Algebra in the Middle School: Developing Functional Relationships Through Quantitative Reasoning

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Abstract Understanding function is a critical aspect of algebraic reasoning, and building functional relationships is an activity encouraged in the younger grades to foster students' relational thinking. One way to foster functional thinking is to leverage the power of students' capabilities to reason with quantities and their relationships. This paper explicates the ways in which reasoning directly with quantities can support middle-school students' understanding of linear and quadratic functions. It explores how building quantitative relationships can support an initial function understanding from a covariation perspective, and later serve as a foundation to build a more flexible view of function that includes the correspondence perspective.

Functions and relations comprise a critical aspect of algebra, and recommendations for supporting students' algebraic reasoning emphasize an early introduction of functional relationships in middle school (NCTM 2000). Students' difficulties in acquiring the function concept is well documented (e.g., Carlson 1998; Carlson et al. 2002; Cooney and Wilson 1996; Monk and Nemirovsky 1994), which highlights the need to better support students' emerging function concepts in ways that are mathematically productive, setting a strong foundation for more formal algebraic reasoning at the high school level. In this chapter I argue that reasoning directly with quantities and their relationships constitutes a powerful way to help students build beginning conceptions of function at the middle-school level. In particular, reasoning with quantities can directly support a covariation approach to function, while also providing a foundation for reasoning more flexibly with functional relationships later on.

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What Is Quantitative Reasoning?

Quantities "are attributes of objects or phenomena that are measurable; it is our *capacity* to measure them—whether we have carried out those measurements or not—that makes them quantities" (Smith and Thompson 2007, p. 101, emphasis original). A quantity is composed of one's conception of an object, a quality of the object, an appropriate unit or dimension, and a process for assigning a numerical value to the quality (Kaput 1995); length, area, speed, and volume are all attributes that can be measured in quantities. When students engage in quantitative reasoning, they operate with quantities and their relationships; quantitative operations are therefore conceptual operations by which one conceives a new quantity in relation to one or more already-conceived quantities (Ellis 2007). For example, one might compare quantities additively, by comparing how much taller one person is to another, or multiplicatively, by asking how many times bigger one object is than another. The associated arithmetic operations would be subtraction and division.

To illustrate the differences between a formal algebraic approach and a quantities-based approach, consider two responses to the following problem about the nature of quadratic growth:

Problem 1 Explain why the "second differences" for a quadratic function $y = ax^2$ are $2a$ for well-ordered tables in which the x-values increase by 1.

This problem emerged from an algebra II classroom in which the students' introduction to non-linear functions included an algorithm for determining the degree of a function based on the finite differences rule (Ellis and Grinstead 2008). The students easily remembered this algorithm, but it was unclear whether anybody understood its origins.

Justification #1: A typical algebraic argument involves relying on variables to represent a general case and writing and manipulating expressions. For instance, one can create a general table for $y = ax^2$ in which the x-values increase by 1:

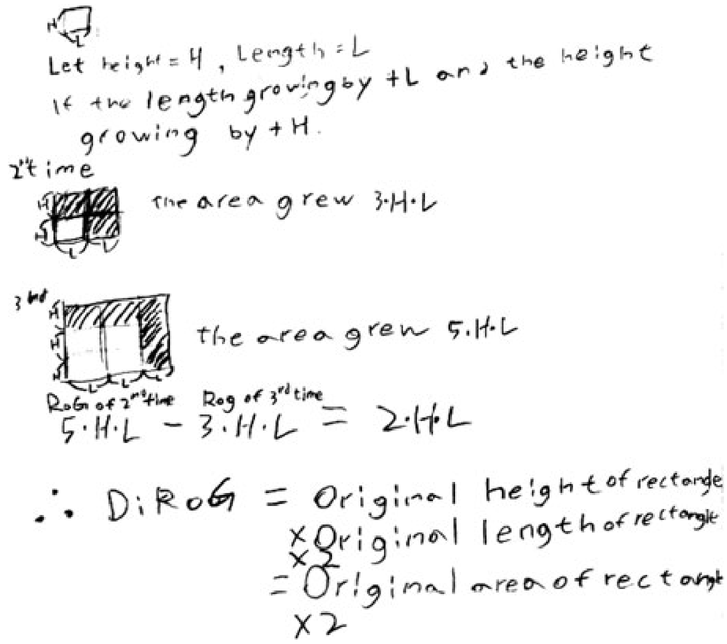
Fig. 1 Table of x- and y-values for $y = ax^2$

x	y
1	a
2	$4a$
3	$9a$
4	$16a$

Calculating the first differences reveals values of $3a$, $5a$, and $7a$. Calculating second differences reveals a constant second difference of $2a$, and this approach can be generalized to any three consecutive entries in the table in which $x = n$, $x = (n + 1)$, and $x = (n + 2)$. Corresponding y-values will be $y = an^2$, $y = a(n + 1)^2$, and $y = a(n + 2)^2$. Calculating the first differences reveals values of $a(2n + 1)$ and $a(2n + 3)$, with the second difference therefore $2a$. This approach lives entirely in the world of symbolic expressions in a manner that is divorced from any realizable situation and its constituent quantities. As a formal algebraic justification it provides a valuable opportunity to generalize beyond specific numbers, but it may fail to support students' understanding of the behavior of quadratic growth, what the second differences can represent, and why they remain constant for quadratic functions.

Justification #2: One group of eighth-grade students created and analyzed tables of quadratic data by exploring the relationships between the lengths, heights, and areas of rectangles that grew while maintaining their length/height ratios. One student attempted a justification by imagining an $H \times L$ rectangle that grew in discrete increments by increasing H units in height and L units in length. The student conceptualized the first differences, which he called the *rate of growth* (RoG), as the growth of the area when the height increased by H units. He conceptualized the second differences as the *difference in the rate of growth* (DiRoG), describing it as the "rate that the increase in the area is increasing:"

Fig. 2 Eighth-grade student's justification



The student reasoned with the relationships between the quantities height, length, and area, engaging in quantitative operations as he compared their differences. He concluded that because he could calculate the difference in the rate of growth as $2HL$ each time the rectangle grew an additional H units in height and L units in length, the second differences must represent twice the original area of the rectangle.

The student's justification contains some limitations, particularly because his drawing only addresses a particular type of growth in which the height and length increase by whole-unit increments of H and L. However, even though the student did not reason about arbitrary increases of H and L, his justification represents a meaningful attempt at a generalized argument. The student's reliance on the relationships between the quantities height, length, and area helped him develop an understanding of what the second differences represented, which provided a springboard for further investigation of why the second differences in well-ordered tables are always constant for quadratic functions.

Steffe and Izsak (2002) argue that quantitative reasoning should be the basis for algebraic reasoning. Focusing on relationships between quantities, rather than on numbers disconnected from meaningful referents, can ground the study of algebra in people's conceptions of their experiential worlds (Chazan 2000). This provides a meaningful starting point for mathematical inquiry, in contrast to taking numbers, shapes, and relationships as givens in their own right (Thompson 1994). I propose that adopting a quantitative reasoning approach can support students' meaningful engagement with algebra in general and with functions in particular. I will present the results from two teaching experiments with middle-school students, the linear functions teaching experiment and the quadratic functions teaching experiment. Excerpts from both teaching experiments demonstrate a number of ways in which students' reasoning with quantities fostered particular types of function understanding.

The Importance of (and Difficulties with) Functional Thinking

The function is a central concept around which school algebra can be meaningfully organized (Kieran 1996; Yerushalmy 2000), and many researchers have argued for the importance of a functional perspective in contrast to the more traditional approach that focuses on algebra as symbolic manipulation (Bednarz et al. 1996; Schliemann et al. 2007). Adopting an approach that places functional relationships at the center of algebra allows us to couch algebraic thinking as the use of a variety of representations in order to make sense of quantitative situations relationally (Kieran 1996). Beyond theoretical considerations, there are also practical reasons for emphasizing a functional approach to algebra. Attaining a deep understanding of function is critical for success in future mathematics courses (Carlson et al. 2003; Romberg et al. 1993) and in courses on scientific inquiry (Farenga and Ness 2005). Many have argued that the function concept is foundational for understanding concepts in advanced mathematics (e.g. Kaput 1992; Rasmussen 2000; Thompson 1994; Zandieh 2000), and as Romberg et al. (1993) argued, "there is general consensus that functions are among the most powerful and useful notions in all mathematics" (p. 1).

Given the widespread agreement on the importance of functions for algebraic reasoning, the value of organizing algebra content around a functions approach, and the need for a deep understanding of functions for further mathematical and scientific inquiry, it is important that we develop ways of helping students successfully understand functional relationships. However, these endeavors have proved difficult: Many studies conducted to investigate students' function understanding suggest that they demonstrate a limited view of the function concept (e.g. Carlson 1998; Sfard and Linchevski 1994; Thompson 1994; Vinner and Dreyfus 1989). In general, students emerge from middle school and high school algebra classes with a weak understanding of function (Carlson et al. 2002; Cooney and Wilson 1996; Monk 1992; Monk and Nemirovsky 1994).

Two examples from my previous studies illustrate some of the common difficulties students experience when encountering functional relationships. The first comes from a problem presenting a direct-ratio situation within the context of a linear functions unit (Ellis 2009):

Problem 2 Say you have a pile with 2 rolls of pennies and a pile with 5 rolls of pennies. If you were to compare their weights, what might you notice?

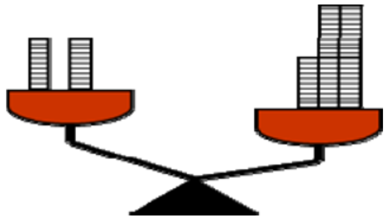


Fig. 3 Picture accompanying the penny-role problem

One eighth-grade student, Juanita, made both additive and multiplicative comparisons across the two piles, noting that the bigger pile had 3 more rolls, and would weigh "2.5 times as much" as the smaller pile. When she investigated the pattern in a tabular form, however, Juanita was unable to recognize the relationship as linear and she could not develop an equation for the data:

Fig. 4 Table of number of rolls and weight values

# of Rolls	Weight
2	9 oz
5	22.5 oz
12	54 oz
16	72 oz

AE: What does this table tell you?
J: It couldn't be a straight line.
AE: How come?
J: (Calculating differences between successive x-values in the table): 3, 7, 4, and that's probably not an even spaced one. Wouldn't be a straight line.
AE: And what's your reason for that?
J: If you made your graph, it doesn't look like it'd be a straight line because it goes up (calculates differences between successive y-values in the table) by 13.5 and 31.5 and then 18.

Juanita created a rough sketch of the data to confirm her belief that the data were not linear:

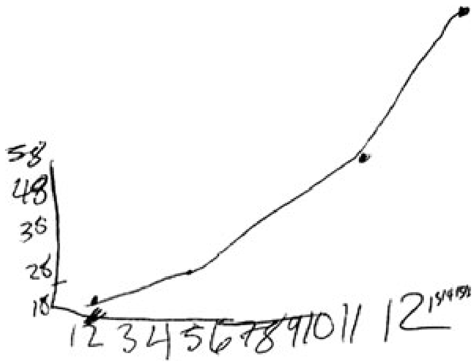


Fig. 5 Juanita's graph of the rolls and their weight

Determined to find a pattern for the data in order to come up with an equation, Juanita continued to search by taking the differences between the rolls and the weight for each table entry, and then taking the differences of those results:

J: 7, 17.5, 42, 56. That's what it goes up by. If you do in between them it's 10.5, 24.5, and then 14. … There's no patterns anywhere!

Juanita's difficulty in recognizing the data as linear and her inability to create an equation mirrors some of the documented difficulties students experience with tables, patterns, and functions. Although students are adept at searching out patterns in tabular representations, many struggle to perceive a functional relationship (MacGregor and Stacey 1993; Mason 1996; Schliemann et al. 2001). Even when students are able to detect patterns, they may not be able to formalize those patterns correctly by writing appropriate equations or algebraic expressions (English and Warren 1995; Orton and Orton 1994; Stacey and MacGregor 1997). Students struggle to correctly translate between tabular, graphical, and algebraic representations of functional relationships, and can become overly dependent on particular artifacts of representations, such as only recognizing a function as linear if its tabular representation has uniformly-increasing x-values (Lobato et al. 2003).

A second example illustrates some of the difficulties students can encounter when approaching non-linear functions. High-school algebra II students encountered the following graph and attempted to find an equation for the parabola (Ellis and Grinstead 2008):

Problem 3 Ravi has 120 meters of fence to make his rectangular rabbit pen. He wants to enclose the largest possible area. Here is Ravi's graph of the relationship between the width and the area:

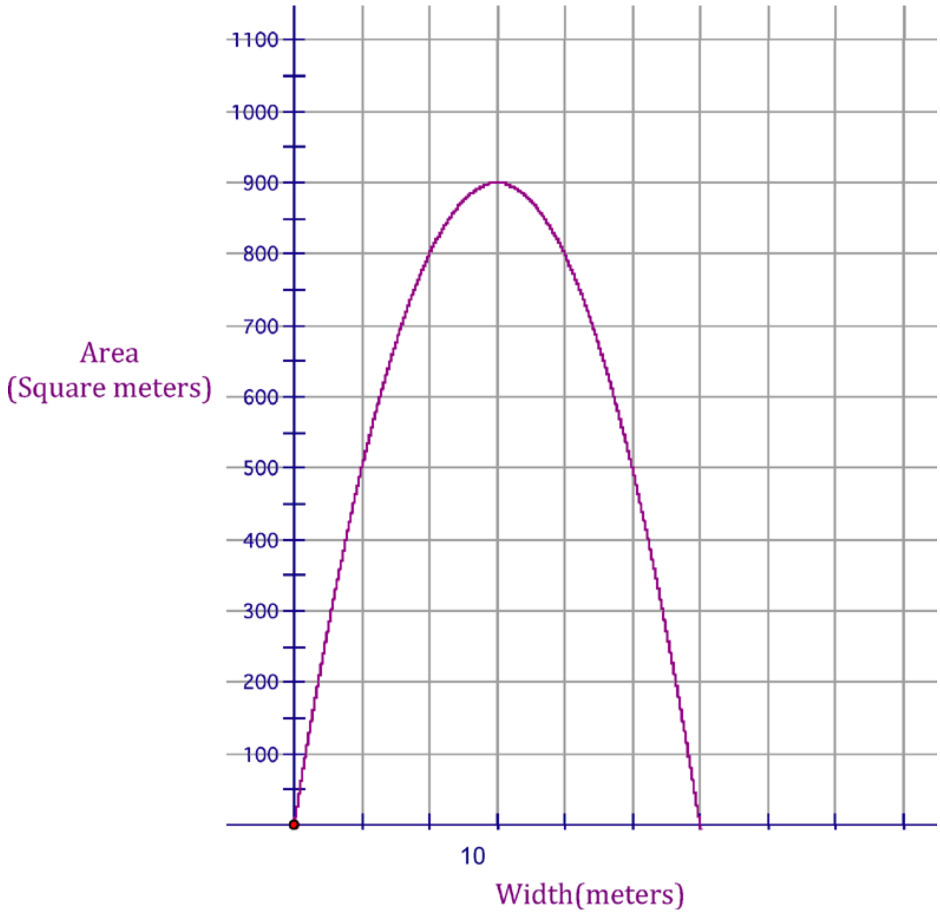


Fig. 6 Graph for the rabbit pen problem

Alexis, a tenth-grade student identified by her teacher as a high performer, determined that the equation for the parabola should have the form $y = -ax^2 + 900$, because the maximum value of the parabola was at $y = 900$. In addition, Alexis knew that the a-value should be negative, because the parabola was "upside down." In order to determine the value of "a", Alexis explained, "you could do this, rise over run." She picked two points, $(10, 500)$ and $(20, 800)$, and then calculated the rise and the run, ignoring the scales on the axes: "So it's 3 over 1, which is basically 3." Alexis concluded that the equation of the parabola should therefore be $y = -3x^2 + 900$.

Alexis' treatment of the graph and development of an equation reflects the research demonstrating students' difficulties understanding the value of "a" in $y = ax^2 + bx + c$ (Dreyfus and Halevi 1991; Zaslavsky 1997). The challenges in connecting algebraic and graphical representations of quadratic functions can further contribute to students' struggles to describe the effects that changing the parameters a, b, and c have on graphs of parabolas (Bussi and Mariotti 1999; Leinhardt et al. 1990; Zazkis et al. 2003). In addition, Alexis' inappropriate adoption of the rise over run method for generating a "slope" mirrors many students' tendencies to generalize from linearity, regardless of the appropriateness of that generalization (Buck 1995; Chazan 2006; Schwarz and Hershkowitz 1999; Zaslavsky 1997).

Given the widespread difficulties students experience as they learn about functions, it is important to develop methods for helping students build a productive understanding of functional relationships from the time that they first experience them in the algebra classroom. Taking a quantities-based approach to informal (and later formal) functional reasoning can support students' initial approaches to functional relationships as they explore coordinated changes between covarying quantities.

An Alternative Approach to Function: Quantities and Covariation

Traditional approaches to function rely on a correspondence or stasis view (Smith 2003), in which one approaches a function as the fixed relationship between the members of two sets. Farenga and Ness (2005) offer a typical correspondence definition of function: "One quantity, y, is a function of another, x, if each value of x has a unique value of y associated with it. We write this as $y = f(x)$, where f is the name of the function" (p. 62). This static view underlies much of school mathematics, particularly in the treatment of functions. Alexis' approach reflects this typical school experience, as she examined the graph and then attempted to build an equation without imagining the two quantities changing together. Instead, Alexis' treatment disconnected the properties of the graph and its associated equation from the contextual situation that referenced the changing relationship between width and area.

In contrast, Smith and Confrey (Smith 2003; Smith and Confrey 1994) describe the covariation approach to functional thinking. Under this approach, one examines a function in terms of a coordinated change of x- and y-values. Confrey and Smith (1992, 1994, 1995) have found that students' initial entry into a problem is typically from the covariational perspective. In addition, they argue that viewing a function as a way of representing the variation of quantities can be a more powerful approach than the correspondence model, particularly in its ability to promote thinking about functions in terms of rates of change (Slavit 1997; Smith and Confrey 1994). As Chazan (2000) argues, the covariation approach can support a view of mathematics as a way of making sense of the phenomena of relationships of dependence, causation, interaction, and correlation between quantities.

Viewing a function as a relationship between covarying quantities is part of a larger idea that acknowledges the importance of the mathematics of change. An emphasis on the mathematics of change can encourage students to examine patterns in relationship to the ways in which they grow or can be extended. Many have suggested that this approach is a critical but overlooked element in the standard U.S. curriculum (Nemirovsky et al. 1993; Mokros et al. 1995). Exploring function as a way to measure change and variation is typically reserved for calculus, thus effectively restricting access to these ideas to the 10% of students who will reach the highest level of high-school mathematics (Roschelle et al. 2000). However, adopting a rate-of-change perspective can be accessible even for beginning algebra students in middle school. One way to foster students' understanding of the mathematics of change is through introducing rich situations that encourage students to construct meaningful relationships between quantities.

For instance, one group of seventh-graders in a linear functions teaching experiment explored constant rates of change by investigating two situations, gear ratios and constant speed (Ellis 2007). The group consisted of 7 pre-algebra students who had not yet studied linear functions or graphs in their mathematics classroom, and a focus of the teaching experiment was to emphasize the activities of generalizing and justifying through meaningful engagement with quantitative referents. The students met for 15 sessions and during the first eight sessions they worked with physical gears to examine different gear ratios. Early in the sessions, the students connected a gear with 8 teeth to a gear with 12 teeth and then spun the gears together, trying to identify ways to simultaneously keep track of the rotations of both gears. By putting small pieces of masking tape on one of the teeth of each of the gears, the students devised a counting system for keeping track of both gears' rotations simultaneously, and ultimately created tables of gear rotation pairs such as the following:

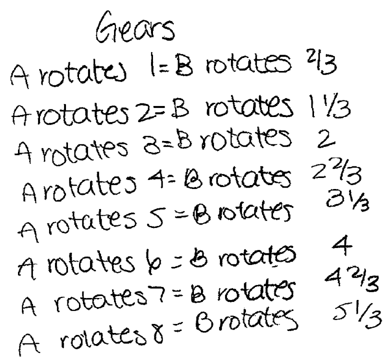


Fig. 7 Maria's table of gear rotations

By working with the physical gears, the students not only found ways to coordinate the rotations of each of the gears, but also developed a covariation language for discussing the nature of the coordinated quantities. For instance, in describing the table in Fig. 7, Dora explained, "For a small turn, the big one goes a two-thirds turn. For the big to turn once, the small one goes one and a half turn."

Carlson and Oehrtman (2005) note that students need to be able to imagine how one variable changes while imagining changes in the other. Relying on situations that involve quantities that students can make sense of, manipulate, experiment with, and investigate can foster their abilities to reason flexibly about dynamically changing events. These experiences were helpful when the linear functions students eventually encountered tables of data referencing multiple rotation pairs, such as the one shown in Problem 4:

Problem 4 The following table contains pairs of rotations for a small and a big gear. Did all of these entries come from the same gear pair, or did some of them come from different gears altogether? How can you tell?

Fig. 8 Table of gear pairs

Small	Big
7 1/2	5
27	18
4 1/2	3
16	10 2/3
1/10	1/15

Dora explained her thinking about the problem by referencing the gears:

D: Think of a gear. When you spin it, the teeth on it pass through. One gear has 8 teeth, the other has 12. When you spin them, teeth pass through each other. For every two-thirds of the teeth passed on the big one, that's 8 teeth, so the small one turns once. If the small one goes 3 turns, the big one will go 2. So if the small one goes 7 and a half times, the big gear will go 5.

A covariation approach can also ultimately support students' abilities to express function relationships algebraically. After hearing Dora's explanation, another student, Larissa, expressed the gear ratio relationship by writing "$s(2/3) = b$", which represents the number of rotations between the small gears and the big gears. Larissa explained, "*s* is the number of small rotations, the number of rotations that the small gear does. And then *b* is the big rotations, the number of rotations that the big gear makes."

Carlson and Oehrtman identified a covariation framework (2005), in which they decompose covariational reasoning into five mental actions. This decomposition has proved useful for promoting covariational reasoning in students. Although the framework evolved in the context of calculus students' reasoning, the first four mental actions described can also apply to algebra students. (The fifth mental action is the coordination of instantaneous rate of change, which is not as applicable to beginning algebra topics.) The gear rotation situation supported students' abilities to coordinate the change of both quantities simultaneously, fostering the first three mental actions in the covariation framework: (1) coordinate the dependence of one variable on another variable, (2) coordinate the direction of change of one variable with changes in the other variable, and (3) coordinate the amount of change of one variable with changes in the other variable. Because quantitative reasoning requires the formation of relationships between quantities, students' activity in constructing these relationships can support the meaningful coordination of variables in function relationships.

The fourth mental action is the coordination of the average rate-of-change of the function with uniform increments of change in the input variable. Exploring phenomena that are linearly related but not in direct proportion can prompt a shift from direct multiplicative comparisons to the creation of ratios of change between coordinated variables. In the gear context, the students examined scenarios in which one gear spun a certain number of times on its own before a second gear was connected to it, at which point they spun together. Although the situation is somewhat contrived from an adult perspective, it was meaningful to students because it described a familiar situation that they could directly imagine. The following table can encourage students to coordinate the rates of change of each of the variables, both because it is not well ordered and because it represents a situation that is not directly proportional (the function described by the ordered pairs is $y = (3/4)x + 5$):

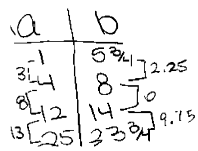
Problem 5 The following table contains pairs of rotations for a big and a small gear. What is the relationship between the two gears?

Fig. 9 Table of gear pairs representing a $y = mx + b$ situation

Small	Big
1	5 3/4
4	8
12	14
25	23.75

One student, Timothy, identified the differences between successive table entries:

Fig. 10 Timothy's calculations with the gear pair table



He explained, "The only thing I found out is that they go up by 3/4, because if you subtract 1 from 4 and 5 and 3/4 from 8, you get 3 and 2.25, and 2.25 over 3 equals 3/4. And that's how I found out that it works for all of them." Pushed to explain why this worked, Timothy said, "B goes up by 3/4 of what A goes up by." When asked to describe what was happening with the gears rotating, he noted, "B had already turned 5 times. And B is like 3/4 the size of A. And so A times 3/4 means that it only goes through 3/4 of its teeth." When Timothy noted that B was 3/4 the size of A, he spoke of the gear's size but appeared to be thinking about the gear's rotations instead; this is consistent with the second half of his statement in which Timothy said that B would only go through 3/4 of its teeth. Dora and several other students expressed this relationship algebraically by writing "$(3/4)a + 5 = b$", and could explain each part of the equation in terms of the relationship between the gears' rotations and number of teeth. Ultimately the students were able to approach new data by calculating the ratio of the change of one variable to the coordinated change in the other variable in order to determine the appropriate relationship between mystery gears.

Students' first approaches to function are typically covariational in nature (Confrey and Smith 1992, 1994, 1995), but it is important to support these initial forays in a manner that supports a meaningful understanding of covarying phenomena, in contrast to the common tendency to engage in recursive pattern seeking with naked numbers. Although students are adept at creating multiple patterns, they can struggle to identify patterns that are algebraically useful and generalizable. Embedding these patterns in meaningful problem situations that require students to identify relationships between covarying quantities can help circumvent the common pattern-seeking traps that sometimes plague students. Quantity-based problem situations can instead "serve as the true source and ground for the development of algebraic methods" (Smith and Thompson 2007, pp. 96–97).

A Flexible Understanding of Functions

Coordinating Covariation and Correspondence Approaches

The prevalence of covariation approaches has been highlighted in the research literature, and this view provides a powerful mechanism for developing an understanding of function as a way of representing variation in coordinated quantities. However, any complete understanding of functional relationships must ultimately include a broader exploration of the relationships between two variables (Carraher and Schliemann 2002). Carlson and Oehrtman (2005) argue that students must be able to understand multiple views of function for success in mathematics: they must develop an understanding of function as a process that accepts inputs and produces outputs, as well as attend to the changing value of output and rate of change as the independent variable is varied.

The shift from a covariation approach to the correspondence view can be difficult for students, but there is evidence that when working directly with quantities, even young children can develop a flexible function understanding (e.g., Nunes et al. 1993; Schliemann et al. 1998, 2003). Working directly with accessible quantitative relationships can aid in beginning algebra students' investigations of functions from multiple perspectives, as well as support their abilities to shift flexibly across different perspectives. The seventh-grade students' experiences with gear ratios (and later constant-speed situations) helped them create algebraic representations such as "$(3/4)a + 5 = b$" that they could ultimately view in terms of both coordinated changes in each gear and as a direct relationships between a and b. These experiences helped the teaching-experiment students make meaningful sense of the pennies problem (Fig. 3) that had caused such difficulty for Juanita, who was not in the teaching experiment. Timothy's response was typical of the teaching-experiment students:

T: [Examining the table in Fig. 4]: Well, let's see. 2 to 9 oz, so that's 4.5 oz per roll. For that. So multiply that by 5. Times 5, equals 22.5. So these (the first two pairs in the table) are both from the same roll. Then multiply it by 12. 4.5 times 12 equals 54, so that's from the same one. And then 16 times 4.5. 16 times 4.5 equals 72, so they're all from the same thing because they all have the same weight for 1 roll.

AE: Do you think the graph is going to be linear or non-linear?
T: It's all going to go on the same line.
AE: Why do you think that would happen?
T: Because whatever the weight is, you can multiply it by 1 over 4.5 to get the number of rolls.

Timothy's reliance on his understanding of the relationship between the number of rolls and the total weight in ounces supported a direct comparison across the x- and y-columns of the table. He noted that whatever the weight is (the input variable), you can multiply it by 1 over 4.5 to get the number of rolls (the output variable); even though this is the reverse of how we might typically approach a table from an input-output perspective, it is correct and enabled Timothy to successfully solve a number of extrapolation and interpretation problems. Moreover, Timothy could move flexibly between the correspondence and covariation approaches, as evidenced by his predictions about the table's graph: "It just looks like to me that all you're doing is going up by 4.5 oz and 1 roll… it's going up by the exact same thing every time."

Reasoning with quantitative relationships can support students' flexible movement between different function approaches for quadratic functions as well. In the quadratic functions teaching experiment (consisting of 15 sessions with 7 eighth-grade students), I introduced quadratic phenomena in terms of the relationships between the lengths, heights, and areas of rectangles that grew while maintaining their length/height ratios. Although none of the students had yet experienced quadratic functions in their normal classrooms, they had all experienced other functional relationships and graphs (such as linear functions) in their algebra or pre-algebra courses. The students worked with a script in Geometer's Sketchpad to explore what happened to the dimensions of a particular rectangle (for instance, a 3 cm by 2 cm rectangle) as it grew and shrank. As predicted, the students made sense of these phenomena from a covariation perspective, imagining what would happen to the area as the length (or width) increased by a uniform amount. The students created their own tables of data to represent the phenomena they observed; a typical table is shown below:

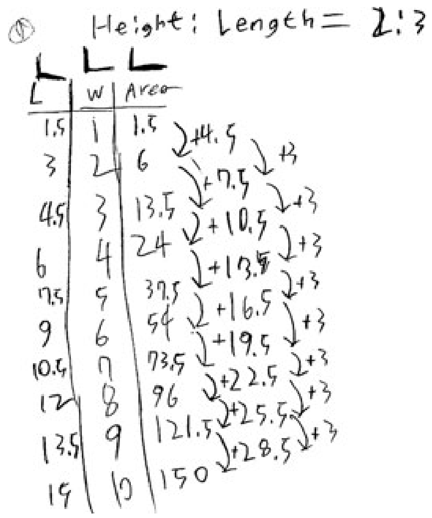


Fig. 11 Student's table of data representing the growing rectangle

In this case, the student was able to coordinate the growth of the length and the area of the rectangle as the width grew in 1-cm increments. He also identified the amount by which the area increased for each additional centimeter in width, as well as their differences.

I introduced a standard far-prediction problem to encourage a shift from the covariation approach to the development of a direct functional relationship between height and area:

Problem 6 Here is a table for the height versus the area of a rectangle that is growing in proportion to itself. What will the area be when the rectangle is 82 units high?

Fig. 12 Table of height/area values for a growing rectangle

Height	Area
2	18
3	40.5
4	72
5	112.5
6	162

The students' initial entry into the problem was from a covariation perspective, in which they coordinated the growth of three quantities: height, length, and area. Each student introduced a third column, length, and noticed that the length increased by 4.5 units each time the height increased by 1 unit:

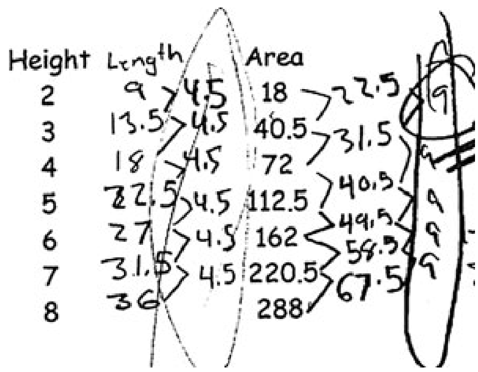


Fig. 13 Student's table with the added length column

One student, Ariel, stated that the area of a rectangle 82 units high would be 30,258 square units. Ariel explained that she found 30,258 by multiplying 82 by 4.5 units to get the corresponding length of 369 units. The area would then be 82 units multiplied by 369 units. Jim relied on his image of the rectangle and the way in which it grew to explain Ariel's reasoning to the class:

J: Well that was the length, the 369, so she has to do height times length equals area. So she had to multiply [the 369 by 82] again.

AE: I see. So how did you know to multiply 82 by 4.5 units to get the length?

J: Because, that's how much the length was going up by every time. So if you, like, made a square, I mean, or a rectangle, and then you moved up 1 unit, it would go over 4.5 for every time you go up the height 1.

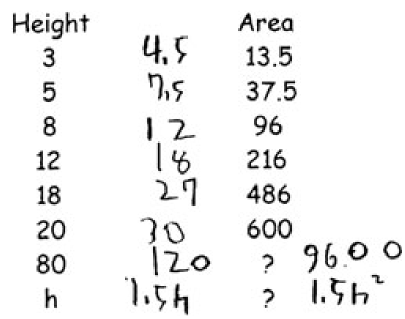
At this stage, the students' thinking relied on an image of the manner in which the rectangle grew in order to coordinate the growth between the height and the length. This supported their understanding from a covariational view, and they capitalized on their understanding of the coordinated growth of the height and the length to determine the area of the rectangle for a large height. However, the students' images of the nature of the rectangle's growth were limited to cases in which the rectangles grew in discrete whole-unit increments, typically increments in which the length or the height increased by 1 unit. Simplifying the nature of the growth initially helped the students coordinate the multiple quantities involved (length, width, area, and increases in each of these quantities), but this was a strategy that would ultimately need to be generalized to encompass the notion of non-unit increments and continuous growth.

In an attempt to encourage the students to think about a direct relationship between the height and the area, I then asked them to compute the area when the height was n units. They quickly produced the formula "area $= 4.5n^2$", and Jim explained his reasoning to another student, Bianca:

J: I put n times 4.5 times n.
B: How did you figure it out?
J: Well, n can be any value...
B: Right.
J: Times 4.5 is your length. Times n again because I do height again, is your area.

Jim simply extended his previous reasoning to determine that the length of a rectangle n units high would be $4.5n$, and thus the area must be the height times the length, or $n(4.5n)$.

The students continued to work with far prediction problems, and the introduction of tables that were not well ordered encouraged the students to conceptualize the (unknown) length in terms of its relationship between the height and the area. Unable to identify the rate of growth of the length, the students instead began to develop third length columns by dividing the area by the height. The inclusion of the length columns also encouraged students to make more explicit connections between the length/height ratio and the "a" in $y = ax^2$, as seen below:



Height		Area	
3	4.5	13.5	
5	7.5	37.5	
8	12	96	
12	18	216	
18	27	486	
20	30	600	
80	120	?	9600
h	1.5h	?	1.5h²

Fig. 14 Student's third length column for a table of height/area values

One student, Tai, explained, "I came up with this equation [area $= 1.5h^2$]. It's like, the number in front of the height squared, is figured out by the area divided by the height squared." Daeshim added, "The number is what you have to multiply the height by to get the length. And then height times length is the area, so that is why it's squared." The norm that students must explain how their equations were related to the quantities in the rectangle supported justifications such as Daeshim's, and encouraged additional connections between features of the equations (such as the value of "a" in $y = ax^2$) and properties of the growing rectangle.

Although the shift from a covariation approach to a correspondence approach was gradual, it was aided by the students' abilities to make direct connections to their images of growing rectangles and their abilities to coordinate relationships between the quantities length, width, and area. Moreover, their reliance on these constructed relationships enabled the students to develop a flexible view of the quadratic function, one in which they frequently shifted between the covariation and correspondence views. In particular, these flexible views helped the students make connections between the value of "a" in $y = ax^2$ and the second differences for area, which the students termed the "difference in the rate of growth of the area", or the DiRoG for area when the width increased in uniform amounts. The students created multiple generalizations about the DiRoG of the area, including the notion that the DiRoG (for tables in which the rectangle's height increases by 1 unit) is twice the value of "a" in $y = ax^2$, the DiRoG is twice the area of the rectangle when the height is 1 unit, and the DiRoG is the value of the rectangle's length when the height is 1 unit.

The students experienced little difficulty when they transitioned to tables that were not well ordered for a number of reasons. First, they were accustomed to picturing the rectangle that was represented in the table's values, so every pattern they developed was solidly grounded in the imagery of length, height, and area and their relationships. This imagery supported the students' abilities to create functional relationships between height and area. In addition, the students had spent so much time focusing on what the DiRoG meant for the rectangle's area in relationship to the equations they built, they became accustomed to moving seamlessly between recursive and functional representations. Because they kept discussing what the values represented in terms of length and area, the students were encouraged to represent those relationships more generally in algebraic forms.

Flexibility Across Forms

Smith and Thompson (2007) remind us that one role of quantitative reasoning is to support thinking that is flexible and general in character. Students in the linear and quadratic functions teaching experiments created many tables and algebraic representations to describe the same phenomena, and could move between them. But what about graphical representations? In both cases, I deliberately refrained from introducing graphs until the students had developed a meaningful understanding of the relationships represented by the graphs. Once that foundation was in place, they began to create their own graphs as a way to justify their conclusions about the quantitative relationships they developed.

For instance, the linear functions students encountered a scenario in which a character walked 5 cm in 4 seconds. They created multiple equivalent ratios to represent the character's speed, and represented these ratios in tables of data. When asked to explain why a speed of 15 cm in 12 seconds was the same as 5 cm in 4 seconds, Timothy asked if he could create a graph:

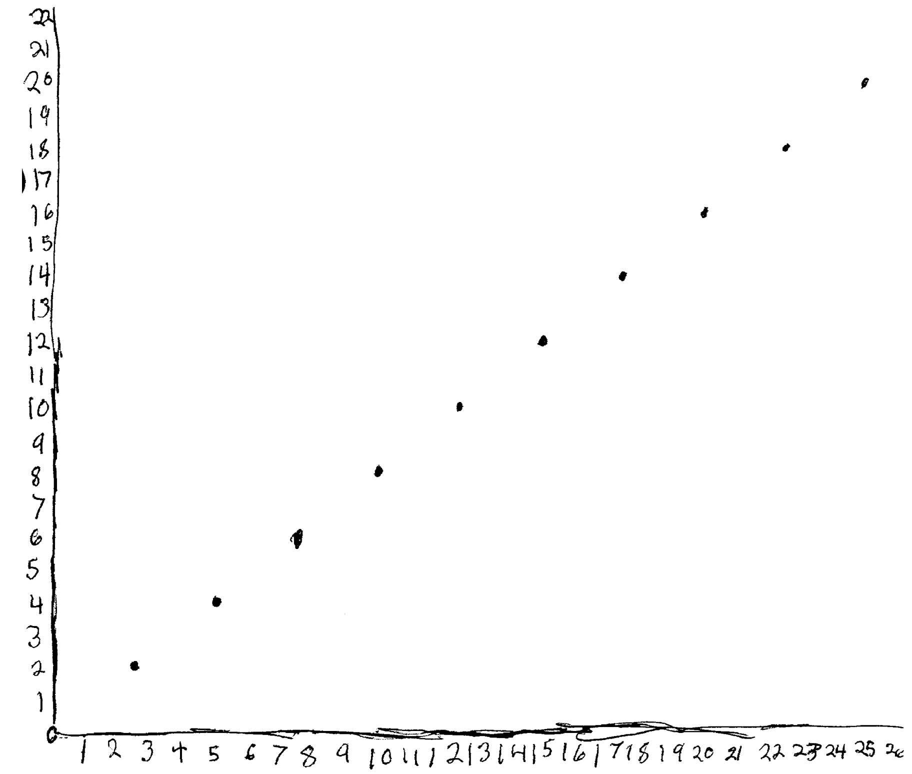


Fig. 15 Timothy's graph of same-speed values

Timothy's partner, Dora, wrote the equation $y = 4/5x$ and explained that the x-axis represented centimeters and the y-axis represented seconds. Timothy explained that he could put a line through the points, if they were appropriately exact:

T: You could put a line there. But it's not a good graph so it's not going to make a straight line.
AE: Okay. You found that the slope of the line was $4/5$. What does that mean?
T: Whatever x is, y is $4/5$ of x. The slope means that whatever x goes up by, $4/5$ of that is how much y goes up by.
AE: And what does $4/5$ have to do with the speed of the clown?
T: It's going basically $4/5$ of a second per centimeter.
AE: Now why is the fact that the clown's speed is $4/5$ of a second per 1 centimeter, why is that the same as the slope being $4/5$? What's the connection?
T: Because for every centimeter it goes, it's going like 4, er, yeah, $4/5$ of a second I think. Every centimeter goes… yeah. Every centimeter it's going $4/5$ of a

second. The slope is $4/5$ because for every centimeter that you add, you add $4/5$ seconds.

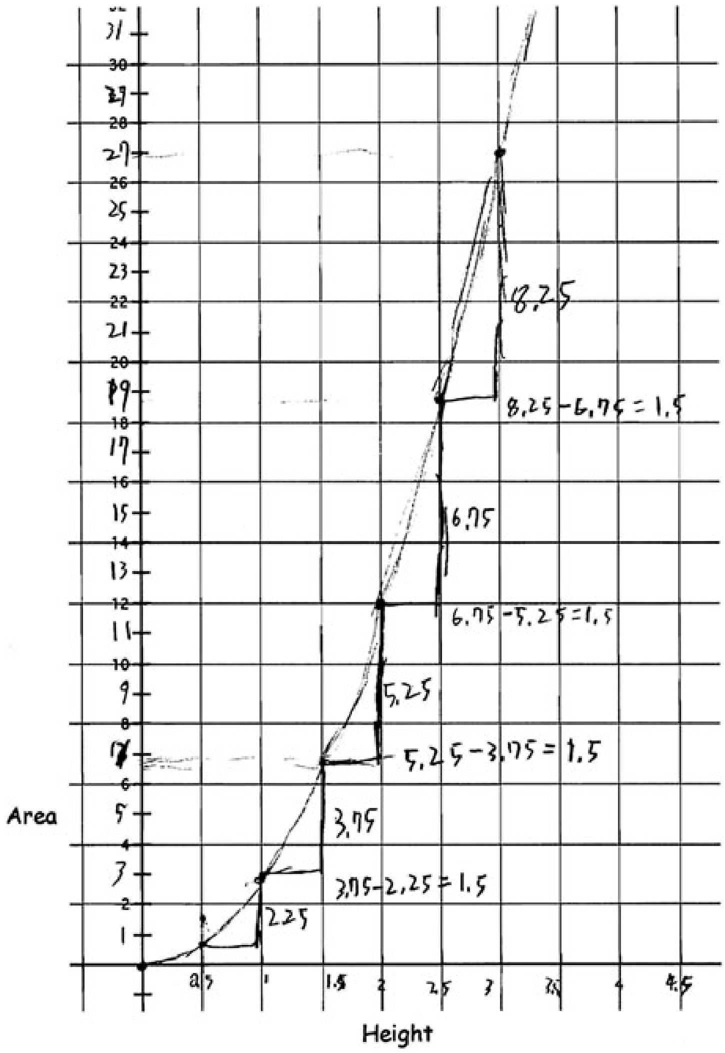
Timothy's understanding of the speed situation, his familiarity with creating same-ratio tables, and his ease with representing these phenomena algebraically all supported his ability to create and make quantitative sense of a linear graph. In addition, Timothy was able to imagine the scenario from a correspondence perspective ("Whatever x is, y is $4/5$ of x") as well as from a covariation perspective ("For every centimeter you add, you add $4/5$ seconds"). Each of these views, as well as Timothy's flexibility with moving across views, was enabled by his understanding of the relationship between the quantities centimeters and seconds to create the phenomenon of constant speed.

The quadratic functions students began to create graphs in the third week of the teaching experiment and ultimately graphed both $y = ax^2$ and $y = ax^2 + c$ situations. Before they produced any graphs, they made predictions about what a graph of the growing rectangle situation might look like:

AE: If you were to graph this one [comparing the height to the area of a square], what do you think the graph would look like?
B: A curve.
S: I thought it would be straight because every time the area's going up by 2.
AE: So what do you think about what Sara's saying? She's saying every time it would go up by 2 so it would be straight.
B: Well, the area's going up by 2 *in between* every time it's going up by a different number, so that makes me think it's going up in a curve because it's, like, staired.

When the students ultimately created graphs, they showed the first and second differences for the area in order to connect their prior emphasis on differences to the graphical representation, and to explain why the graph must be curved instead of straight. For instance, Daeshim's graph identified the constant second differences as 1.5 cm^2 when the height increased in 0.5 cm increments, and he showed this by calculating the increase in area for each 0.5-cm increase in the height, and then showing the difference between each successive area increase to be 1.5:

Fig. 16 Daeshim's graph of $A = 3h^2$



The students' quantitative understanding of the rectangle situation enabled them to make accurate predictions about the nature of graphs and interpret new graphs by thinking about the value of each point in relationship to a hypothetical rectangle. The students correctly predicted, for instance, that the parabola for $y = 5x^2$ would be narrower than the parabola for $y = 0.5x^2$, because the former represented a larger rectangle that was adding much more area with each height increase than the latter. They also made sense of graphs with non-zero y-intercepts by imagining rectangles with a constant number of extra square units tacked on. While students' later forays into features of graphs and families of functions will likely rely less on quantitative images, reasoning directly with the quantities can provide a critical sense-making foundation for their initial investigations of graphical representations.

In both the linear and the quadratic case, the students made use of different representations (tabular, algebraic, and graphical) to describe and make sense of the quantitative situations involving gear ratios, speed, or growing rectangles. Since each representation was a way of describing the quantitative phenomena, rather than an instructor-introduced artifact divorced from any referents, the connections across the representations were natural ones that enabled seamless transitions. Depending on the questions at hand, the students made use of the type of representation that they found most helpful for describing quantitative phenomena.

Fostering a Focus on Quantities

The situations with gear ratios and growing rectangles were optimal contexts for exploring linear and quadratic functions because the phenomena were precisely, rather than approximately, linear and quadratic. Some problem situations involve contexts in which the data are not exact; for instance, students may gather real-world quadratic data from rolling balls down inclined planes, or explore contrived problems presenting supposedly linear relationships between the number of surf boards sold and the temperature for a given day. The contrived nature of some contexts may interfere with students' natural sense making, and realistic situations with messy data may prevent students from directly manipulating quantities in order to form the necessary conceptual relationships that embody the phenomenon in question (Ellis 2007). While approximate or messy data are fully appropriate data to investigate, particularly in terms of highlighting the power of mathematical models for making sense of real-world situations, these contexts may not be ideal for middle school students who are exploring functional relationships for the first time.

Instead students will benefit from opportunities to explore the nature of linear (or quadratic) relationships by directly manipulating quantities: for instance, examining how changing time or distance independently affects the emergent quantity speed, creating two-number ratios and then iterating them and partitioning them to form equivalent ratios, and otherwise investigating how the constituent quantities affect the functional relationship at hand. The students in the linear and quadratic functions teaching experiments had opportunities to manipulate and explore physical artifacts (for the gears) or run experiments with computer software (for the speed situation and the growing rectangles situation). However, even in cases in which physical artifacts or computer simulations are not available, students can investigate how changing a particular quantity can affect the others related to it. Teachers may have to take care to support students' engagement with these problems, particularly because the tendency to extract numbers and focus on pattern-seeking activities appears to be a strong pull for middle-school students. In these cases an instructor's intervention can draw students' attention back to the quantitative referents of numbers and patterns. For instance, if a student describes a pattern in a table such as "each time x goes up by 4, y goes up by 5", a teacher could ask students to describe what this means in terms of the gears rotating.

Students' unique interactions with and interpretations of real-world situations remind us that these contexts are not a panacea. Introducing a quantitatively-rich situation does not guarantee that students will build quantitative relationships; a quantity is, after all, a person's conception of a measurable attribute, rather than the attribute itself. Students may focus on any number of features in a problem situation, and this focus may not always include productive relationships between quantities. Therefore teachers play an important role in shaping a classroom discussion, posing appropriate questions, inserting new information, and otherwise guiding students to develop the quantitative operations that will support the formation of functional relationships. A common refrain in the quadratic functions teaching experiment was "what does this mean in terms of the rectangle?" because this reminder encouraged the students to develop pattern generalizations that were meaningfully grounded rather than arbitrary and unproductive.

Students' initial learning of functions is particularly critical because it sets the foundation for future work in algebra at the high school level and beyond. Supporting students' abilities to make sense of functions from a quantitatively meaningful stance can foster a function understanding that is productive, grounded, and flexible in nature. A focus on numbers, relationships, and functional behaviors in absence of quantitative referents is certainly appropriate for mathematics students and, in the long term, necessary as students explore increasingly abstract ideas. However, I argue that for middle school students' first introduction to functional relationships, a grounding in quantities, relationships, and meaningful situations can ultimately support the eventual shift to more formal algebraic practices in high school.

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