Brief introduction to metamaterials

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1 Dispersion models of natural materials

In this report, we want to drawn an introduction to metamaterials. With the terms *metamaterials*, we intend an engineered material, with electromagnetic characteristics beyond to that of the natural materials. In order to understand the metamaterials and their behavior, we start with a brief description of the most common modesl for the permittivity and the permeability of the natural materials.

As it is well known, when the electromagnetic radiation interacts with a material, can be defined a polarization vector, $\underline{P}$, for the electric interaction, and a magnetization vector, $\underline{M}$, for the magnetic interaction. The permittivity and the permeability of a linear medium can be defined as follows:

$$\underline{D} = \epsilon_0 \underline{E} + \underline{P} = \epsilon_0 \underline{E} + \epsilon_0 \chi_e \underline{E} = \epsilon \underline{E} \tag{1.1}$$

$$\underline{B} = \mu_0 \underline{H} + \underline{M} = \mu_0 \underline{H} + \mu_0 \chi_m \underline{H} = \mu \underline{H} \tag{1.2}$$

where χ_e and χ_m are the electric and magnetic susceptibility, respectively. Therefore, to define the permittivity or the permeability of a medium, we have to find the linear relation between the electric field and the polarization, or the magnetic field and the magnetization, respectively.

There are many mathematical model to describe the dispersion of natural media. Here, we briefly discuss the following models:

- Lorentz model.
- Debye model.
- Drude model
- Magnetize ferrite model

1. Lorentz mode:

   The Lorentz model describes a non polar dielectric, i.e., a dielectric in which the molecules have not its own electric dipole. In this media the polarization is due to the deformation of the atoms. When an external electric field $\underline{E}$ is imposed, the positive and negative charges of each atom move away from each other and each atom gains its own electric dipole moment $\underline{p} = e\underline{d}$, where e is the elementary charge, and $\underline{d}$ is the oriented distance between the positive and negative charges. The forces involved in this process are: the external force $e\underline{E}$, the elastic force of the atom that tends to bring the charge in their original positions, $-k\underline{d}$, with k the equivalent elastic coefficient of the atom, and the frictional forces that hinder the motion of the charges, $-\beta\underline{v}$, whit $\underline{v}$ the velocity of the charges and β the viscous frictional coefficient. By applying the Newton's law, we can describe the Lorentz model with the following differential equation:

   $$m\frac{\mathrm{d}^2\underline{d}}{\mathrm{d}t^2} = -k\underline{d} - \beta\frac{\mathrm{d}\underline{d}}{\mathrm{d}t} + e\underline{E} \tag{1.3}$$

   reminding that the polarization is $\underline{P} = N\underline{p}$, we can multiply each side of the equation to Ne, obtaining:

   $$m\frac{\mathrm{d}^2\underline{P}}{\mathrm{d}t^2} + k\underline{P} + \beta\frac{\mathrm{d}\underline{P}}{\mathrm{d}t} = Ne^2\underline{E} \tag{1.4}$$

   the solution of this equation in the frequency domain gives the value of the electric susceptibility:

   $$\underline{P} = \chi_e \underline{E} = \frac{Ne^2}{m}\frac{1}{\omega_0^2 - \omega^2 + j\omega\alpha} \tag{1.5}$$

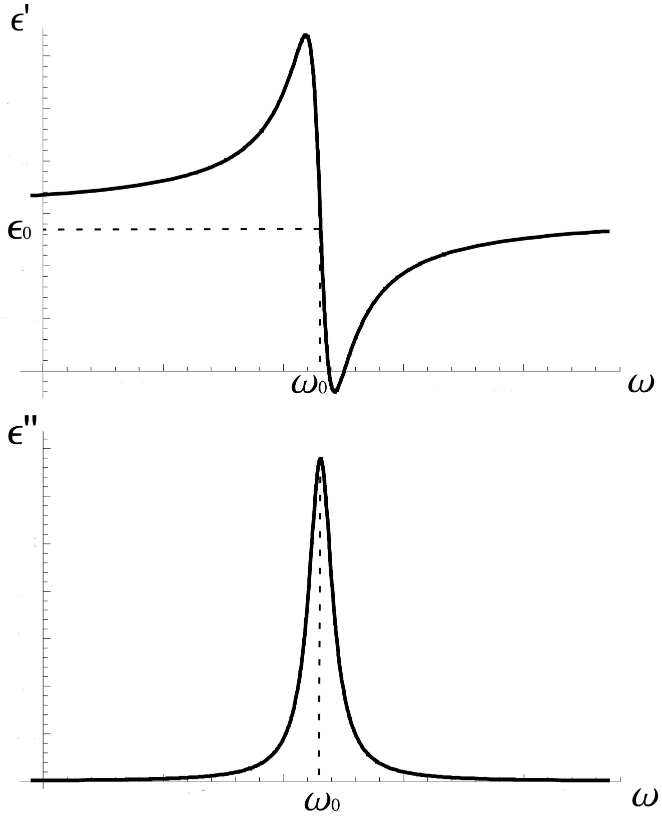

Figure 1: Real and imaginary parts of the Lorentz's permittivity.

where $\omega_0^2 = k/m$ and $\alpha = \beta/m$. From this expression the permittivity of the dielectric can be easily computed:

$$\underline{\epsilon} = \epsilon_0 + \frac{Ne^2}{m} \frac{1}{\omega_0^2 - \omega^2 + j\omega\alpha} \tag{1.6}$$

In Fig. 2, there are the qualitative behavior of the real and imaginary parts of this permittivity. As can be see, the imaginary part have the peculiar bell shape, with a peaked maximum in correspondence of the resonance frequency. On the other hand, the real part grows up to a maximum value in a neighborhood of the resonance frequency and after that it suddenly decreases to a minimum value and from it start to increase again and tends asymptotically to the permittivity of a vacuum. We want to remind that in the intervals where the permittivity is growing the frequency behavior is called *normal dispersion*, and in the intervals where the permittivity decreases are called of *anomalous dispersion*. In the case of negligible losses, it can be shown that the maximum tends to plus infinity and the minimum to minus infinity and the dispersion becomes normal at all frequencies.

2. Debye model:

We shown the Lorents model as first because the other models are simplification of it. The Debye model describe the behavior of a polar dielectric, where the polarization is due to the orientation of the electric dipoles by rotation. Because of the great complications in the mathematical construction of this model, we consider here only the final result. The permittivity of this model is the following:

$$\epsilon = \epsilon_0 + \frac{\Delta\epsilon}{1 + j\omega\tau} \tag{1.7}$$

where: $\Delta\epsilon$ is the difference between the permittivity in DC and the permittivity of a vacuum, and τ is the relaxation time constant, i.e., the reciprocal of the relaxation frequency. The behaviors of the real and imaginary parts of the permitticity in the Debye model are similar to that of the Lorentz model, where the real part does not present any maximum or minimum. It can be seen the typical relaxation effect, i.e., the

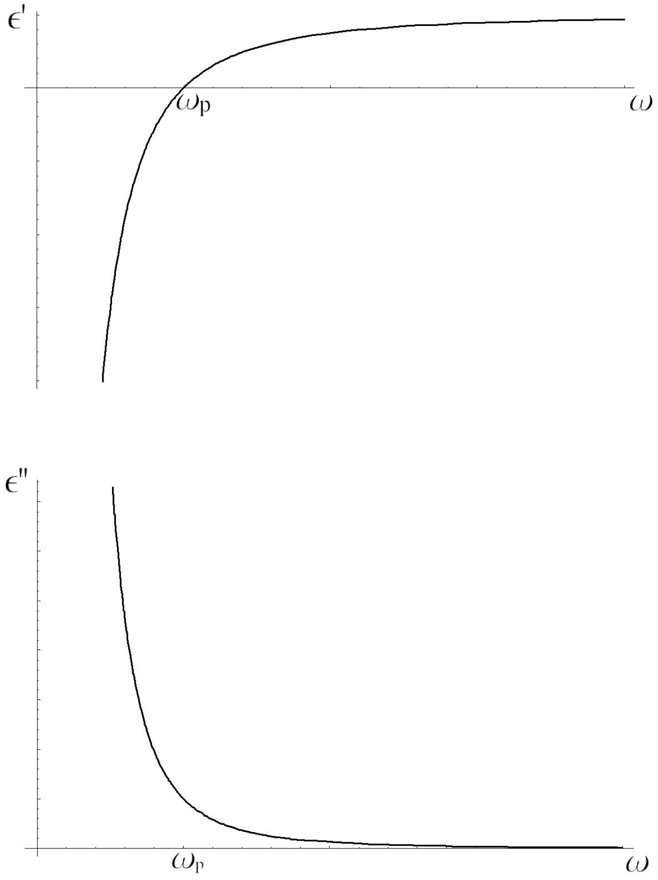

Figure 2: Real and imaginary parts of the Drude's permittivity.

decreasing of the real part of the permittivity at a certain frequency, related to a maximum of the imaginary part at the same frequency. Moreover, it must be said that the main difference between this model and the Lorentz's model is that the Debye permittivity presents always an anomalous dispersion at all frequencies.

3. Drude Model:

   The Drude model is a good mathematical model to describe the polarization of a material where there are free charges, e.g., a plasma or an electric conductor. In this case, we can consider the same differential equation of the Lorentz model, where is not present the elastic force, because the electrons are considered free in the material. In this case, the permittivity of the material is:

   $$\epsilon = \epsilon_0 \left(1 - \frac{\omega_p^2}{\omega^2 - j\omega\gamma} \right) \tag{1.8}$$

   where, ω_p is the plasma angular frequency, depending on the charge of the free particles, their mass and their volume density, while γ takes into account the losses in the material. In this case, the behavior of the real and imaginary parts of the permittivity is shown in Fig. **??**. As we can see, the real part of the permittivity goes to infinity in the static case. It is the behavior of the metal as perfect electric conductor. When the frequency approach the plasma frequency, the metal (or the plasma) behaves as a dielectric with negative permittivity, while for frequencies greater than the plasma frequency it behaves as a dielectric with a positive permittivity, but with relative permittivity less than one.

4. Magnetized ferrite model:

   These models concern to the permittivity of materials. In the same way, we could consider the magnetic properties of some natural materials. For example, the magnetic permeability of the magnetized ferrite, that

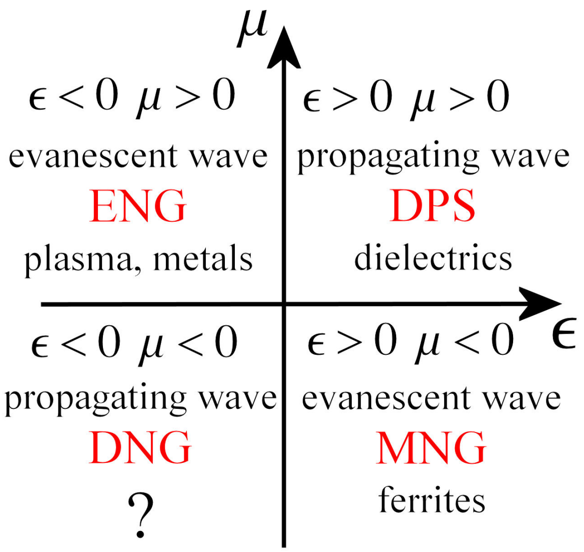

Figure 3: ϵ-μ diagram. It is a Cartesian plane where the abscissa represent the electric permittivity of a material and the ordinate its magnetic permeability. On this plane we can find all the possible isotropic materials. This diagram was firstly proposed by veselago in his paper on the DNG materials in 1968.

is a wide used material in the microwave components. The permeability of the magnetized ferrite, in a model where the losses are neglected, can be expressed as follows:

$$\underline{\underline{\mu}} = \mu_0 \begin{pmatrix} \mu_1 & -j\mu_2 & 0 \\ j\mu_2 & \mu_1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{1.9}$$

where:

$$\mu_1 = 1 + \frac{\omega_0^2 \rho}{\omega_0^2 - \omega^2} \tag{1.10}$$

$$\mu_2 = \frac{\omega \omega_0 \rho}{\omega_0^2 - \omega^2} \tag{1.11}$$

As we can see, the components of the permeability tensor are very similar to the expression of the permittivity in the Lorentz model.

2 The negative refractive index materials

The models of the electric permittivity and magnetic permeability show how these properties can present many different behaviors. Therefore, an interesting question can be: "How many different possibility do exist?". A possible answer is that, talking from a theoretical point of view, any combination of sign is possible. Therefore, both the permittivity and the permeability can be positive, we call these materials Double-Positive (DPS), both the constants can be negative, we call these materials Double-Negative (DNG), or only one of the two constants can be negative, we call these materials Epsilon-Negative (ENG) and Mu-Negative (MNG), respectively. In Fig. 3, a ϵ-μ diagram is shown. This diagram was proposed by Veselago in his paper on DNG materials in 1968. In the first quadrant we find the DPS materials, i.e., the ordinary isotropic dielectrics. In these materials, as it is well known, the propagation of electromagnetic waves is allowed, in fact, the refractive index of these materials is purely real[1]: $n = \sqrt{\mu_r \epsilon_r}$[2]. In the second quadrant, we find the ENG materials, as metals or plasmas below the plasma frequency. In these media, the refractive index is purely imaginary, $n = -j\sqrt{\mu_r |\epsilon_r|}$. Therefore, the propagation of waves is not allowed, and we can find only evanescent waves in these media. The same we can say about the fourth

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[1] It is well understood that we are neglecting the losses in this analysis.

[2] Here we are using the subscript "r" to indicate the relative permittivity and permeability.

quadrant. About the third quadrant, we find a very strange behavior, in fact in this case the refractive index of the material is again real. Therefore, the electromagnetic propagation is allowed in these materials. From this analysis, it is natural to ask what is the behavior of an electromagnetic wave in this material. In order to understand this behavior, we can consider the Maxwell equations:

$$\nabla \times \underline{E} = -j\omega\mu\underline{H} \tag{2.1}$$

$$\nabla \times \underline{H} = j\omega\epsilon\underline{E} \tag{2.2}$$

if we looking for a plane wave solution $\exp\left(-j\underline{k}\cdot\underline{r}\right)$, we can write the first equation as follows:

$$\underline{k} \times \underline{E} = \omega\mu\underline{H} \tag{2.3}$$

moreover, let us consider the Poynting vector in the frequency domain:

$$\underline{S} = \frac{1}{2}\underline{E} \times \underline{H}^* \tag{2.4}$$

From equation (2.3), we see that, in a DPS material, the three vectors $\underline{k}$, $\underline{E}$, and $\underline{H}$ are right-handed. Similarly, from equation (2.4), we see that the three vectors $\underline{S}$, $\underline{E}$, and $\underline{H}$ are right-handed. On the other hands, if we suppose that the medium is DNG, from equation (2.3), we find that the three vectors $\underline{k}$, $\underline{E}$, and $\underline{H}$ are left-handed, while the three vectors $\underline{S}$, $\underline{E}$, and $\underline{H}$ remain right-handed. Therefore, in a DNG material the propagation vector and the Poynting vector are anti-parallel, i.e., they lie on the same line, but they point in opposite directions. As we will see, this fact bring many consequences. First of all, we want to give a mathematical formulation of this fact. Let us call $\underline{s}_0$ the unit vector parallel to the Poynting vector:

$$\underline{S} = S\ \underline{s}_0 \tag{2.5}$$

where S is the magnitude of the Poynting vector. Than, we can always write the propagation vector as follows:

$$\underline{k} = n\ k_0\ \underline{s}_0 \tag{2.6}$$

where $k_0 = \omega\sqrt{\mu_0\epsilon_0}$ is the vacuum wave number, and n is the refractive index of the material. From this definition, we can see that in a DPS material the refractive index must be positive, while in a DNG material it must be negative. Therefore, we can define the refractive index as follows:

$$n = \begin{cases} \sqrt{\mu_r\epsilon_r} & \text{in a DPS material} \\ \\ -\sqrt{\mu_r\epsilon_r} & \text{in a DNG material} \end{cases} \tag{2.7}$$

In a way, it is as if the to cases of DPS and DNG materials correspond, in a mathematical point of view, to the two determinations of the square root.

Following to the previous discussion, we can understand the names used in the literature to refer to the DNG materials. One of them is "left-handed materials," to indicate that in these materials the three vectors $\underline{k}$, $\underline{E}$, and $\underline{H}$ are left-handed. Another name widely used is "Negative Refractive Index (NRI) materials", to indicate that in these materials the refractive index is negative. Another name used sometimes is "backward-wave materials", to indicate that in these materials the phase change in the opposite direction with respect to the energy.

The definition of a negative refractive index it is useful in order to understand the nature of a DNG material, but it is useful also in order to obtain some interesting behavior of these media. Let us make some examples.

Let us consider as a first example the Doppler effect. As it is well known, when an electromagnetic source generates an electromagnetic radiation at frequency ω_0, and it is in relative motion with respect to an observer, than the observer "sees" an electromagnetic wave at a frequency different from ω_0, in particular, he sees a frequency:

$$\omega = \omega_0\left(1 - n\ \frac{v}{c}\right) \tag{2.8}$$

where v is the relative velocity between the source and the observer, while c is the speed of light in a vacuum. In a DPS medium, we see that, if the observer is going away from the source, than the velocity is positive, and the observed frequency is less than the source frequency, i.e., the wavelength is larger, while if the observer is going toward the source, than the velocity is negative and the observed frequency is greater than the source frequency, i.e., the wavelength is smaller. Now in a DNG material, we find exactly the opposite behavior, in fact, being the refractive index negative, when the observer goes away from the source the observed wavelength is smaller than the

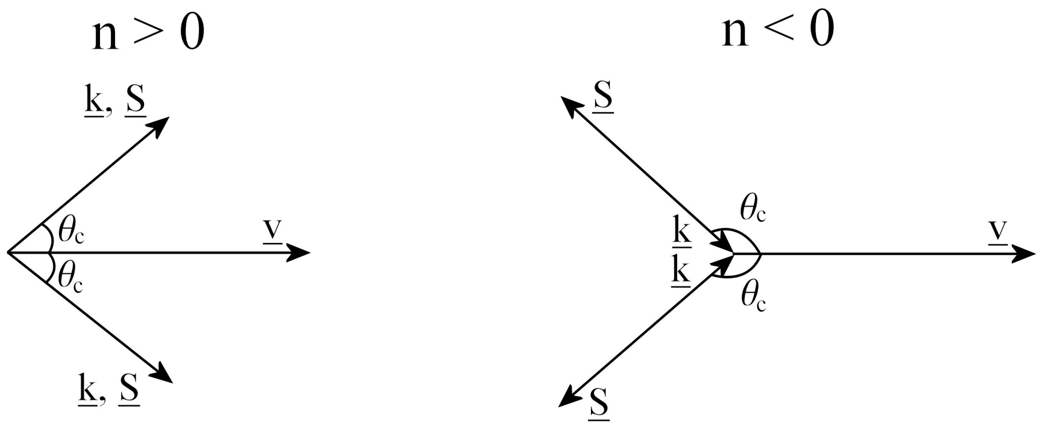

Figure 4: Cerenkov radiation in DPS and DNG materials.

source wavelength, while when the observer goes toward the source, the observed frequency is larger than the source wavelength. This behavior is amazing and counterintuitive. We can understand it if we think that the Doppler effect is a consequence of the relative motion between the observer and the radiation phase. Therefore, we have not to think to the relative motion with respect to the source, but with respect to the phase. Since the phase velocity, in the DPS and in the DNG materials has opposite velocities, it is obvious that the Doppler effect returns opposite behaviors in the two cases. Another surprising fact, is that we obtained this result with out any calculation. We just considered the formula of the Doppler effect, and we considered what happen when the refractive index becomes negative. For this reason it is very important to have found the model of negative refractive index to describe the DNG materials.

Another interesting example is related to the Cherenkov radiation. It is the electromagnetic radiation emitted by a charged particle that is moving in a material with a velocity v greater than the speed of light in the material, i.e., $v > c/n$. The radiation is emitted with an angle θ_C with respect to the direction of propagation of the particle, see Fig. 4. The expression of this angle is the following:

$$\cos\left(\theta_C\right) = \frac{c}{n\ v} \tag{2.9}$$

In a DPS material, we find always an angle $\theta_C \in [0, \pi/2]$. In a DNG, this angle is $\theta_C \in [\pi/2, \pi]$. it means that, the radiation is emitted backward with respect to the speed direction, see Fig. 4. It is important to note that, this time, we are talking about of the direction of both the phase and the energy. Of course, in the DNG, the phase and the energy remain anti-parallel.

Another well known effect that can be considered is the light refraction at the interface between two materials. If we consider an interface between a vacuum and a material with refractive index n, an electromagnetic wave that impinges on this interface with an angle θ_i, is refracted with an angle θ_t given by the Snell's law:

$$\sin\left(\theta_t\right) = \frac{1}{n} \sin\left(\theta_i\right) \tag{2.10}$$

from the Snell's law, we see that, if $n > 0$, the transmitted angle is between $\theta_t \in [0, \pi/2]$. On the other hand, we see that, if $n < 0$, it is between $\theta_t \in [-\pi/2, 0]$, i.e., we have the so called *negative refraction*, see Fig. 5.

The effect of the negative refraction is very important, and it has been one of the most surprising result in the research on the metamaterials. One of the most important application of this effect is the so called *perfect lens*. In optics, a lens is an instrument that focus the field emitted from a source S in an image point I, see Fig. 6. The emitted field by the source is not a plane wave, but it has a plane wave spectrum $V\left(k_z\right)$, where k_z is the component, parallel to the optic axis, of the propagation vector of the elementary plane wave of the spectrum. Therefore, the emitted field by the source is:

$$V\left(\underline{r}\right) = \frac{1}{2\pi} \int\limits_{-\infty}^{+\infty} V\left(k_z\right) e^{k_x x + k_y y + k_z z} dk_z \tag{2.11}$$

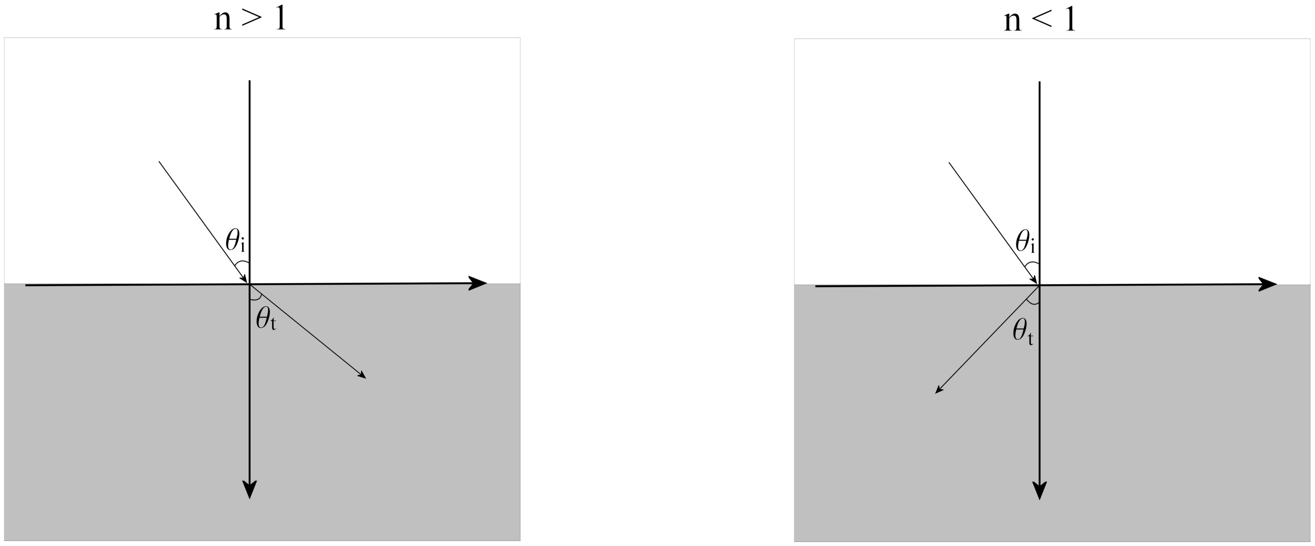

Figure 5: Refraction at the interface in the case of a DPS and a DNG material.

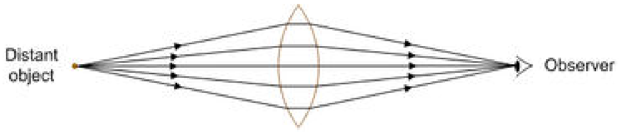

Figure 6: Schematic operation of a focusing optics lens.

for each elementary plane wave holds the dispersion equation: $k_x^2 + k_y^2 + k_z^2 = k^2$. Therefore, in order to have a propagating wave, i.e., $k_z \in \mathbb{R}$, it must be $k_t^2 = k_x^2 + k_y^2 < k^2$. Where k_t is the transverse propagation constant, and its interval of variation is connected to the interval of variation of the focused rays of the source field. This constrains is connected to the minimum observable size of the source object i.e., it is connected with the diffraction limit. A perfect lens, is a structure that allow us to focus the field emitted by a source without any diffraction limit. The perfect lens proposed in the literature is composed by a slab of a material with refractive index $n = -1$. Now, we want to obtain the reflection and transmission coefficients of such slab.

Let us consider a plane wave propagating in a vacuum, in the (x, z) plane, that impinges on an interface with a medium with $n = -1$, placed in $z = 0$. The medium has thickness t, and in $z = t$ presents another plane interface with a vacuum, see Fig. 7. The incident field is supposed TEM, with the following electric field:

$$\underline{E}_i = \underline{y}_0 E_0 e^{-jk_x x - jk_z z} \tag{2.12}$$

so the incidence is in E-polarization (p-polarized). The related magnetic field will be:

$$\underline{H}_i = \zeta_0 E_0 \underline{k}_0 \times \underline{y}_0 e^{-jk_x x - jk_z z} \tag{2.13}$$

where $\underline{k}_0$ is the unit vector of the propagation vector. Similarly, the fields transmitted in the slab will be the following:

$$\underline{E}_t = T \underline{y}_0 E_0 e^{-jk_x x - jk'_z z} \tag{2.14}$$

$$\underline{H}_t = T \zeta E_0 \underline{k}_0 \times \underline{y}_0 e^{-jk_x x - jk'_z z} \tag{2.15}$$

where ζ is the impedance of the slab, and with:

$$k_z = \sqrt{k_0^2 - k_x^2} \tag{2.16}$$

$$k'_z = \sqrt{n^2 k_0^2 - k_x^2} \tag{2.17}$$

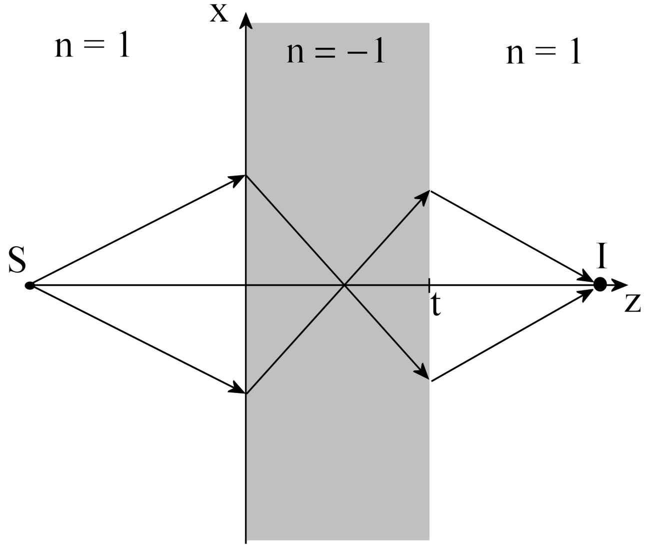

Figure 7: The field generated by an electromagnetic source S impinges on a slab of a material with refractive index $n = -1$. The radiation is totally focused on a image point I.

and where, R and T are the Fresnel coefficients:

$$R = \frac{\mu k_z - k'_z}{\mu k_z + k'_z} \tag{2.18}$$

$$T = \frac{2\mu k_z}{\mu k_z + k'_z} \tag{2.19}$$

while the Fresnel coefficients for the other interface, from the slab to a vacuum, are the following:

$$R' = \frac{k'_z - \mu k_z}{k'_z + \mu k_z} \tag{2.20}$$

$$T' = \frac{2k'_z}{k'_z + \mu k_z} \tag{2.21}$$

In order to obtain the total transmission coefficient of the slab, T_S, we have to consider the multiple reflections. We obtain the following expression:

$$T_S \; = T\;T' e^{-jk'_z t} \; + \; R'^2\;T\;T' e^{-3jk'_z t} \; + \; R'^4\;T\;T' e^{-5jk'_z t} + \; ... = T\;T' e^{-jk'_z t} \sum_{n=0}^{+\infty} R'^{2n} e^{-2njk'_z t} \tag{2.22}$$

We can see that the first element of the sum is the direct wave from the first interface to the second one, the second term is the wave reflected by the second interface, reflected again from the first interface, and transmitted by the second one, and go on. We recognize in this summation a geometric series, so the transmission coefficient can be written as follows:

$$T_S \; = \; \frac{T\;T' e^{-jk'_z t}}{1 - R'^2 e^{-2jk'_z t}} \tag{2.23}$$

by using the expression of the reflection and transmission coefficients, we obtain:

$$T_S \; = \frac{2\mu k_z}{\mu k_z + k'_z} \frac{2k'_z}{k'_z + \mu k_z} \frac{e^{-jk'_z t}}{1 - \left(\frac{k'_z - \mu k_z}{k'_z + \mu k_z}\right)^2 e^{-2jk'_z t}} = \frac{4\mu k_z k'_z e^{-jk'_z t}}{\left(k'_z + \mu k_z\right)^2 - \left(k'_z - \mu k_z\right) e^{-2jk'_z t}} \tag{2.24}$$

At this point, we have to consider the limit of this expression for $\epsilon, \mu \to -1$, in this case we find:

$$\lim_{\epsilon,\mu \to -1} k'_z = k_z \tag{2.25}$$

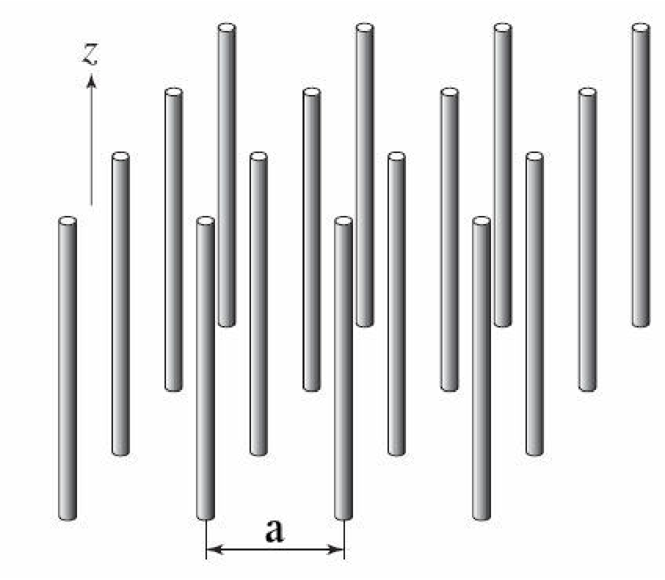

Figure 8: A sketch of a wire medium, the wires are aligned in $z-$direction and they are spaced of a distance a.

Therefore, the transmission coefficient becomes:

$$\lim_{\epsilon,\mu\to-1} T_S \;= e^{jk_zt} \tag{2.26}$$

We see that the transmission coefficient of the slab is only a phase factor, it means that the transmitted field has the same magnitude of the incident one, but it is phase shifted. We can calculate, following similar steps, that the reflection coefficient of the slab is identically zero. Moreover, we presented the result in E polarization, but, being $\epsilon = \mu$, it can be shown, for duality, that in H polarization the same result is obtained.

Let us consider again expression (2.26). We can see that the ordinary waves are shifted in phase, on the other hand, the evanescent waves, i.e., when $k_x^2 > k_0^2$ and $k_z = -j\sqrt{k_x^2 - k_0^2}$, are amplified by the slab. This amplification is the reason because the slab can focus an image exactly equal to the source, because the evanescent waves, that decay exponentially from the source, are amplified in the slab.

3 Realization of a metamaterial

We shown many interesting effects and applications of the left-handed materials. The question now is if it is possible to realize such materials! In the literature, especially in the last decade, many techniques have been proposed in order to obtain a DNG material with configurable properties. The possibilities are several and the topic is hard to be covered in its whole entirety. Here, we aim to draw the ideas behind some of these techniques.

One of the simplest way to achieve a negative permittivity with configurable properties is the so called *wire medium*. It is an array of conducting wires, see Fig. 8. The wires have a radius r, and they are arranged in a square array of side a. If we call σ the conductivity of the wires, their resistance for unit length can be written as follows:

$$R = \frac{1}{\pi r^2 \sigma} \tag{3.1}$$

About the effective inductance for unit length of the wires, it is a little bit more complicate to calculate, and we simply give the final result:

$$L = \frac{\mu_0}{2\pi} \ln\left(\frac{a}{r}\right) \tag{3.2}$$

At this point, we know that the relation between the voltage and the current, between unit length, in the unit cell is the following:

$$V = (R + j\omega L)\, I \tag{3.3}$$

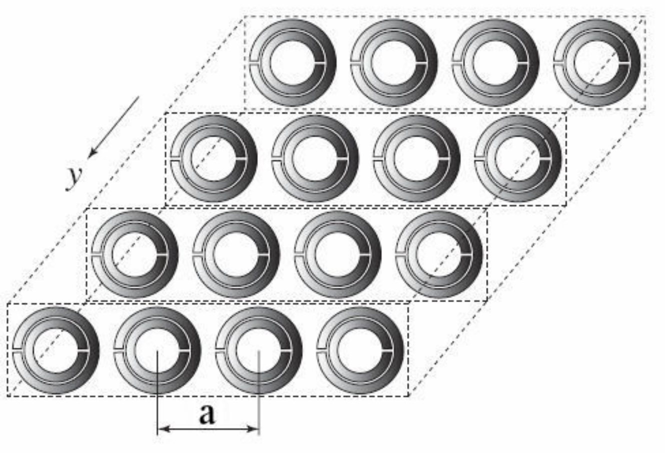

Figure 9: A sketch of a SRR medium, Split rings are aligned in $y-$direction and they are spaced of a distance a.

where V and I are the voltage and the current between unit length, respectively. The voltage per unit length is related to the electric field parallel to the wires E_z, on the other hand, the displacement current is related to the electric field as follows:

$$D_z = \left(\epsilon_0 - j\frac{\sigma}{\omega}\right) E_z = \epsilon_0 E_z + P_z \tag{3.4}$$

hence:

$$P_z = -j\frac{\sigma}{\omega} E_z = \frac{J_z}{j\omega} = \frac{I}{j\omega a^2} \tag{3.5}$$

where we defined the current in the unit cell as $I = J_z a^2$, in the hypothesis that the unit cell is much smaller than the wavelength, i.e., $a << \lambda_0$. Obtaining I from the last equation, and inserting in (3.3), we obtain:

$$E_z = j\omega a^2 \left(R + j\omega L\right) P_z \tag{3.6}$$

being:

$$P_z = \epsilon_0 \left(\epsilon_r - 1\right) E_z \tag{3.7}$$

we obtain:

$$\epsilon = \epsilon_0 + \frac{1}{a^2\left(-\omega^2 L + j\omega R\right)} = \epsilon_0 \left(1 - \frac{\omega_p^2}{\omega^2 - j\omega\alpha}\right) \tag{3.8}$$

with:

$$\omega_p^2 = \frac{1}{\epsilon_0 a^2 L} \tag{3.9}$$

$$\alpha = \frac{R}{L} \tag{3.10}$$

We recognize in (3.8) an effective permittivity as in the Drude model. In this expression, we can design, by varying the dimensions of the wires array, the plasma frequency, i.e., we can obtain negative permittivities at a very low frequency. The wire medium allow us to design an artificial medium with a configurable permittivity.

At this point, we must show how to obtain a medium with a configurable permeability. We consider the so called *Split Ring Resonator (SRR) medium*, see Fig. 9. A rigorous analysis of the SRR is not a simple task and it leaves aside the purposes of this report. Here, we give a very simple model of the SRR just to understand its principle of operation. The SRR is characterized by a resistance, due to the conductive losses, an inductance, due to the ring shape, and by a capacitance, due to the split on the ring, Therefore, the relation between the voltage and the current of the single ring is the following:

$$V = \left(R + j\omega L + \frac{1}{j\omega C} \right) I \tag{3.11}$$

The relation between the voltage and the magnetic field applied on the SRR, in the hypothesis of magnetic field orthogonal to the plane of the SRR, is the following:

$$V = \int_a \underline{E} \cdot d\underline{s} = \int_S \nabla \times \underline{E} \cdot d\underline{S} = -j\omega\mu_0 \int_S \underline{H} \cdot d\underline{S} = -j\omega\mu_0 S H_y \tag{3.12}$$

where S is the effective surface looking at the SRR as an elemental current loop. On the other hand, the magnetic dipole moment of the SRR between unit volume can be obtained as following:

$$M_y \tau = IS \tag{3.13}$$

where τ is the effective volume. As a consequence, we can obtain the relation between the magnetic dipole moment and the magnetic field:

$$M_y = \frac{-j\omega\mu_0 S^2}{\tau \left(R + j\omega L + \frac{1}{j\omega C} \right)} H_y = \frac{\omega^2 F}{\omega_0^2 - \omega^2 + j\omega\alpha} \tag{3.14}$$

with:

$$F = \frac{S^2 \mu_0}{L\tau} \tag{3.15}$$

$$\omega_0^2 = \frac{1}{LC} \tag{3.16}$$

$$\alpha = \frac{R}{L} \tag{3.17}$$

reminding the relation between the magnetic dipole moment and the magnetic field:

$$\underline{M} = (\mu_r - 1)\, \underline{H} \tag{3.18}$$

we obtain[3]:

$$\mu_r = 1 + \frac{\omega^2 F}{\omega_0^2 - \omega^2 + j\omega\alpha} \tag{3.20}$$

we found here an expression for the permeability very similar to the one of a magnetized ferrite. Again, with this medium we can obtain a configurable value of the permeability.

To obtain a DNG medium, we can make a double array of wires and SRRs, see Fig 10.

In the above presentation, we supposed that the electric field, in a wire medium, is parallel to the wires, while the magnetic field in a SRR medium is perpendicular to the plane where the SRRs lie. It means that such media behaves as we modeled us only for a fixed angle of incidence of the electromagnetic radiation. If, for example, we consider the DNG depicted in Fig. 10, its permittivity and permeability follow the dispersion rules in (3.8) and (3.20), respectively only if the incident plane wave has the electric field parallel to the wires, and the magnetic field orthogonal to the SRR, i.e., only if the propagation vector is parallel to the SRR and perpendicular to the wires. If the wave propagates in a different direction, the dispersion behaviors of the permittivity and permeability would be different, i.e., the material is anisotropic. This is a first disadvantage of the proposed structures: they are intrinsically anisotropic. Another important disadvantage of these structures is the high losses, in fact, both

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[3] A more accurate analysis returns the following expression:

$$\mu_r = 1 + \frac{\omega^2 \frac{\pi r^2}{a^2}}{\frac{3a}{\pi\mu_0\epsilon_0 \ln\left(\frac{2w}{\delta}\right) r^3} - \omega^2 + j\omega \frac{2a\sigma}{r\mu_0}} \tag{3.19}$$

, where a is the unit cell dimension, r is the internal radius of the SRR, σ is the conductivity of the sheets, w is the width of the sheets and δ is the distance between the two concentric rings.We can see that this expression has the same dependence on frequency of the one obtained in this report.

Figure 10: The realization of a DNG material at microwave frequencies with an array of wires and SRRs.

in the wire medium and in the SRR medium there is a lot of metal, it means there are many conducting currents, i.e., high Ohm losses. Finally, these structures are critical in the dispersion. To demonstrate this last point, let us consider the electric and magnetic energy density of the homogenized material. If we suppose that in a narrow band the material can be considered non-dispersive, the energy density would be the following:

$$W = \frac{1}{2}\left(\epsilon|E|^2 + \mu|H|^2\right) \tag{3.21}$$

we have to remind that the energy density is a quadratic quantity, i.e., it must be positive. Now, if we suppose that the material behaves as a DNG material, we can easily see that the energy density is surely negative, that is impossible. It means that for the DNG materials we cannot make the assumption of non dispersive materials! The left-handed materials must be always considered dispersive. The electric and magnetic energy density in a dispersive material can be written as follows:

$$W = \frac{1}{2}\left[\frac{d\left(\omega\epsilon\right)}{d\omega}|E|^2 + \frac{d\left(\omega\mu\right)}{d\omega}|H|^2\right] = \frac{1}{2}\left[\left(\epsilon|E|^2 + \mu|H|^2\right) + \left(\frac{d\epsilon}{d\omega}|E|^2 + \frac{d\mu}{d\omega}|H|^2\right)\right] \tag{3.22}$$

from this expression we see that the permittivity and the permeability can be simultaneously negative as long as their derivatives with respect to the frequency are sufficiently greater than zero, i.e., in a DNG we need a high dispersion! In other words, it is possible to obtain a left-handed material only with a high dispersion in frequency. The main consequence of this constrains is on the resonance behavior of the permittivity and permeability in frequency, in fact, a high dispersion is connected to a sudden change of the parameter in frequency, and a sudden change is usually connected to resonating behaviors.

Just a last consideration. We saw that the DNG materials are in general anisotropic, and we asserted that this is a disadvantage of the material. However, it is not always true. In fact, in many applications we can make use of the anisotropy in order to obtain particular effects. Moreover, the metamaterials are not only anisotropic, but they can be also bi-isotropic, or bi-anisotropic. With these terms, we refer to materials where the electric field is linearly related to the magnetic field and vice-versa:

$$\underline{D} = \underline{\underline{\epsilon}} \cdot \underline{E} + \underline{\underline{\xi}} \cdot \underline{H} \tag{3.23}$$

$$\underline{B} = \underline{\underline{\mu}} \cdot \underline{H} + \underline{\underline{\zeta}} \cdot \underline{E} \tag{3.24}$$

these constitutive relations describe a bi-anisotropic material[4]. Many materials has been proposed in the literature in order to obtain such relations, i.e., Tellegen medium or omega medium.

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[4] A bi-isotropic material respect the same relations, where $\underline{\underline{\epsilon}}$, $\underline{\underline{\mu}}$, $\underline{\underline{\xi}}$, and $\underline{\underline{\zeta}}$ are scalars.