoidable contradictions within the concept of infinity.* ("Infinity" historical case study [D/4])

The concept of infinity, one of the most fundamental concepts in mathematics, has undergone considerable debate throughout the history of mathematics. Because of our tendency to extend to the infinite our limited experience of the finite alone, our intuitive concept of infinity contains some implicit contradictions that cannot be totally eliminated. It took mathematicians a long time to come to appreciate this fact as well as some of its disturbing consequences—such as the fact that alterna-

tive resolutions of these contradictions could be equally reasonable and, therefore, could not be decided on purely logical grounds. This historical case study presents an interesting example of how it may not always be possible to eliminate certain perceived errors (because they are the consequence of contradictions inherent to our intuitions or to mathematics itself) and how the analysis and resolution of such errors may require the consideration of elements such as purposes, values, and context—elements that are not usually associated with mathematical activity and even less with mathematical research.

Error case study E: *Progressive refinements of Euler's theorem on the "characteristic" of polyhedra.* ("Euler theorem" historical case study [E/4])

In this error case study I report the key points of Lakatos' historical analysis of how the tentative theorem initially proposed by Euler as "In a polyhedron the relationship between the number of faces (F), vertices (V), and edges (E) satisfies the equation $V + F - E = 2$ " was criticized and revised by several mathematicians through a process of successive "proofs and refutations." This historical example suggests that, at least to a certain extent, all mathematical results are tentative and liable to continuous refinement and, consequently, errors should be considered an integral component of the construction of mathematical knowledge.

Error case study F: $\frac{1}{2} = \frac{1}{4}$ —How can such a crazy simplification work? ("My $\frac{1}{2} = \frac{1}{4}$ " case study [F/5])

Realizing that an obviously incorrect procedure such as canceling the sixes in $16/64$ yielded a correct result really puzzled me and, in turn, invited me to explore how this could have happened. The inquiry that resulted involved some interesting arithmetic and algebraic explorations involving the solution of unusual equations and the unexpected use of some concepts and results from number theory. In this error case study I report on the results and the development of this inquiry so as to show how errors can provide even people who are not research mathematicians with the stimulus for posing and solving worthwhile mathematical problems.

Error case study G: *Incorrect definitions of circle—A Gold mine of opportunities for inquiry.* ("My definitions of circle" case study [G/5])

My in-depth analysis of a collection of about 50 mostly incorrect definitions of circle (collected from various mathematics teachers, high school students, and college students) resulted in a number of valuable mathematical activities—including a categorization of these definitions with respect to the mathematical properties of circle they identified and the identification and categorization of the reasons why specific definitions did not seem to me appropriate to characterize circles. The results of this analysis, as reported in this error case study, provided me

with a better understanding of the concept of circle as well as of the notion of mathematical definition itself. I have included this case study to provide further evidence of the potential of errors to provide a deeper understanding both of mathematical content and of more general mathematical notions—such as *circle* and *definition*, respectively, in this case.

Error case study H: *The unexpected value of an unrigorous proof.*
 ("My unrigorous proof" case study [H/5])

Having generated a very "loose" proof for the evaluation of an infinite expression as a student in a mathematics education class, I felt uneasy about my procedure, despite the fact that I did not doubt the validity of its outcome. This uneasiness, in turn, compelled me to justify the result reached more rigorously. The analysis of my first tentative proof led not only to the identification of some missing steps (and, then, to the production of a more acceptable proof), but also to a generalization of the procedure originally used. This result, in turn, invited me to apply the generalized procedure to the evaluation of other infinite expressions and led me to some new mathematical results. Last but not least, the experience provided me with a firsthand experience, and hence some new insights, about the process of creating mathematical proofs. The report of this personal experience was included to show how guesses and unrigorous proofs can often be capitalized on, as they can suggest alternative, and perhaps more efficient and productive, approaches for the solution of a problem, and offer the starting point for a reflection on the notion of mathematical proof itself.

Error case study I: *Students' analysis of incorrect definitions of circle.*
 ("Students' definitions of circle" case study [I/6])

This error case study reports on how two eleventh-grade students responded to an activity based on the analysis of a list of incorrect definitions of circle (generated by them and some of their peers), an error activity inspired by my own experience as reported in "My definitions of circle" case study [G/5]. This episode took place at the beginning of a teaching experiment on the nature of mathematical definition and was intended to initiate the students' inquiry on this complex meta-mathematical notion. This experience shows the value for students of engaging in an explicit analysis of errors at different levels of mathematical discourse, even when assuming a remediation stance.

Error case study J: *Debugging an unsuccessful homework assignment.*
 ("Students' homework case study [J/6])

During the same teaching experiment on mathematical definitions, one of the students asked me to examine her unsuccessful attempts to solve the assigned problem of finding the circle passing through three given

points. The analysis of her work soon allowed us to identify and correct the error she had made, and thus helped the student reach the correct solution. At the same time, the error she had made (using the coordinates of only two of the given points to set up the system of equations which, once solved, should have provided the radius and center of the circle we were trying to identify) provided even myself with new insights about circles and with the answer to a number of interesting mathematical questions that I had never thought of asking (such as: How can we find the circle(s) passing through two given points? What is the relationship of these circles with the locus of all the points equidistant from two given points?). This instructional experience provides a good example of how the study of an error does not always need to be initiated by the teacher nor to occur within a planned activity and, also, of how the analysis of an error may lead to unexpected results and to the identification of interesting problems other than the one initially assigned by the teacher.

Error case study K: *Students using errors constructively when developing a theorem about polygons.*
 ("Students' polygon theorem" case study [K/6])

As part of the same teaching experiment, the two students engaged in the verification and refinement of the following (incorrect) theorem proposed by the instructor: "In a polygon, the sum of the interior angles is equal to 180° times the number of sides." False steps and tentative conjectures, as well as the realization that the initial statement of the theorem was incorrect, all played an important role throughout this activity (which turned out to provide the students with a novel and challenging "problem" to solve). This case study is a good illustration of how students can use errors constructively in a variety of ways in the process of solving a mathematical problem.

Error case study L: *Students dealing with an unresolvable contradiction—The case of 0^0 .*
 ("High school students' 0^0 " case study [L/6])

Toward the end of the teaching experiment on mathematical definitions from which the previous three error case studies were also derived, the students engaged in a reconstruction of the rules to operate with negative and fractional exponents. After having been initially quite successful in this enterprise, they were very surprised when one of the methods they had productively used thus far yielded two different values for 0^0 , especially as it soon became clear that there was no way of resolving that apparent contradiction. In the effort to make sense of this puzzling situation, one of the students was led to articulate and reflect on her conception of mathematics and to explicitly recognize the value of perceiving this discipline in a more humanistic light.

Error case study M: Building on errors to construct the formula for the probability of "A or B" in a middle school class. ("Students' P(A or B)" case study [M/8])

This error case study is based on a series of three lessons, conducted within a probability unit in a ninth-grade mathematics class, which were almost fully devoted to resolving the controversy generated by the proposal of two contrasting solutions to the problem "What is the probability of drawing a jack or a diamond from a standard deck of cards?" This case study provides an example of how an error can really trigger curiosity and consequently initiate mathematical activities that are meaningful and interesting to middle-school students. This instructional vignette also illustrates the kind of classroom dynamics and discourse that accompany error activities of this kind.

Error case study N: Students dealing creatively with errors when doing geometric constructions. ("Students' geometric constructions" case study [N/8])

This case study relates a classroom experience conducted within a unit on geometric constructions in a tenth-grade mathematics course. As students engaged in the creative activity of drawing a triangle given a side, an adjacent angle and a nonadjacent angle (AAS) by using only an unmarked ruler and a compass, they used their own errors constructively in a variety of ways as they tried to complete the set task. This experience well illustrates the dynamics of pursuing mathematical inquiries stimulated by errors in the context of a regular classroom and the important role that the teacher needs to play as a facilitator in these situations.

Error case study O: Problems encountered when discussing the "crazy" simplification $\frac{1}{2} = \frac{1}{4}$ in a secondary classroom. ("Students' $\frac{1}{2} = \frac{1}{4}$ " case study [O/8])

In this error case study I report on the surprising outcomes of my opening lesson for an eleventh-grade experimental course that I had planned to teach making a consistent use of errors as springboards for inquiry. The lesson consisted mainly of a series of guided explorations around the error $\frac{1}{2} = \frac{1}{4}$ (taking inspiration from my own prior analysis of this situation—see "My $\frac{1}{2} = \frac{1}{4}$ " case study [F/5]). Although at the end of the class I felt quite satisfied with the results of this lesson, to my surprise most of the students did not share my enthusiasm and even threatened to drop the course! A talk with the class the next day revealed some crucial differences in the students' and my expectations of what makes a mathematical activity valuable, and between this classroom experience and my own inquiry around the same error. This experience illustrates the importance of taking into account students' mathematical and pedagogical beliefs when introducing a use of errors

as springboards for inquiry. It also points out some of the prerequisites for a successful implementation of the suggested strategy in today's mathematics classrooms.

Error case study P: College students dealing with "undefined" results in mathematics. ("College students' 00" case study [P/8])

The error activity discussed here, like the one previously reported in the "High school students' 00" case study [L/6], was organized around the puzzling result that 00 must remain undefined, because contrasting but equally plausible values can be derived for this expression. This time, however, the activity took place in the context of a mini-course offered to a few college students with a weak mathematics background and low mathematical self-esteem. As it had happened with the high school students, trying to grapple with some puzzling contradictions and their implications led this group of college students, too, to reveal some of their expectations about mathematics and then discuss them explicitly. This case study further confirms that the realization that some perceived errors are actually limitations inherent to mathematics itself may trigger reflections on the nature of mathematics challenging the students' conception of the discipline, even at the college level.

Error case study Q: Teachers' analysis of incorrect definitions of circle. ("Teachers' definitions of circle" case study [Q/9])

A list of mostly incorrect definitions of circle (including many of those discussed earlier in "My definitions of circle" case study [G/5]) was also used in a graduate mathematics education course within a series of activities designed to stimulate reflection on the nature of mathematical definitions and to enable the participants to appreciate the potential of errors of this kind to refine one's understanding of a mathematical concept. The report of how these activities developed in our class meetings, along with the more in-depth discussion of the written work of one of the participants, shows the remarkable reflections and growth that this experience stimulated for the participating teachers. This error case study also illustrates how mathematics teachers could benefit from engaging as learners in error activities dealing with mathematical content belonging to the K-12 curriculum.

Error case study R: Teachers' reflections and problem-solving activities around an unrigorous proof. ("Teachers' unrigorous proof" case study [R/9])

The experience reported in this error case study was developed in the same graduate mathematics education course from which the previous "Teachers' definitions of circle" case study [Q/9] was derived. This time, however, the participating teachers were asked to deal with mathematical content that was challenging for them, as they were asked to examine critically and generatively my "unrigorous" proof for the re-

sult $\sqrt{2 + \sqrt{2 + \dots + \sqrt{2 + \sqrt{2}}}} = 2$ (discussed earlier in "My unrigorous proof" case study [H/5]). Within this error activity, the participating teachers engaged in some genuine and demanding mathematical inquiry, as they tried to understand my initial proof and its shortcoming, explored how the procedure could be modified to evaluate other infinite expressions, and reflected on the implications of this experience for their notion of mathematical proof. This activity enabled the participating teachers to experience firsthand the potential of errors to stimulate worthwhile mathematical explorations and reflections and, consequently, helped them come to realize the accessibility of problem posing for naive mathematicians as well as the potential value of using errors as springboards for inquiry in mathematics instruction.

Error Case Study S: "Numbering Systems Without Zero"—A Teacher-generated exploration (by Richard Fasse). ("Numbers without zero" case study [S/9])

The last three error case studies reported in the book were initiated, developed, and written by mathematics teachers and show how teachers can come to internalize the use of errors as springboards for inquiry they themselves have experienced in activities such as those described in the previous "Teachers' definitions of circle" [Q/9] and "Teachers' unrigorous proof" [R/9] case studies, so that they can generate similar activities on their own—a critical premise to planning error activities in their own classes. More specifically, the study reported in this error case study consists of a mathematical investigation motivated by the realization that "zero" is the source of many computational errors for students. Knowing that zero was not "invented" until relatively late in the history of mathematics, the author of this case study decided to explore the implications of working with numeration systems without zero and did so by inventing one such system and developing algorithms for addition, subtraction, and multiplication within it. This error case study well illustrates how errors may enable teachers to initiate and develop novel "technical" explorations and thus experience the challenge and satisfaction of engaging in genuine mathematical inquiry.

Error Case Study T: "Beyond straight lines": A teacher's reflections and explorations into the history of mathematics (by John R. Sheedy). ("Beyond straight lines" case study [T/9])

This teacher-generated investigation differs considerably from the one reported in the previous "Numbers without zero" case study [S/9], because it mostly involves the historical exploration of some key events in the history of mathematics, an exploration that was motivated by its author's realization of the inadequacy of his initial view of mathemat-

ics as a dualistic discipline. This error case study illustrates how this kind of investigation, and the reflections it is likely to stimulate, can have a great impact on teachers' conceptions of mathematics and, consequently, on their attitudes toward the discipline and its teaching.

Error Case Study U: Building on probability misconceptions—A student activity created by a college teacher (by Barbara Rose). ("Probability misconceptions" case study [U/9])

This last error case study reports the design and major results of an "error project" developed by a college mathematics teacher. This project consisted of the construction of a multiple-choice questionnaire based on common misconceptions about the interpretation of some simple probability results and, then, its nontraditional use as a tool to initiate discussion on the meaning of probability results with students enrolled in a college-level probability course. This case study shows how a teacher processed what she learned in the course about using errors as springboards for inquiry so as to be able to implement the proposed strategy with her students.

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Author Index

- A**
- Alexandrov, A. D., 87, 301
- Ashlock, R., 7, 8, 39, 301
- B**
- Balacheff, N., 25, 30, 42, 43, 301
- Baroody, A. J., 20, 301
- Baursfeld, H., 22, 301
- Bell, A., 41, 301
- Bishop, A., 17, 22, 301
- Blumer, 22
- Bolzano, B., 56, 301
- Borasi, R., 5, 8, 15, 18, 24, 25, 33, 46, 70, 71, 81, 98, 100, 105, 111, 119, 120, 135, 138, 146, 149, 150, 151, 165, 167, 169, 179, 203, 205, 206, 209, 215, 224, 236, 256, 260, 275, 278, 282, 301, 302, 303, 307
- Boyet, J., 31, 304
- Brette, G., 41, 301
- Brown, A., 31, 303
- Brown, C., 255, 303
- Brown, J. S., 4, 140, 303
- Brown, S. I., 25, 70, 116, 144, 202, 210, 255, 256, 278, 302, 303, 304
- Bruckheimer, M., 7, 307
- Burke, C., 26, 304
- Burton, R., 4, 303
- Butler, M., 22, 305
- C**
- Callahan, L., 70, 210, 303
- Cantor, G., 57, 303
- Carey, R. F., 18, 27, 28, 307
- Carr, E. G., 31, 32, 305
- Carver, S. M., 35, 303
- Cobb, P., 20, 22, 23, 42, 202, 278, 303, 307, 308
- D**
- Collins, A., 140, 303
- Confrey, J., 19, 21, 25, 31, 36, 37, 38, 39, 40, 41, 303
- Connolly, P., 33, 303
- Cooney, T. J., 255, 256, 303, 304
- Copes, L., 255, 304
- Coxford, A. F., 270, 308
- E**
- Dauben, J. W., 55, 57, 58, 304
- Dawson, C., 36, 307
- Dewey, J., 19, 304
- Dupont, P., 45, 49, 304
- F**
- Ernig, J., 33, 304
- Engelhardt, J., 5, 304
- Ernest, P., 20, 304
- F**
- Fischbein, E., 46, 304
- Flavell, J. H., 31, 304
- Fosnot, C. T., 272, 307
- Fulwiler, T., 33, 308
- G**
- Gajary, M., 32, 304
- Gallie, G., 56, 304
- Gardner, H., 304
- Garofalo, J., 31, 304
- Gere, A. R., 33, 304
- Gertzog, W. A., 36, 306
- Ginsburg, H. P., 20, 301, 304
- Goldin, A. G., 21, 304
- Graeber, A. O., 5, 31, 37, 38, 39, 41, 304
- Grouws, D. A., 2, 304

H
 Harter, J., 26, 304
 Helin, H., 40, 306
 Herszkowitz, R., 7, 307
 Hess, P., 46, 304
 Hewson, P. W., 36, 304, 306
 Hofstadter, D., 90, 304

I
 Imbelder, B., 31, 304

J
 Johnson, E., 71, 76, 304
 Johnson, M. L., 4, 5, 31, 37, 38, 39, 41, 304, 305
 Jones, B. F., 31, 32, 305

K
 Kilpatrick, J., 5, 20, 21, 305, 307
 Kline, M., 19, 45, 48, 49, 51, 52, 53, 244, 247, 248, 305

Kolmogorov, A. N., 87, 301
 Kroll, 31
 Kuhn, T., 19, 28, 31, 305
 Kuhn, G., 270, 305

L
 Lakatos, I., 19, 28, 29, 45, 59, 60, 61, 62, 107, 127, 155, 236, 305

Lakoff, G., 4, 305
 Lamon, S. J., 270, 305
 Lampert, M., 25, 26, 42, 202, 278, 305
 Lankford, F., Jr., 7, 39, 305
 Laurent'ev, M. A., 87, 301
 Lave, J., 22, 305, 306
 Leder, G., 270, 305
 Lerman, S., 17, 305
 Lesh, R., 270, 305
 Lester, F. K., 31, 304
 Lester, N. B., 33, 305

M
 Marshall, S. P., 270, 305
 Maurana, H., 22, 305
 Maurer, S. B., 4, 305
 Mayher, J. S., 33, 305
 Melamed, U., 46, 304
 Mental, G., 42, 202, 278, 308
 Meyerson, L., 255, 306
 Michaelsen, J., 8, 302
 Miller, P. H., 31, 37, 306

N
 Newman, S. E., 140, 303
 Niss, M., 270, 306
 Novak, J. D., 40, 306
 Novick, S., 36, 306
 Nussbaum, J., 36, 306

O
 Ogle, D. S., 31, 32, 305
 Ortony, A., 4, 306

P
 Palinscar, A. S., 31, 32, 305
 Papert, S., 33, 34, 35, 88, 306
 Perret-Clermont, A. N., 22, 306
 Piaget, J., 20, 306
 Posner, G. J., 36, 306
 Pradl, G. M., 33, 305
 Purdy, D., 41, 301

R
 Radatz, H., 37, 38, 306
 Resnick, L., 20, 23, 306
 Richards, J., 20, 25, 306, 307
 Rogoff, B., 22, 306
 Romberg, T. A., 270, 306
 Rowell, J., 36, 307

S
 Schifter, D., 272, 307
 Schlesinger, B., 20, 307
 Schoenfeld, A. H., 18, 20, 23, 25, 32, 33, 146, 206, 270, 307
 Schon, D. A., 4, 207, 307
 Short, K., 26, 304
 Siegel, M., 15, 18, 27, 28, 150, 302, 307
 Silver, E. A., 20, 307
 Simon, M. A., 272, 307
 Sinclair, H., 31, 304
 Skagstad, P., 19, 307
 Smith, D., 255, 303
 Smith, S., 22, 305
 Steffe, L. P., 20, 307
 Strike, K. A., 36, 306
 Swan, M., 41, 301, 307

T
 Thompson, A. G., 255, 270, 305, 307
 Tirosch, D., 46, 304

V
 Van Lehn, K., 4, 303

Vinner, S., 7, 307
 von Glasersfeld, E., 19, 20, 307, 308
 Vygotsky, L. S., 21, 308

W
 Walter, M. I., 25, 116, 144, 303
 Webb, N. L., 270, 308
 Welch, W., 16, 308

Wenger, E., 22, 305
 Wheatley, G., 42, 202, 278, 308
 White, A., 17, 308
 Wood, T., 22, 25, 42, 202, 278, 303, 308
Y
 Yackel, E., 22, 25, 42, 202, 278, 303, 308
 Young, A., 33, 308

Subject Index

- A**
- Abduction, 27—28
 - Accommodation, 31, 36
 - Affinities, 85
 - Algebra, 13, 170
 - Anomalies
 - abductive reasoning, 28
 - definition, 27
 - errors and, 28
 - mathematics instruction/education, 264
 - science and, 28, 36
 - Appunti di storia dell'analisi infinitesimale*, 45
 - Assimilation, 31, 36
- B**
- Beyond straight lines case study, 211—212, 242—248, 262
 - abstract, 198—199
 - chess, 242—243
 - Euclidean geometry, 245—247
 - learning opportunities, 256—257, 273
 - mathematics
 - humanistic view of, 248
 - nature of, 244—245
 - Bolyai, János, 246
 - Bolzano, Bernhard, 56
- C**
- Calculus
 - derivative, 47
 - error case studies, 13, 170
 - historical case study, 46—51, 64—65, 260—261
 - abstract, 292
 - infinite series, 49—50, 233—234
 - integrals, 47
 - area computation, 48
 - lack of rigor, 46—47, 47—51, 64, 108
 - leap of faith, 48—49
 - my unrigorous proof case study, 70, 229—233
 - teachers' unrigorous proof case study, 70, 228—238
 - Calculus historical case study, 46—51, 64—65, 108, 260—261
 - abstract, 292
 - incorrect result error, 110
 - Cantor, Georg, 56, 247
 - cardinal numbers, 57—58
 - set theory, 57
 - Cauchy, Augustin Louis, 51
 - Euler theorem historical case study, 59—60
 - Children and learning, 31
 - Circles, 79—97
 - construction, 219
 - curvature, 87—88, 95—96, 217—218
 - definitions, 79—81, 213—214
 - analytic geometry, 85—87, 123, 221—223
 - circularity, 95—96
 - differential geometry, 87—88, 221
 - generative, 221
 - kind of error, 82, 90—96
 - mathematical content, 82—90, 214
 - metric definitions, 82—83, 123, 220—221
 - non-exclusiveness, 92—94
 - non-inclusiveness, 94
 - polygonal, 223
 - redundancy, 94—95
 - rotation definitions, 87
 - tactical geometry, 83—84, 95
 - teaching experiment, 152—153, 159—161

- terminology, 90—92
 - topological and metric, 85, 221
 - topological-projective, 83—85, 93, 221
 - visual descriptions, 88, 222
 - weird definitions, 89—90, 222
 - formula, 218—219
 - and infinity, 90
 - locus, 218
 - LOGO computer program, 88, 221
 - my definitions of circle case study, 79—97, 110
 - properties, 82, 223
 - equidistance property, 83, 86
 - students' definition of circle case study, 120—124
 - students' homework case study, 120, 124—127
 - teachers' definition of circle case study, 211—228
 - College students' 0^0 case study, 171, 193—202, 262
 - abstract, 297
 - contradictions, 197, 200—201, 264
 - discovery approach, 203
 - exponentiation, 194—196, 201
 - mathematics
 - dualistic views, 197
 - humanistic views, 199—201, 203
 - patterns, 196
 - planned errors, 206
 - process goals, 204
 - small-group discussion, 194
 - Computer programming
 - errors, debugging, 33—34
 - resistance to, 34
 - LOGO, 34—35
 - Conflict teaching, 41—42
 - Contradictions, 30, 64
 - college students' 0^0 case study, 197, 200—201, 264
 - high school students' 0^0 case study, 133
 - infinity, 54, 58—59, 64
 - historical case study, 110, 116—117
 - knowledge development, 64
 - mathematics, 64
 - types of errors, 110, 113—114
 - Counterexamples, 60, 64
 - Euler theorem historical case study, 60—64, 110, 114
 - global, 28—29, 61—63
 - local, 28—29, 61
 - types of errors, 110, 114—115
- D**
- Deduction, 27—28
 - Dewey, John, 19
 - Diamond, definition, 159
 - Discovery learning, 21, 25
- E**
- Einstein, Albert, 247
 - Equations
 - my $1/6/4=1/4$ case study, 70—79
 - necessary conditions, 77—78
 - power of, 77
 - students' $1/6/4=1/4$ case study, 70, 189—192
 - Error activities, 30, 149—150
 - sources of errors, 109—110, 163, 280
 - experts, 109, 141
 - math-inherent error, 109, 142, 200
 - students, 109, 141
 - teachers, 109, 141, 206
 - stances of learning, 137—138, 162, 203, 278—279
 - teaching experiment, 150—168
 - analysis, 161—164
 - evaluation, 164—168
 - learning opportunities, 164—165
 - overview, 150
 - stances of learning, 162
 - telegraphic descriptions, 151—161
 - types of errors, 110—111, 163—164, 280
 - contradictions, 110, 113—114
 - counterexamples, 110, 114—115
 - incorrect assumptions, 110
 - incorrect definitions, 110, 112—113
 - incorrect procedure, 110—111
 - incorrect results, 110—112
 - planned, 142, 206
 - unexpected, 142, 206
 - Error analysis
 - assumptions, 38
 - diagnostic tests, 38—39
 - fractiions, 39
 - remediation, 38—39, 137—138
 - research, 37—39, 282—284
 - rigorization, 51
 - student errors, 38
 - Error case studies, 12—14, 278, 291—299
 - algebra, 13, 170
 - arithmetic, 170
 - calculus, 13, 170
 - college mathematics classes, 169—207
 - experienced by teachers, 209—257
 - beyond straight lines case study, 211—212, 242—248
 - numbers without zero case study, 211, 238—242
 - probability misconceptions case study, 211—212, 248—253
 - teachers' definition of circle case study, 211—228
 - teachers' unrigorous proof case study, 211, 228—238
 - fractions, 71—79, 187—193
 - geometry, 13, 170
 - historical, 45—63, 260—261
 - logic, 170
 - mathematical definitions, 120—148
 - numbers, 13
 - probability, 13, 170
 - secondary mathematics classes, 169—207
 - Error making
 - in context, 6, 30
 - getting lost metaphor, 5—7, 207
 - alternative routes scenario, 5—6
 - time constraint scenario, 5
 - tourist scenario, 6
 - medical metaphor, 4—5
 - programming bugs metaphor, 4, 6, 33—34
 - Errors, *see also* Mathematical errors
 - alternative views on, 27—43
 - anomalies and, 28
 - capitalizing on, 4, 11, 65, 202
 - analysis, 119—148
 - benefits of, 143—148, 207, 253—257
 - classroom examples, 169—207
 - instructional content, 202
 - instructional strategy, 135—142, 202—203
 - teaching experiment, 149—168
 - learning opportunities, 3—4, 6—7, 137—138, 281
 - communication, 145—146
 - experiencing doubt, 143
 - experiencing initiative, 145
 - humanistic mathematics, 145
 - mathematical issues, 143, 254—255
 - monitoring/justifying, 144
 - nature of mathematics reflection, 145
 - problem-solving, 144, 206, 254—255
 - pursuing explorations, 143—144
 - negative attitudes influence, 3—6, 35
 - students' P(A or B) case study, 206
 - test grades, 37, 270
 - positive role, 3, 31—32, 205—206
 - misconceptions, 40
- F**
- Fasse, Richard, 211
 - Fernal, Pierre de, 47—48
 - FOCUS: *On Learning Problems in mathematics*, 210
 - Fonzi, Judi, 178—179
 - Fractions
 - error analysis, 39
 - my $1/6/4=1/4$ case study, 71—79, 111
 - ratios case study, 8—9
 - simplification, 70—79
 - students' $1/6/4=1/4$ case study, 170, 187—193
- G**
- Galliei, Galileo, 55—56
 - Gauss, Karl, 246
- radical constructivism**, 29—30
- as springboards for inquiry**, 7—14, 277—282
- assumptions**, 260—267
- constructivist epistemology**, 27—29
- error case studies**, 7—10, 12—14
- historical error case studies**, 63—67, 260—261
- instructional strategy**, 108—117
- mathematical knowledge development**, 11, 64—67
- taxonomy of uses**, 135—139, 279
- teacher education**, 209—257
- theoretical support**, 11, 30—31
- value of**, 207
- catalysts of change**, 40—41
- source of information**, 4
- in writing instruction**, 32—33
- Essentiality**, 122, 165, 224
- Euler theorem historical case study**, 29, 46, 59—63, 261
- abstract**, 293
- conjecture**, 59, 110, 114
- counterexamples**, 60—64, 110, 114
- definition modification**, 94
- polygons**, 127
- Everybody counts**, 17
- Exponentiation**, 166
- college students' 0^0 case study**, 194—196, 201
- definition**, 156—157
- extension**, 160
- fractional exponents**, 158
- high school students' 0^0 case study**, 132

- Geometry
 beyond straight lines case study, 242–248
 error case studies, 13, 170
 Euclidean, 53, 245–247
 non-Euclidean historical case study, 46, 51–54
 parallel postulate axiom, 51–54
 student's geometric constructions case study, 170, 178–187
 taxicab, 83–84, 95, 158–159, 166
 transformations, 84–85
- H**
 High school students' $\varnothing^0$ case study, 121, 132–135, 201, 262
 abstract, 295
 contradictions, 133, 136, 264
 exponentiation, 132
 learning opportunity
 humanistic mathematics, 145, 198
 learning opportunity
 communication, 145–146
 doubt, 143
 teaching experiment, 167
- I**
 Induction, 27–28
 my unrigorous proof case study, 100–103
 teachers' unrigorous proof case study, 234
 Infinity, 260–261
 actual, 55–56
 and circles, 90
 contradictions, 54, 58–59, 64
 historical case study, 46, 54–59, 64, 260
 abstract, 292–293
 infinite numbers, 55–58
 in context, 58–59
 infinite sets, 54
 one-to-one correspondence, 54–57
 part-whole principle, 54, 56
 potential, 55
 Infinity historical case study, 46, 54–59, 64, 260
 abstract, 292–293
 contradictions, 110, 116–117
 error sources, 109
- K**
 Kaya, error case study student, 120–148, 150–168
 Knowledge
 dynamic view of, 18–19
- as inquiry process, 19, 23
 uncertainty, role of, 25, 27
- L**
 Lakatos, I.
 conjecture, 28
 imitations, 29
 method of monster-barring, 62
 proofs and refutations, 28–29, 63
 conjecture, 59, 62, 114
 counterexamples, 60–62, 114
 Euler theorem historical case study, 59–63, 114
 lemma, 61–63
- Learning
 accommodation, 31
 accumulation of knowledge, 16
 assimilation, 31
 behaviorist
 assumptions, 20
 view, 37
 cognitive dissonance and, 31
 community of practice, 22, 269
 constructivist, 20–23
 implications, 30–32
 Piagetian, 22, 30
 social, 22
 discovery learning, 21, 25
 meaning making process, 24
 stances of, 137–139, 162, 203
 discovery, 137, 279
 inquiry, 137, 203–204, 279
 remediation, 137, 278–279
 teaching experiment, 162
Learning Mathematics Through Inquiry, 24, 120, 150
 Lemmas, *see* Counterexamples - local
 Lhuillier, 61, 63
 Lobachevski, Nikolaus, 246
 Logo computer program, 34–35
 circles, 88
- M**
 Markham, Tracy, 171
 Mary, error case study student, 120–148, 150–168
 Mathematical discourse
 levels, 115–117, 135–139, 278
 mathematical concept, 116, 136
 mathematical task, 115, 136, 203
 nature of mathematics, 116, 136, 203
 Mathematical errors, *see also* Errors
 computational, 33
 definition of, 66–67
 history, 45–67, 260–261
 students' geometric constructions, 33
 students' polygon theorem, 33
 successive draft approach, 33
 Mathematics
 definitions, 96–97
 attributes, 224, 226
 concept isolation, 123, 152, 165, 224
 error case studies, 120–148
 essentiality, 122, 165, 224
 my definitions of circle case study, 70, 79–97
 nature of, 223–228
 precision, 152, 165, 224
 students' definitions of circle case study, 70, 120–124
 teachers' definitions of circle case study, 70, 223–228
 teaching experiment, 154
 tentativeness, 132, 166
 understandability, 225
 usefulness, 225
 domain of application, 64–65
 dualistic view, 197, 255
 history, 45–67
 beyond straight lines case study, 242–248
 humanistic view, 15, 17–18, 23, 199–200, 248
 beyond straight lines case study, 248
 college students' $\varnothing^0$ case study, 199–201
 high school students' $\varnothing^0$ case study, 145, 198
 learning opportunities, 145
 students' polygon theorem case study, 145
 teaching experiment, 166
 uncertainty, role of, 25, 27
 instruction/education, 14–26, 225–226
 anomalies, 264
 community of practice, 22, 269
 conflict teaching, 41–42
 errors, uses of, 36–43, 210
 journals, 211, 274
 making sense, 24, 263–264
 mathematics education course, 210–212
 reconceptualization, 15–26, 263–266
 rethinking, 263–267
- socialization process, 23, 264–266
 transmission model, 15
 knowledge development, 64–67
 contradictions, 64
 counterexamples, 64
 errors, use of, 64–65
 nature of, 9–10, 14
 established facts, 16
 rethinking, 260–263
 proofs, 105–107, 234–238
 teaching experiment, 154–155
 radical constructivism, 19, 25
 rigorization, 64–66
 my unrigorous proof case study, 97–108
 as social interaction, 19–20, 23, 263
 truth, 260–263
 absolute, 260–261
 in context, 261–263
 Mathematics, school
 classroom
 community of practice, 269
 discourse, 269
 learning environment, 2, 7, 25–26, 259–275
 organization, 269
 curriculum, 69–117
 diverse mathematical content, 70
 goals, 267–268
 error activities, 69, *see also* Error activities
 error case studies, 69–117, *see also* Error case studies
 evaluation, 270
 goals, 146–148
 doing mathematics, 147, 168
 mathematical content, 147, 168
 mathematics as a discipline, 146–147, 167–168
 process goals, 204
 using mathematics, 147, 168
 inquiry approach, 11, 15–26, 203–204
 assumptions, 23–24, 260–267
 elements, 23–26
 implications, 267–272
 NCM Standards, 24, 26, 268
 problem solving, 42
 research, 24, 282–284
 teacher support, 272–275
 problem-solving, 2, 25
 reform, 1, 15
 beliefs and expectations, 2
 New Math projects, 1

- positive role for errors, 3
- teachers, 2—3, 11, *see also* Teachers
- inquiry environment, 25—26, 203—204, 259—275
 - role of, 181, 198
- transmission model, 11, 15—17
- assumptions, 16
- constructivist learning, 20—23
- critiques, 17—23
- dynamic view of knowledge, 18—19
- standardized tests, 16—17, 270
- technological change, 17
- Mathematics: The Loss of Certainty*, 45, 247
- Metacognition, 31—32
- Metaphors
 - error making
 - getting lost, 5—7
 - medical, 4—5
 - programming bugs, 4, 6, 33—34
 - walking on a bog, 18—19
- Multiplication, 9, 156—157, 166
- college student's 0^0 case study, 196
- numbers without zero case study, 241
- My $16/64=1/4$ case study, 70, 71—79, 170, 187
 - abstract, 293
 - educational value, 76—79
 - problem posing, 77, 79
 - problem solving, 77, 79, 192
 - incorrect procedure, 110—111
 - mathematical discourse level, 116
 - new inquiry avenues, 74—76
 - simplification, 111
 - solving original puzzle, 71—74
- My definitions of circle case study, 79—97, 110, 262
 - abstract, 293—294
 - error sources, 109
 - incorrect definitions error, 110, 113
 - list of definitions, 79—81
 - mathematical discourse level, 116
 - metric definitions, 82—83
 - properties of circles, 82
- My unrigorous proof case study, 97—108
 - abstract, 294
 - alternative proof, 98—100
 - analysis, 229—233
 - commentary, 107—108
 - concept of limit, 99—100, 107
 - convergence, 100, 103, 107, 230—231
 - error sources, 107—108
 - incorrect procedure, 110—111
 - infinite nested radicals, 103—105

mathematical discourse levels, 115—116

mathematical proofs, 105—107

natural numbers, 103—105

nature of sequences, 99, 107, 230—231

proof by induction, 100—103

remediation, 138

N

National Council of Teachers of Mathematics, 24

Agenda for Action, 1

National Research Council, 24

NCTM Evaluation and Curriculum Standards for School Mathematics, 24, 26, 268

process goals, 77

New Math projects, 1

Non-Euclidean geometry historical case study, 46, 51—54, 65, 260—262

abstract, 292

error sources, 109

incorrect assumptions, 110

incorrect results error, 112

Numbers

cardinal, 56—58

composite, 78

definition of, 59

error case studies, 13

infinite, 55—58

natural, 54—56

my unrigorous proof case study, 103—105

ordinal, 56—57

prime, 75, 78

Numbers without zero case study, 134, 211, 238—242, 261

abstract, 298

addition algorithm, 239—240

alternate representation, 239

computer metaphor, 241—242

learning opportunities, 254—257, 273

multiplication algorithm, 241

subtraction algorithm, 240—241

Number systems, 9, 238—242

Number theory, 78

P

Polygons

definitions, 155

students' polygon theorem case study, 120, 127—132, 135

teaching experiment, 155

Polyhedra

definition, 59, 61—63

Euler theorem historical case study, 59—63

simple, 63

Probability

error case studies, 13, 170

probability misconceptions case study, 211—212, 248—253

abstract, 299

learning opportunities, 256—257, 273

procedures, 249—250

small-group discussion, 250—253

students' P(A or B) case study, 170

Proof, mathematical, 105—107

teachers' unrigorous proof case study, 234—238

teaching experiment, 154—155

Proofs and Refutations, 45, 236

R

Radical constructivism, 19, 25

errors, 29—30

Ratios, number system, 9

Ratios case study, 7—10, 262

abstract, 291

baseball batting average, 8

fractions, 8—9

game results, 8

incorrect results error, 110, 112

Reasoning, 27—28

Reimann, Bernhard, 246

Remediation

error analysis, 38—39, 137—138

my unrigorous proof case study, 138

stances of learning, 137, 162

students' $16/64=1/4$ case study, 192

students' homework case study, 138

Rose, Barbara, 193, 212

S

Seachuri, 52—53

Science

anomalies and, 28

conceptual change approach, 36

development of, 28

error analysis, 28, 36

misconceptions, 36

Sheedy, John, 212

Simplification, 70

my $16/64=1/4$ case study, 71—79, 111

Small-group discussion

college students' 0^0 case study, 194

probability misconceptions case study,

250—253

students' P(A or B) case study, 171, 174, 178, 205

Stances of learning, 278—279

discovery, 137, 279

college students' 0^0 case study, 203

students' P(A or B) case study, 177, 203

remediation, 137, 278—279

students' $16/64=1/4$ case study, 192

Students

error analysis/resolution, 26, 34, 138

expectations, 192—193, 198, 206—207, 270—272

involvement levels, 139—141, 205, 280—281

independent student inquiry, 140, 204, 265—266

teacher-led student inquiry, 140, 204, 266—267

teacher modeling, 140, 201

teaching experiment, 162, 166—167

misconceptions

conflict phase, 41

intuitive phase, 41

reasonableness, 40

research, 39—41, 282—284

resolution phase, 41

retrospective phase, 42

science, 36

risk taking, 26

Students' $16/64=1/4$ case study, 170, 187—193, 264, 266

abstract, 296—297

counterexample, 188

equations, 189—192

planned errors, 193, 206

process goals, 204

remediation stance of learning, 192

variables, 188—189, 192

Students' definition of circle case study, 120—124, 264

abstract, 294

error debugging, 136

incorrect definitions errors, 121

learning opportunity

communication, 145—146

monitoring/justifying, 144

reflecting on mathematics, 145

mathematical discourse levels, 135—136

teaching experiment, 152—153, 166

Students' geometric constructions case study,

170, 178—187, 264, 266

- abstract, 296
- alternative procedure, 183—184, 186
- cumulative results, 185
- level of involvement, 186, 204—205
- mathematical discourse level, 186
- mathematical task, 203
- mathematical errors, 33
- problem-solving process, 181—182, 186—187, 203
- process goals, 204
- role of the teacher, 181, 205
- stances of learning, 186, 203
- triangles, 179—185
- unexpected errors, 206
- Students' homework case study, 120, 124—127, 139, 264
- abstract, 294—295
- error debugging, 136
- error sources, 142
- learning opportunity
 - doubt, 143
 - explorations, 144
 - problem-solving, 144
 - partial results, 125
 - planning, 266—267
 - remediation, 138
- teaching experiment, 153
- Students' P(A or B) case study, 170—178, 264
- abstract, 296
- discovery stance of learning, 177, 203
- goals, 204
- level of involvement, 205
- mathematical discourse levels, 177
- mathematical task, 203
- math journal, 171, 178, 205
- planned errors, 206
- probability of a disjunction, 171—172
- algorithm, 175—178
- results justification, 177
- small-group discussion, 171, 174, 178, 205
- source of errors, 177
- Students' polygon theorem case study, 120, 127—132, 139, 262, 264
- abstract, 295
- counterexamples, 128
- justification, 130
- learning opportunity
 - communication, 145—146
 - doubt, 143
 - explorations, 144
 - humanistic mathematics, 145
 - initiative, 145
- monitoring/justifying, 144
- problem-solving, 144
- mathematical errors, 33
- problem-solving, 136, 170
- student involvement, 140
- students as initiators, 130—131
- teaching experiment, 135, 166—167
- tenativeness, 132, 136

T

Teachers

- education, 209—257
- error case studies, 209—257
- learning opportunities, 254—257, 273
- mathematics
 - dualistic view, 255
 - school, 2—3, 11
 - views of, 255—256
- modeling, 140, 201
- role, 181, 205, 266—267
- sources of errors, 109, 141, 206
- and student inquiry, 140, 204, 266—267
- support, 272—275
- professional development, 272—273
- Teachers' definition of circle case study, 211—228, 273
- abstract, 297
- categorization, 215—220
- acceptability, 215—217
- mathematical content, 217—219
- usefulness, 219—220
- concept of circle, 220—223
- learning opportunities, 254—256
- Teachers' unrigorous proof case study, 211, 228—238, 273
- abstract, 297—298
- detailed analysis, 231—233
- convergence, 230—233
- limits, 232—233
- sequences, 230—233
- learning opportunities, 254—256
- proof
 - debugging, 235—237
 - by induction, 234, 237
 - nature of, 234—238

Teaching

- stimulation and support, 24
- transmission of knowledge, 16
- Transformations, geometrical, 84—85
- Triangles
 - student's geometric constructions case study, 179—185

V

- Variables, 152, 156
- Viète, 97
- W
 - Weierstrass, Karl Theodor, 51
 - What-not-if strategy, 116, 144
- Writing
 - errors, role of, 32—33
 - writing to learn approach, 33

Z

- Zero, 238—239
- college students' 0⁰ case study, 171, 193—202
- high school students' 0⁰ case study, 132—135, 146
- numbers without zero case study, 134, 211, 238—242