PHYS 263

Exercises with solutions

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Part I

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Chapter 1

Vector relations

Show that:

$$\bigtriangledown \times (\bigtriangledown \times \mathbf{A}) = \bigtriangledown(\bigtriangledown \cdot \mathbf{A}) - \bigtriangledown^2 \mathbf{A} \tag{1.1}$$

Solution:

We consider the x component on each side of (1.1), and start with the left-hand side

$$\begin{aligned}
[\bigtriangledown \times (\bigtriangledown \times \mathbf{A})]_x = \left| \begin{array}{ccc} \hat{\mathbf{e}}_x & \hat{\mathbf{e}}_y & \hat{\mathbf{e}}_z \\ \bigtriangledown_x & \bigtriangledown_y & \bigtriangledown_z \\ (\bigtriangledown \times \mathbf{A})_x & (\bigtriangledown \times \mathbf{A})_y & (\bigtriangledown \times \mathbf{A})_z \end{array} \right|_x &= \bigtriangledown_y(\bigtriangledown \times \mathbf{A})_z - \bigtriangledown_z(\bigtriangledown \times \mathbf{A})_y \\
&= \frac{\partial}{\partial y}(\bigtriangledown \times \mathbf{A})_z - \frac{\partial}{\partial z}(\bigtriangledown \times \mathbf{A})_y
\end{aligned} \tag{1.2}$$

Considering the z component of $(\bigtriangledown \times \mathbf{A})$, we have

$$(\bigtriangledown \times \mathbf{A})_z = \left| \begin{array}{ccc} \hat{\mathbf{e}}_x & \hat{\mathbf{e}}_y & \hat{\mathbf{e}}_z \\ \bigtriangledown_x & \bigtriangledown_y & \bigtriangledown_z \\ A_x & A_y & A_z \end{array} \right|_z = \bigtriangledown_x A_y - \bigtriangledown_y A_x = \frac{\partial}{\partial x} A_y - \frac{\partial}{\partial y} A_x \tag{1.3}$$

Similarly, we get for the y component of $(\bigtriangledown \times \mathbf{A})$:

$$(\bigtriangledown \times \mathbf{A})_y = \bigtriangledown_z A_x - \bigtriangledown_x A_z = \frac{\partial}{\partial z} A_x - \frac{\partial}{\partial x} A_z \tag{1.4}$$

We substitute these two expressions into (1.2) and obtain for the x component on the left-hand side of (1.1):

$$\begin{aligned}
[\bigtriangledown \times (\bigtriangledown \times \mathbf{A})]_x &= \frac{\partial}{\partial y}[\frac{\partial}{\partial x} A_y - \frac{\partial}{\partial y} A_x] - \frac{\partial}{\partial z}[\frac{\partial}{\partial z} A_x - \frac{\partial}{\partial x} A_z] \\
&= \frac{\partial^2 A_y}{\partial x \partial y} - \frac{\partial^2 A_x}{\partial y^2} - \frac{\partial^2 A_x}{\partial z^2} + \frac{\partial^2 A_z}{\partial x \partial z}
\end{aligned} \tag{1.5}$$

For the right-hand side of (1.1) we have:

$$\begin{aligned}
[\bigtriangledown(\bigtriangledown \cdot \mathbf{A}) - \bigtriangledown^2 \mathbf{A}]_x &= \bigtriangledown_x(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}) - (\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2})A_x \\
&= \frac{\partial^2 A_x}{\partial x^2} + \frac{\partial^2 A_y}{\partial x \partial y} + \frac{\partial^2 A_z}{\partial x \partial z} - \frac{\partial^2 A_x}{\partial x^2} - \frac{\partial^2 A_x}{\partial y^2} - \frac{\partial^2 A_x}{\partial z^2} \\
&= \frac{\partial^2 A_y}{\partial x \partial y} + \frac{\partial^2 A_z}{\partial x \partial z} - \frac{\partial^2 A_x}{\partial y^2} - \frac{\partial^2 A_x}{\partial z^2} \\
&= [\bigtriangledown \times (\bigtriangledown \times \mathbf{A})]_x \qquad (1.6)
\end{aligned}$$

Thus, as far the x component of (1.1) is concerned, the right-hand side is equal to the left-hand side. The proof for the y or z component of (1.1) follows in a similar manner.

Chapter 2

Variable change in the wave equation

In chapter 4.1 in the lecture notes [?] the equation

$$\frac{\partial^2 V}{\partial p \partial q} = 0 \tag{2.1}$$

constitutes an important part of the derivation of a general solution of the wave equation.

Show that

$$\frac{\partial^2 V}{\partial \zeta^2} - \frac{1}{v^2}\frac{\partial^2 V}{\partial t^2} = 0 \tag{2.2}$$

can be written in the form:

$$\frac{\partial^2 V}{\partial p \partial q} = 0 \tag{2.3}$$

where $p = \zeta - vt$ and $q = \zeta + vt$.

Solution:

$$p = \zeta - vt \quad ; \quad q = \zeta + vt. \tag{2.4}$$

$$\begin{aligned} \frac{\partial p}{\partial \zeta} = \frac{\partial q}{\partial \zeta} &= 1 \\ \frac{\partial p}{\partial t} &= -v \\ \frac{\partial q}{\partial t} &= v \end{aligned} \tag{2.5}$$

The first term in 2.2 becomes:

$$\begin{aligned}\frac{\partial V}{\partial \zeta} &= \frac{\partial V}{\partial p}\frac{\partial p}{\partial \zeta} + \frac{\partial V}{\partial q}\frac{\partial q}{\partial \zeta} = \frac{\partial V}{\partial p} + \frac{\partial V}{\partial q} \\ \frac{\partial^2 V}{\partial \zeta^2} &= \frac{\partial}{\partial p}\left[\frac{\partial V}{\partial p} + \frac{\partial V}{\partial q}\right]\frac{\partial p}{\partial \zeta} + \frac{\partial}{\partial q}\left[\frac{\partial V}{\partial p} + \frac{\partial V}{\partial q}\right]\frac{\partial q}{\partial \zeta} = \frac{\partial^2 V}{\partial p^2} + \frac{\partial^2 V}{\partial q^2} + 2\frac{\partial^2 V}{\partial p \partial q} \end{aligned} \tag{2.6}$$

The second term in 2.2 becomes:

$$\begin{aligned}\frac{\partial V}{\partial t} &= \frac{\partial V}{\partial p}\frac{\partial p}{\partial t} + \frac{\partial V}{\partial q}\frac{\partial q}{\partial t} = -v\left[\frac{\partial V}{\partial p} - \frac{\partial V}{\partial q}\right] \\ \frac{\partial^2 V}{\partial t^2} &= -v\left\{\frac{\partial}{\partial p}\left[\frac{\partial V}{\partial p} - \frac{\partial V}{\partial q}\right]\frac{\partial p}{\partial t} + \frac{\partial}{\partial q}\left[\frac{\partial V}{\partial p} - \frac{\partial V}{\partial q}\right]\frac{\partial q}{\partial t}\right\} \\ &= -v\left\{\frac{\partial}{\partial p}\left[\frac{\partial V}{\partial p} - \frac{\partial V}{\partial q}\right](-v) + \frac{\partial}{\partial q}\left[\frac{\partial V}{\partial p} - \frac{\partial V}{\partial q}\right]v\right\} \\ \Rightarrow -\frac{1}{v^2}\frac{\partial^2 V}{\partial t^2} &= -\frac{\partial^2 V}{\partial p^2} - \frac{\partial^2 V}{\partial q^2} + 2\frac{\partial^2 V}{\partial p \partial q} \end{aligned} \tag{2.7}$$

Adding 2.6 and 2.7, we get:

$$\frac{\partial^2 V}{\partial \zeta^2} - \frac{1}{v^2}\frac{\partial^2 V}{\partial t^2} = 4\frac{\partial^2 V}{\partial p \partial q} = 0 \tag{2.8}$$

which was to be proven.

Chapter 3

Laplacian operator for spherically symmetric functions

Show that the Laplacian operator has the following form in spherical co-ordinates for a function with spherical symmetry (this form is used in section 4.2 in the lecture notes [?]):

$$\bigtriangledown^2 f(r) = \frac{1}{r}\frac{\partial^2}{\partial r^2}[r f(r)] \tag{3.1}$$

where

$$r = \sqrt{x^2 + y^2 + z^2}. \tag{3.2}$$

Solution:

Consider first the left-hand side of (3.1), from which we have:

$$\bigtriangledown^2 f(r) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) f(r) \tag{3.3}$$

Considering the first term in (3.3), we have:

$$\frac{\partial}{\partial x} f = \frac{\partial f}{\partial r}\frac{\partial r}{\partial x} = \frac{x}{r}\frac{\partial f}{\partial r} \tag{3.4}$$

where we have used that

$$\frac{\partial r}{\partial x} = \frac{\partial}{\partial x}\sqrt{x^2 + y^2 + z^2} = \frac{2x}{2\sqrt{x^2 + y^2 + z^2}} = \frac{x}{r}. \tag{3.5}$$

Differentiating once more, we have

$$\begin{aligned}\frac{\partial^2}{\partial x^2} f &= \frac{\partial}{\partial x}\left[x \cdot \frac{1}{r}\frac{\partial f}{\partial r}\right] \\ &= \frac{1}{r}\frac{\partial f}{\partial r} + x\frac{\partial}{\partial x}\left[\frac{1}{r}\frac{\partial f}{\partial r}\right]\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{r}\frac{\partial f}{\partial r} + x\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial f}{\partial r}\right]\cdot\frac{\partial r}{\partial x} \\
&= \frac{1}{r}\frac{\partial f}{\partial r} + \frac{x^2}{r}\left[-\frac{1}{r^2}\frac{\partial f}{\partial r} + \frac{1}{r}\frac{\partial^2 f}{\partial r^2}\right] \\
&= \frac{1}{r}\frac{\partial f}{\partial r} + \frac{x^2}{r^2}\left[\frac{\partial^2 f}{\partial r^2} - \frac{1}{r}\frac{\partial f}{\partial r}\right]. \qquad (3.6)
\end{aligned}$$

In a similar manner we find that

$$\frac{\partial^2}{\partial y^2}f = \frac{1}{r}\frac{\partial f}{\partial r} + \frac{y^2}{r^2}\left[\frac{\partial^2 f}{\partial r^2} - \frac{1}{r}\frac{\partial f}{\partial r}\right] \qquad (3.7)$$

$$\frac{\partial^2}{\partial z^2}f = \frac{1}{r}\frac{\partial f}{\partial r} + \frac{z^2}{r^2}\left[\frac{\partial^2 f}{\partial r^2} - \frac{1}{r}\frac{\partial f}{\partial r}\right]. \qquad (3.8)$$

By adding (3.6), (3.7) and (3.8), we obtain:

$$\begin{aligned}
\bigtriangledown^2 f &= \frac{3}{r}\frac{\partial f}{\partial r} + \frac{x^2+y^2+z^2}{r^2}\left[\frac{\partial^2 f}{\partial r^2} - \frac{1}{r}\frac{\partial f}{\partial r}\right] \\
&= \frac{2}{r}\frac{\partial f}{\partial r} + \frac{\partial^2 f}{\partial r^2}. \qquad (3.9)
\end{aligned}$$

Consider next the right-hand side of (3.1), from which we have:

$$\begin{aligned}
\frac{1}{r}\frac{\partial}{\partial r}(rf) &= \frac{1}{r}\left[f + r\frac{\partial f}{\partial r}\right] \\
\frac{1}{r}\frac{\partial^2}{\partial r^2}(rf) &= \frac{1}{r}\frac{\partial}{\partial r}\left[f + r\frac{\partial f}{\partial r}\right] \\
&= \frac{1}{r}\left[\frac{\partial f}{\partial r} + \frac{\partial f}{\partial r} + r\frac{\partial^2 f}{\partial r^2}\right] \\
&= \frac{2}{r}\frac{\partial f}{\partial r} + \frac{\partial^2 f}{\partial r^2}. \qquad (3.10)
\end{aligned}$$

Comparing (3.9) with (3.10), we see that the left-hand side of (**??**) is equal to the right-hand side, which was to be proven.

Chapter 4

Waves and wave packets

4.1 Superposition of two harmonic plane waves

Two harmonic, plane waves of equal amplitudes propagate in the positive z direction. One of the waves has angular frequency ω_1, wave number, k_1, and phase constant δ_1, and the other has angular frequency ω_2, wave number, k_2, and phase constant δ_2.

Find a formula for the sum of the two waves expressed in terms of the

- Average frequency: $\overline{\omega} = \frac{1}{2}(\omega_1 + \omega_2)$
- Average wave number: $\overline{k} = \frac{1}{2}(k_1 + k_2)$
- Average phase: $\overline{\delta} = \frac{1}{2}(\delta_1 + \delta_2)$
- Difference frequency: $\triangle\omega = \omega_1 - \omega_2$
- Difference wave number: $\triangle k = k_1 - k_2$, and
- Difference phase: $\triangle\delta = \delta_1 - \delta_2$.

Solution:

$$V_1 = a\cos\left(k_1 z - \omega_1 t + \delta_1\right) \tag{4.1}$$

$$V_2 = a\cos\left(k_2 z - \omega_2 t + \delta_2\right) \tag{4.2}$$

$$V_R = V_1 + V_2 = a\left[\cos(k_1 z - \omega_1 t + \delta_1) + \cos\left(k_2 z - \omega_2 t + \delta_2\right)\right]. \tag{4.3}$$

We use the following identity

$$\cos x + \cos y = 2\cos\frac{x+y}{2}\cos\frac{x-y}{2} \tag{4.4}$$

where we let

$$\begin{aligned} x &= k_1 z - \omega_1 t + \delta_1 \\ y &= k_2 z - \omega_2 t + \delta_2 \end{aligned} \tag{4.5}$$

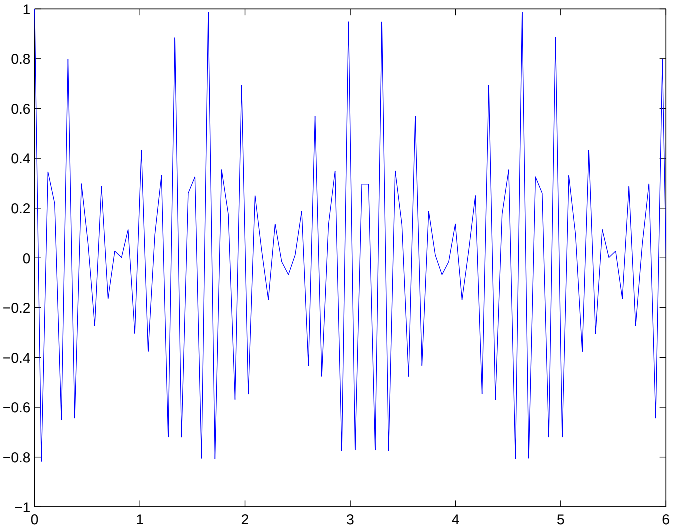

Figure 4.1: The amplitude $A(z,t)$ in (4.7) varies slowly compared with the factor $\cos(\overline{k}z - \overline{\omega}t + \overline{\delta})$.

so that

$$\begin{aligned}
\frac{1}{2}(x+y) &= \frac{1}{2}(k_1+k_2)z - \frac{1}{2}(\omega_1+\omega_2)t + \frac{1}{2}(\delta_1+\delta_2) \\
&= \overline{k}z + \overline{\omega}t + \overline{\delta} \\
\frac{1}{2}(x-y) &= \frac{1}{2}(k_1-k_2)z - \frac{1}{2}(\omega_1-\omega_2)t + \frac{1}{2}(\delta_1-\delta_2) \\
&= \frac{1}{-}2(\triangle kz + \triangle\omega t + \triangle\delta) \\
\Rightarrow V_R &= 2a\cos(\overline{k}z - \overline{\omega}t + \overline{\delta})\cos\frac{1}{2}(\triangle kz - \triangle\omega t + \triangle\delta). \qquad (4.6)
\end{aligned}$$

4.2 Group velocity and phase velocity

Show that when $\Delta\omega/\overline{\omega} << 1$, the sum V_R of the two waves in (4.6) can be interpreted as a harmonic plane wave that propagates in the positive z direction with phase velocity $v = \overline{\omega}/\overline{k}$ and with a slowly varying amplitude that propagates in the positive z direction with group velocity $v_g = \Delta\omega/\Delta k$. Sketch the sum of the two waves.

Solution:

The expression in (4.6) can be written:

$$V_R(z,t) = A(z,t)\cos(\overline{k}z - \overline{\omega}t + \overline{\delta}) \qquad (4.7)$$

where

$$A(z,t) = 2a\cos[\frac{1}{2}(\Delta kz - \Delta\omega t + \Delta\delta)]. \qquad (4.8)$$

When $\Delta\omega << \overline{\omega}$, the amplitude $A(z,t)$ will vary slowly comapred with $\cos(\overline{k}z - \overline{\omega}t + \overline{\delta})$, as shown in Fig. 4.1.

We find the phase velocity for the sum of the two waves, which has period $\overline{T} = 2\pi/\overline{\omega}$ and wavelenghth $\overline{\lambda} = 2\pi/\overline{k}$, by letting the phase be equal to a constant (since we consider a wave front) and differentiating with respect to time:

$$\begin{aligned} \overline{k}z(t) - \overline{\omega}t + \overline{\delta} &= \text{constant} \\ &\Downarrow \\ \overline{k}z'(t) - \overline{\omega} &= 0 \\ &\Downarrow \\ v = z'(t) = \frac{\overline{\omega}}{\overline{k}}. \end{aligned} \tag{4.9}$$

We find the propagation velocity of the amplitude $A(z,t)$ or the group velocity in the same manner. The result is:

$$v_g = \frac{\Delta\omega}{\Delta k}. \tag{4.10}$$

4.3 Dispersion and energy propagation

Show from the results in Exercise 4.2 that

1. $v_g \neq v$ in a *dispersive* medium.
2. $v_g = v$ in a *non-dispersive* medium.

Solution:

In a dispersive meduim k as a function of ω is given by:

$$k(\omega) = \frac{\omega}{v(\omega)} = \frac{\omega}{c} \cdot \frac{c}{v(\omega)} = \frac{\omega}{c} n(\omega) \tag{4.11}$$

$$\Rightarrow k_1 = k(\omega_1) = \frac{\omega_1}{c} n(\omega_1) \tag{4.12}$$

$$k_2 = k(\omega_2) = \frac{\omega_2}{c} n(\omega_2) \tag{4.13}$$

$$\Rightarrow \overline{k} = \frac{1}{2}(k_1 + k_2) = \frac{1}{2c}[\omega_1 n(\omega_1) + \omega_2 n(\omega_2)] \tag{4.14}$$

$$\Rightarrow \Delta k = k_1 - k_2 = \frac{1}{c}[\omega_1 n(\omega_1) - \omega_2 n(\omega_2)] \tag{4.15}$$

We find the phase velocity and the group velocity from (4.9) and (4.10):

$$v = \frac{\overline{\omega}}{\overline{k}} = \frac{c[\omega_1 + \omega_2]}{\omega_1 n(\omega_1) + \omega_2 n(\omega_2)} \tag{4.16}$$

$$v_g = \frac{\Delta\omega}{\Delta k} = \frac{c[\omega_1 - \omega_2]}{\omega_1 n(\omega_1) - \omega_2 n(\omega_2)}. \tag{4.17}$$

From these two results we see that:

1. $v_g \neq v$ when $n(\omega_1) \neq n(\omega_2)$, i.e. in a *dispersive* medium.
2. In a non-dispersive medium $n(\omega_1) = n(\omega_2) = n$, so that

$$v_g = \frac{c}{n} = v. \tag{4.18}$$

Chapter 5

Connection between group velocity, phase velocity, and refractive index

5.1 Group velocity as a function of phase velocity and refractive index

Show that the group velocity v_g can be expressed in terms of the phase velocity v and the refractive index $n(\omega)$ in the following manner:

$$\frac{1}{v_g} = \frac{1}{v} + \frac{\omega}{c}\frac{dn(\omega)}{d\omega}. \tag{5.1}$$

Solution:

$$v_g = \frac{d\omega}{dk} \Rightarrow \frac{1}{v_g} = \frac{dk}{d\omega} \tag{5.2}$$

$$\begin{aligned} k &= \frac{\omega}{v} = \frac{\omega}{c}\frac{c}{v} = \frac{\omega}{c}n = \frac{\omega}{c}n(\omega) \\ \Rightarrow \frac{1}{v_g} &= \frac{d}{d\omega}(\frac{\omega}{c}n(\omega)) = \frac{n(\omega)}{c} + \frac{\omega}{c}\frac{dn}{d\omega} \\ \Rightarrow \frac{1}{v_g} &= \frac{1}{v} + \frac{\omega}{c}\frac{dn}{d\omega}. \end{aligned} \tag{5.3}$$

5.2 The group velocity is less than the speed of light in vacuum

Given that

$$n = \sqrt{1 - \frac{B}{\omega^2 - {\omega_0}^2}} \quad ; \quad B > 0 \tag{5.4}$$

show that $v_g < c$ when $\omega < \omega_0$.

Solution:

$$n^2 = 1 - \frac{B}{\omega^2 - {\omega_0}^2} \qquad ; \qquad B > 0 \tag{5.5}$$

$$\Rightarrow 2n\frac{dn}{d\omega} = \frac{B \cdot 2\omega}{(\omega^2 - {\omega_0}^2)^2}$$

$$\Rightarrow \frac{dn}{d\omega} = \frac{B\omega}{n(\omega^2 - {\omega_0}^2)^2}. \tag{5.6}$$

We substitute this result in (5.3) to obtain

$$\Rightarrow \frac{1}{v_g} = \frac{1}{v} + \frac{\omega}{c}\frac{B\omega}{n(\omega^2 - {\omega_0}^2)^2}$$

$$\Rightarrow \frac{c}{v_g} = n + \frac{B\omega^2}{n(\omega^2 - {\omega_0}^2)^2}. \tag{5.7}$$

Because $B > 0$, we see that $\frac{c}{v_g} > n$. But when $\omega < \omega_0$, it follows from (5.5) that $n > 1$. Thus, we have $\frac{c}{v_g} > 1$ or $v_g < c$, which was to be proven.

Chapter 6

Propagation in a dispersive medium

Consider a polychromatic plane wave that propagates in the z direction in a dispersive medium, so that

$$u(z,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{u}(0,\omega)e^{i[k(\omega)z-\omega t]}d\omega. \tag{6.1}$$

It follows from (6.1) that in the plane $z = 0$

$$u(0,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{u}(0,\omega)e^{-i\omega t}d\omega \tag{6.2}$$

so that

$$\tilde{u}(0,\omega) = \int_{-\infty}^{\infty} u(0,t)e^{i\omega t}dt \tag{6.3}$$

where $\tilde{u}(0,\omega)$ is the frequency spectrum of the plane wave in the plane $z = 0$. Let the frequency spectrum $\tilde{u}(0,\omega)$ have its maximum value at $\omega = \omega_0$, and let $\tilde{u}(0,\omega)$ fall off rapidly from this value, so that we may represent $k(\omega)$ by the first two terms in a Taylor series around ω_0:

$$k(\omega) = k(\omega_0) + \left.\frac{dk}{d\omega}\right|_{\omega=\omega_0}(\omega-\omega_0). \tag{6.4}$$

6.1 Propagation of a quasi-monochromatic wave

Show that when we neglect terms of higher order than those retained in (6.4), we can express $u(z,t)$ in (6.1) as follows:

$$u(z,t) \approx e^{i\omega_0 z\left(\frac{1}{v_0}-\frac{1}{v_{g0}}\right)}u\left(0,-\frac{z}{v_{g0}}+t\right) \tag{6.5}$$

where

$$v_0 = \frac{\omega_0}{k(\omega_0)} \qquad ; \qquad v_{g0} = \left.\frac{d\omega}{dk}\right|_{\omega=\omega_0}. \tag{6.6}$$

Solution:

By using the information given in the exercise, we may write:

$$k(\omega) = k(\omega_0) + \left(\frac{1}{\frac{d\omega}{dk}}\right)\Bigg|_{\omega=\omega_0} (\omega - \omega_0) = \frac{\omega_0}{v_0} - \frac{\omega_0}{v_{g0}} + \frac{\omega}{v_{g0}} \tag{6.7}$$

so that

$$k(\omega)z - \omega t \approx \omega_0 z\left(\frac{1}{v_0} - \frac{1}{v_{g0}}\right) + \omega\left(\frac{z}{v_{g0}} - t\right) \tag{6.8}$$

which upon substitution in (6.1) gives:

$$u(z,t) \approx e^{i\omega_0 z\left(\frac{1}{v_0} - \frac{1}{v_{g0}}\right)} \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{u}(0,\omega) e^{-i\omega\left(-\frac{z}{v_{g0}}+t\right)} d\omega. \tag{6.9}$$

Because of the Fourier transform relation given in (6.2), (6.9) becomes

$$u(z,t) \approx e^{i\omega_0 z\left(\frac{1}{v_0} - \frac{1}{v_{g0}}\right)} u\left(0, -\frac{z}{v_{g0}} + t\right). \tag{6.10}$$

6.2 The shape and speed of the wave

Give a physical interpretation of the result in (6.10).

Solution:

v_0 and v_{g0} are the phase velocity and group velocity, respectively, at angular frequency ω_0. The result in (6.10) shows that when $\tilde{u}(0,\omega)$ has its maximum value at ω_0 and falls off rapidly from its maximum value of $\tilde{u}(0,\omega_0)$, then, except for a phase factor, $u(t)$ will <u>not</u> change its form, and $u(t)$ will propagate at the group velocity.

6.3 Alternative way to proceed

Show that the same result as in (6.10) can be obtained by considering

$$u(z,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{u}(0,\omega) e^{i\frac{z}{c}f(\omega)} d\omega \tag{6.11}$$

where

$$f(\omega) = \omega[n(\omega) - \theta] \qquad ; \qquad \theta = \frac{ct}{z} \tag{6.12}$$

and expanding $f(\omega)$ in a Taylor series around ω_0 to the first order, i.e.

$$f(\omega) = f(\omega_0) + f'(\omega_0)(\omega - \omega_0). \tag{6.13}$$

Solution:

$$
\begin{aligned}
f(\omega) &= \omega[n(\omega)-\theta] \qquad ; \qquad \theta = \frac{ct}{z} \\
&\Downarrow \\
f'(\omega) &= n(\omega) - \theta + \omega \frac{dn}{d\omega} \\
&= n(\omega) - \theta + \omega n'(\omega)
\end{aligned} \tag{6.14}
$$

Thus, we have

$$
f'(\omega_0) = n(\omega_0) - \theta + \omega_0 n'(\omega_0) \tag{6.15}
$$

so that (6.13) gives

$$
\begin{aligned}
f(\omega) &= f(\omega_0) + f'(\omega_0)(\omega - \omega_0) \\
&= \omega_0[n(\omega_0) - \theta] + [n(\omega_0) - \theta + \omega_0 n'(\omega_0)](\omega - \omega_0) \\
&= -\omega_0^2 n'(\omega_0) + \omega[n(\omega_0) - \theta + \omega_0 n'(\omega_0)].
\end{aligned} \tag{6.16}
$$

Further, we have:

$$
\begin{aligned}
v_g = \frac{d\omega}{dk} \Rightarrow \frac{1}{v_g} &= \frac{dk}{d\omega} = \frac{d}{d\omega}\left(\frac{\omega}{c} n(\omega)\right) = \frac{n(\omega)}{c} + \frac{\omega}{c} n'(\omega) \\
\Rightarrow \frac{1}{v_{g0}} &= \frac{n(\omega_0)}{c} + \frac{\omega_0}{c} n'(\omega_0) = \frac{1}{v_0} + \frac{\omega_0}{c} n'(\omega_0).
\end{aligned} \tag{6.17}
$$

from which we obtain:

$$
\omega_0 n'(\omega_0) = c\left[\frac{1}{v_{g0}} - \frac{1}{v_0}\right] \quad ; \quad v_{g0} = v_g(\omega_0) \quad ; \quad v_0 = v(\omega_0). \tag{6.18}
$$

By combining (6.16) with (6.18), we get

$$
f(\omega) = -\omega_0^2 \frac{c}{\omega_0}[\frac{1}{v_{g0}} - \frac{1}{v_0}] + \omega\left\{n(\omega_0) - \theta + \omega_0 \frac{c}{\omega_0}[\frac{1}{v_{g0}} - \frac{1}{v_0}]\right\} \tag{6.19}
$$

$$
= -\omega_0 c[\frac{1}{v_{g0}} - \frac{1}{v_0}] + \omega\left\{n(\omega_0) - \theta + c[\frac{1}{v_{g0}} - \frac{1}{v_0}]\right\}. \tag{6.20}
$$

By substituting the expression for $f(\omega)$ in (6.20) into the expression for $u(z,t)$ in (6.11), we get

$$
\begin{aligned}
u(z,t) &= e^{iz\omega_0[\frac{1}{v_0} - \frac{1}{v_{g0}}]} \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{u}(0,\omega) e^{i\omega \frac{z}{c}\left\{\frac{c}{v_0} - \frac{ct}{z} + c[\frac{1}{v_{g0}} - \frac{1}{v_0}]\right\}} d\omega \\
&= e^{iz\omega_0[\frac{1}{v_0} - \frac{1}{v_{g0}}]} \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{u}(0,\omega) e^{-i\omega[-\frac{z}{v_{g0}} + t]} d\omega.
\end{aligned} \tag{6.21}
$$

Finally, we get from (6.21) and 6.2

$$
u(z,t) = e^{iz\omega_0[\frac{1}{v_0} - \frac{1}{v_{g0}}]} u\left(0, -\frac{z}{v_{g0}} + t\right). \tag{6.22}
$$

Chapter 7

Polarisation and rotation of co-ordinate system

7.1 Maximum and minimum values

Show that the ellipse (Eq. (15), page 25 in Born and Wolf [**?**]):

$$\frac{x^2}{{a_1}^2} + \frac{y^2}{{a_2}^2} - 2\frac{x}{a_1}\frac{y}{a_2}\cos\delta = \sin^2\delta \tag{7.1}$$

has the following maximum and minimum values:

$$\begin{array}{rcl} y_{maks} = a_2 & \text{for} & x = a_1\cos\delta \\ y_{min} = -a_2 & \text{for} & x = -a_1\cos\delta \\ x_{maks} = a_1 & \text{for} & y = a_2\cos\delta \\ x_{min} = -a_1 & \text{for} & y = -a_2\cos\delta. \end{array} \tag{7.2}$$

Solution:

We consider y as a function of x and define the function $F(x, y(x))$ as follows:

$$F(x,y(x)) = 0 \quad ; \quad F(x,y(x)) = \frac{y^2}{a_2^2} + \frac{x^2}{{a_1}^2} - 2\frac{x}{a_1}\frac{y}{a_2}\cos\delta - \sin^2\delta. \tag{7.3}$$

By implicit differentiation (as in chapter 13.7 in [**?**]) we have:

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0 \quad \Rightarrow \quad \frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}. \tag{7.4}$$

Thus, we get

$$\frac{dx}{dy} = 0 \qquad \text{for} \qquad y = \frac{\partial F}{\partial y} = 0. \tag{7.5}$$

Further, we have that $\frac{\partial F}{\partial y} = 2\frac{a_2}{a_1}x\cos\delta$, implying that x has a maximum or minimum value for $y = \frac{a_2}{a_1}x\cos\delta$. By substituting this in (7.1), we find that the minimum or maximum value x_e^2 for x^2 is given by:

$$\frac{{x_e}^2}{{a_1}^2} + \frac{[\frac{a_2}{a_1}x_e\cos\delta]^2}{{a_2}^2} - 2\frac{x_e}{a_1}\frac{\left[\frac{a_2}{a_1}x_e\cos\delta\right]}{a_2}\cos\delta = \sin^2\delta \tag{7.6}$$

which gives

$${x_e}^2 = {a_1}^2 \tag{7.7}$$

or

$$\begin{aligned} x_{maks} &= a_1 \qquad \text{for} \qquad y = a_2\cos\delta \\ x_{min} &= -a_1 \qquad \text{for} \qquad y = -a_2\cos\delta. \end{aligned} \tag{7.8}$$

We can find the minimum and maximum values for y by solving (7.1) with respect to x^2 and then differentiating with respect to y and following the same procedure as above. But due to the symmetry in (7.1) with respect to the exchange of x and y as well as of a_1 and a_2, we obtain the desired result by making these exchanges in the result given above:

$$\begin{aligned} y_{maks} &= a_2 \qquad \text{for} \qquad x = a_1\cos\delta \\ y_{min} &= -a_2 \qquad \text{for} \qquad x = -a_1\cos\delta. \end{aligned} \tag{7.9}$$

7.2 Rotation of co-ordinate system

Show that by rotating the co-ordinate system an angle Ψ, so that

$$x = \xi\cos\Psi - \eta\sin\Psi \qquad ; \qquad y = \xi\sin\Psi + \eta\cos\Psi \tag{7.10}$$

the coefficients in front of the $\xi\eta$ term in the equation obtained from (7.1) as a result of this rotation will disappear if Ψ satisfies the relation

$$\tan(2\Psi) = \tan(2\alpha)\cos\delta \qquad ; \qquad \tan\alpha = \frac{a_2}{a_1}. \tag{7.11}$$

Solution:

We write (7.1) in the form:

$${a_2}^2x^2 + {a_1}^2y^2 - 2a_1a_2xy\cos\delta = {a_1}^2{a_2}^2\sin^2\delta. \tag{7.12}$$

By letting

$$x = \xi\cos\Psi - \eta\sin\Psi \qquad ; \qquad y = \xi\sin\Psi + \eta\cos\Psi \tag{7.13}$$

we get

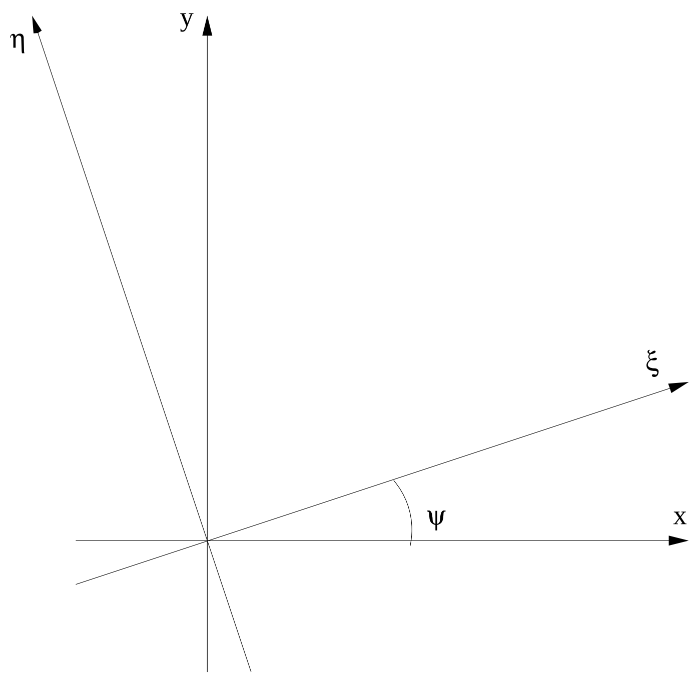

Figure 7.1: Co-ordinate system rotated an angle Ψ.

$$\begin{aligned}
{a_2}^2 x^2 &= {a_2}^2[\xi^2\cos^2\Psi + \eta^2\sin^2\Psi - 2\xi\eta\sin\Psi\cos\Psi] \\
{a_1}^2 y^2 &= {a_1}^2[\xi^2\sin^2\Psi + \eta^2\cos^2\Psi + 2\xi\eta\sin\Psi\cos\Psi] \\
-2a_1a_2xy\cos\delta &= -2a_1a_2\cos\delta[(\xi^2-\eta^2)\sin\Psi\cos\Psi + \xi\eta(\cos^2\Psi - \sin^2\Psi)].
\end{aligned} \tag{7.14}$$

By substituting (7.14) into (7.12), we see that the coefficients in front of the cross term will disappear provided

$$\begin{aligned}
({a_1}^2 - {a_2}^2)2\sin\Psi\cos\Psi &- 2a_1a_2\cos\delta(\cos^2\Psi - \sin^2\Psi) = 0 \\
&\Downarrow \\
({a_1}^2 - {a_2}^2)\sin(2\Psi) &- 2a_1a_2\cos\delta\cos(2\Psi) = 0.
\end{aligned} \tag{7.15}$$

This gives:

$$\tan(2\Psi) = \frac{2a_1a_2\cos\delta}{{a_1}^2 - {a_2}^2}. \tag{7.16}$$

We now introduce $\tan\alpha = a_2/a_1$, so that

$$\tan(2\alpha) = \frac{2\tan\alpha}{1-\tan^2\alpha} = \frac{2\frac{a_2}{a_1}}{1-(\frac{a_2}{a_1})^2} = \frac{2a_1a_2}{{a_1}^2 - {a_2}^2}. \tag{7.17}$$

Thus, we have

$$\tan(2\Psi) = \tan(2\alpha)\cos\delta. \tag{7.18}$$

7.3 Simplification of the expression for the ellipse

Show that in (ξ, η) co-ordinates (7.1) can be written as follows:

$$b^2\xi^2 + a^2\eta^2 = a^2b^2 \tag{7.19}$$

where

$$\begin{aligned} a^2 &= {a_2}^2 \sin^2\Psi + {a_1}^2 \cos^2\Psi + 2a_1a_2\cos\delta\sin\Psi\cos\Psi \\ b^2 &= {a_2}^2 \cos^2\Psi + {a_1}^2 \sin^2\Psi - 2a_1a_2\cos\delta\sin\Psi\cos\Psi. \end{aligned} \tag{7.20}$$

Solution:

By substituting (7.14) into (7.12) and choosing Ψ such that the cross term disappears, we obtain:

$$\begin{aligned} & \xi^2[{a_2}^2 \cos^2\Psi + {a_1}^2 \sin^2\Psi - 2a_1a_2\cos\delta\sin\Psi\cos\Psi] \\ + \; & \eta^2[{a_2}^2 \sin^2\Psi + {a_1}^2 \cos^2\Psi + 2a_1a_2\cos\delta\sin\Psi\cos\Psi] \\ = \; & {a_1}^2{a_2}^2\sin^2\delta. \end{aligned} \tag{7.21}$$

Thus, we have

$$b^2\xi^2 + a^2\eta^2 = {a_1}^2{a_2}^2\sin^2\delta \tag{7.22}$$

where

$$\begin{aligned} b^2 &= {a_2}^2 \cos^2\Psi + {a_1}^2 \sin^2\Psi - 2a_1a_2\cos\delta\sin\Psi\cos\Psi \\ a^2 &= {a_2}^2 \sin^2\Psi + {a_1}^2 \cos^2\Psi + 2a_1a_2\cos\delta\sin\Psi\cos\Psi. \end{aligned} \tag{7.23}$$

It remains to show that $a^2b^2 = {a_1}^2{a_2}^2\sin^2\delta$. By use of the formulas

$$\cos^2\Psi = \frac{1+\cos(2\Psi)}{2} \quad ; \quad \sin^2\Psi = \frac{1-\cos(2\Psi)}{2} \tag{7.24}$$

we get

$$2b^2 = {a_2}^2(1+\cos(2\Psi)) + {a_1}^2(1-\cos(2\Psi)) - 2a_1a_2\cos\delta\sin(2\Psi) \tag{7.25}$$

which gives

$$b^2 = \frac{1}{2}[a_1^2 + a_2^2 + A] \tag{7.26}$$

where

$$A = ({a_2}^2 - {a_1}^2)\cos(2\Psi) - 2a_1a_2\cos\delta\sin(2\Psi). \tag{7.27}$$

Similarly, we find that

$$a^2 = \frac{1}{2}[a_1^2 + a_2^2 - A] \tag{7.28}$$

so that

$$a^2b^2 = \frac{1}{4}[(a_1^2 + a_2^2)^2 - A^2]. \tag{7.29}$$

From Exercise 7.1 we have

$$\tan(2\Psi) = \frac{2a_1a_2\cos\delta}{{a_1}^2 - {a_2}^2} \tag{7.30}$$

which gives

$$\sin(2\Psi) = \frac{2a_1a_2\cos\delta}{\sqrt{B}} \qquad ; \qquad \cos(2\Psi) = \frac{a_1^2 - a_2^2}{\sqrt{B}} \tag{7.31}$$

where

$$B = ({a_1}^2 - {a_2}^2)^2 + (2a_1a_2\cos\delta)^2. \tag{7.32}$$

By substituting (7.31) into (7.27), we obtain

$$\begin{aligned} A &= ({a_2}^2 - {a_1}^2)\frac{{a_1}^2 - {a_2}^2}{\sqrt{B}} - 2a_1a_2\cos\delta\frac{2a_1a_2\cos\delta}{\sqrt{B}} \\ &= -\frac{B}{\sqrt{B}} \\ &= -\sqrt{B} \end{aligned} \tag{7.33}$$

which upon substitution in (7.29) gives:

$$\begin{aligned} a^2b^2 &= \frac{1}{4}[({a_1}^2 + {a_2}^2)^2 - B] \\ &= \frac{1}{4}[({a_1}^2 + {a_2}^2)^2 - ({a_1}^2 - {a_2}^2)^2 - (2a_1a_2\cos\delta)^2] \\ &= \frac{1}{4}[4{a_1}^2{a_2}^2(1 - \cos^2\delta)] \\ &= {a_1}^2{a_2}^2\sin^2\delta. \end{aligned} \tag{7.34}$$

Substitution of this result in (7.22) gives:

$$b^2\xi^2 + a^2\eta^2 = a^2b^2 \tag{7.35}$$

where a^2 and b^2 are given in (7.23).

Chapter 8

Phase velocity and group velocity for surface waves on water

The propagation of linear, harmonic surface waves in water of constant depth is governed by the equations

$$\nabla^2\phi(x,y,z) = 0 \qquad ; \qquad d < y < 0 \tag{8.1}$$

$$-\omega^2\phi(x,y,z) + g\frac{\partial\phi(x,y,z)}{\partial y} = 0 \qquad ; \qquad y = 0 \tag{8.2}$$

$$\frac{\partial\phi(x,y,z)}{\partial y} = 0 \qquad ; \qquad y = -d \tag{8.3}$$

$$\eta(x,z) = \frac{i\omega}{g}\phi(x,0,z). \tag{8.4}$$

The symbols in the equations above have the following meaning:

- $\omega = 2\pi/T$; T is the period,
- $d =$ water depth,
- $\phi =$ velocity potential,

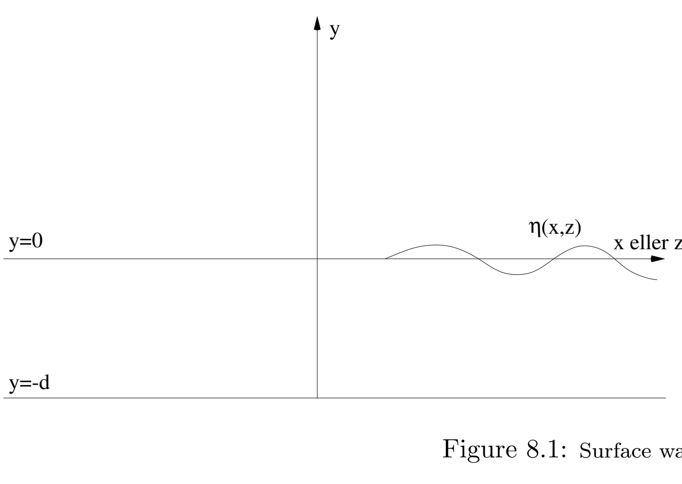

Figure 8.1: Surface waves

- g = acceleration of gravity, and
- η = displacement of the water surface from the position $y = 0$, which is its position at rest.

The velocity $\mathbf{v}$ of a water "particle" is given as

$$\mathbf{v} = \bigtriangledown\phi. \tag{8.5}$$

8.1 Separation of variables

Use separation of variables to find the solution of (8.1). Thus, express ϕ as a product

$$\phi = A(y)B(x,z) \tag{8.6}$$

where A only depends on y and B only depends on x and z, and show by substitution into (8.1) that

$$\frac{\partial^2 A(y)}{\partial^2 y} - k^2 A(y) = 0 \tag{8.7}$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} + k^2\right) B(x,z) = 0 \tag{8.8}$$

where k^2 is a separation constant.

Solution:

$$\begin{aligned}
\bigtriangledown^2\phi &= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) A(y)B(x,z) \\
&= A(y)\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right] B(x,z) + B(x,z)\frac{\partial^2}{\partial y^2}A(y) = 0 \\
\Rightarrow \frac{1}{A(y)}\frac{\partial^2 A(y)}{\partial y^2} &= -\frac{1}{B(x,z)}\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right] B(x,z) = k^2 \qquad (8.9)
\end{aligned}$$

where k^2 (the separation constant) must be a constant because the left-hand side of the equation depends on y only, while the right-hand side depends on x and z. Thus, both sides must be equal to the same constant, which we here have denoted by k^2. Equation (8.9) gives the following two equations:

$$(\frac{\partial^2}{\partial y^2} - k^2)A(y) = 0 \tag{8.10}$$

$$(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k^2)B(x,z) = 0. \tag{8.11}$$

8.2 Dispersion relation

Show that

$$A(y) = C\cosh[k(y+d)] \quad ; \quad \text{C = constant} \tag{8.12}$$

satisfies (8.3) and (8.10). Substitute this solution into (8.2) and show that the dispersion relation, i.e. the relation between ω and k is as follows:

$$\omega^2 = gk\tanh(kd). \tag{8.13}$$

Solution:

First we show that $A(y)$ satisfies (8.10). From (8.6) we have for ϕ:

$$\begin{aligned} \phi &= A(y)B(x,z) = C\cosh[k(y+d)]\cdot B(x,z) \\ \Rightarrow \frac{\partial\phi}{\partial y} &= C\sinh[k(y+d)]B(x,z)|_{y=-d} = 0 \end{aligned} \tag{8.14}$$

which was to be proven.

The general solution of (8.10) is

$$A(y) = C_1 e^{ky} + C_2 e^{-ky}. \tag{8.15}$$

By choosing

$$C_1 = \frac{1}{2}Ce^{kd} \quad \text{og} \quad C_2 = \frac{1}{2}Ce^{-kd} \tag{8.16}$$

we get

$$A(y) = \frac{1}{2}C\left(e^{k(d+y)} + e^{-k(d+y)}\right) = C\cosh[k(y+d). \tag{8.17}$$

Since $A''(y) = k^2 A(y)$, which follows by differentiating the expression above twice, we see that (8.10) is satisfied. By substitution of the expression for ϕ above into (8.2), we get

$$\left[-\omega^2 B(x,z)\cdot\cosh[k(y+d)] + gB(x,z)\cdot Ck\sinh[k(y+d)]\right]\Big|_{y=0} = 0 \tag{8.18}$$

which gives

$$-\omega^2\cosh(kd) + gk\sinh(kd) = 0 \tag{8.19}$$

or

$$\omega^2 = gk\tanh(kd). \tag{8.20}$$

8.3 Phase velocity and group velocity

Find the phase velocity and the group velocity.

Solution:

Phase velocity:

$$\begin{aligned} v &= \frac{\omega}{k} = \frac{1}{k}\sqrt{gk\tanh(kd)} \\ &\Rightarrow \quad v = \sqrt{\frac{g}{k}\tanh(kd)}. \end{aligned} \tag{8.21}$$

We obtain an alternative expression by writing

$$\begin{aligned} v &= \frac{\omega}{k} = \frac{\omega^2}{\omega k} = \frac{gk\tanh(kd)}{\omega k} \\ &= \frac{g\tanh(kd)}{\omega} \\ &= \frac{gT\tanh(kd)}{2\pi}. \end{aligned} \tag{8.22}$$

Differentiating (8.20), we get

$$2\omega\frac{d\omega}{dk} = g\left[\tanh(kd) + k\frac{1}{\cosh^2(kd)}d\right] \tag{8.23}$$

which gives the group velocity

$$v_g = \frac{d\omega}{dk} = \frac{1}{2}\frac{g}{\omega}\left[\tanh(kd) + \frac{kd}{\cosh^2(kd)}\right]. \tag{8.24}$$

8.4 The phase velocity of surface waves in water of infinite depth

Show that when the water depth increases, so that $kd \to \infty$, then the phase velocity approaches the following limiting value

$$v \to v_0 = \frac{g}{2\pi}T \tag{8.25}$$

whereas the group velocity approaches the limiting value

$$v_g \to v_{g0} = \frac{1}{2}v_0. \tag{8.26}$$

Solution:

When $kd \to \infty$, we have

$$\lim_{kd\to\infty} \tanh(kd) = 1 \tag{8.27}$$

$$\lim_{kd\to\infty} \frac{kd}{\cosh^2(kd)} = 0 \tag{8.28}$$

so that we get

$$v = v_0 \quad = \quad \sqrt{\frac{g}{k_0}} = \underline{\frac{g}{\omega}} \tag{8.29}$$

$$v_g = v_{g0} \quad = \quad \frac{1}{2}\frac{g}{\omega} = \underline{\frac{1}{2}v_0}. \tag{8.30}$$

Since $\omega = \frac{2\pi}{T}$, we get $v_0 = \frac{g}{2\pi}T$, which was to be proven.

8.5 The phase velocity of surface waves in water of finite depth

Show that $v_g < v$ also when the depth is finite.

Solution:

From (8.24) we have

$$v_g = \frac{1}{2}\frac{g}{\omega}\left[\tanh(kd) + \frac{kd}{\cosh^2(kd)}\right] \quad = \quad \frac{1}{2}\frac{g}{\omega}\tanh(kd)\left[1 + \frac{kd}{\frac{\sinh(kd)}{\cosh(kd)}\cosh^2(kd)}\right]$$

$$= \quad \frac{1}{2}\frac{g}{\omega}\tanh(kd)[1 + \frac{2kd}{2\sinh(kd)\cosh(kd)}]. \tag{8.31}$$

Since [cf. (8.20)]

$$v^2 = \frac{\omega^2}{k^2} = \frac{g}{k}\tanh(kd) \tag{8.32}$$

we have

$$\frac{g}{\omega}\tanh(kd) = \frac{k}{\omega}\frac{g}{k}\tanh(kd) = \frac{v^2}{\omega/k} = v. \tag{8.33}$$

Also, $2\sinh(x)\cosh(x) = \sinh(2x)$, so that we get

$$v_g = \frac{1}{2}v\left[1 + \frac{1}{\frac{\sinh(x)}{x}}\right] \quad ; \quad x = 2kd. \tag{8.34}$$

Since $\sinh(x)/x > 1$, as shown below:

$$\frac{sinh(x)}{x} = \frac{1}{x}\cdot\sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)!} = 1 + \sum_{n=1}^{\infty}\frac{x^{2n}}{(2n+1)!} > 1 \tag{8.35}$$

we find that $\underline{v_g < v}$, which was to be proven.

8.6 The refractive index

The refractive index for water waves is defined as

$$n = \frac{v_0}{v} \tag{8.36}$$

where v_0 is the phase velocity in water of infinite depth given by (8.30). Note the similarity with light waves, in which case v_0 corresponds to the speed of light in vacuum. Show that n can be expressed as

$$n = \coth(nk_0 d) \tag{8.37}$$

where

$$k_0 = \frac{\omega}{v_0} = \frac{\omega^2}{g} = \frac{4\pi^2}{gT^2}. \tag{8.38}$$

(Hint: Use that $k = \frac{\omega}{v} = \frac{\omega}{v_0}\frac{v_0}{v} = k_0 n$.)

Determine n numerically for $T = 12$ s and $d = 100$ m and for $T = 12$ s and $d = 25$ m.

Solution:

$$n = \frac{v_0}{v} = \frac{\sqrt{\frac{g}{k_0}}}{\sqrt{\frac{g}{k}\tanh kd}}. \tag{8.39}$$

With $k = k_0 n$ we get

$$n^2 = \frac{\frac{g}{k_0}}{\frac{g}{k}\tanh kd} = \frac{n}{\tanh(nk_0 d)} \tag{8.40}$$

or

$$\underline{n = \coth(nk_0 d)} \tag{8.41}$$

which was to be proven.

To determine n numerically from the transcendental equation in (8.41), we let $x = n$ and $\alpha = k_0 d$, so that (8.41) becomes

$$g(x) = x - \coth(\alpha x) = \frac{x}{\tanh(\alpha x)} f(x) = 0 \tag{8.42}$$

which implies that

$$f(x) = \tanh(\alpha x) - \frac{1}{x} = 0. \tag{8.43}$$

We may use Newton's iterative method, illustrated in Fig. 8.2, to solve (8.43). Starting with $x = x_1$, we see from Fig. 8.2 that

$$f'(x_1) = \frac{f(x_1)}{x_1 - x_2} \tag{8.44}$$

or

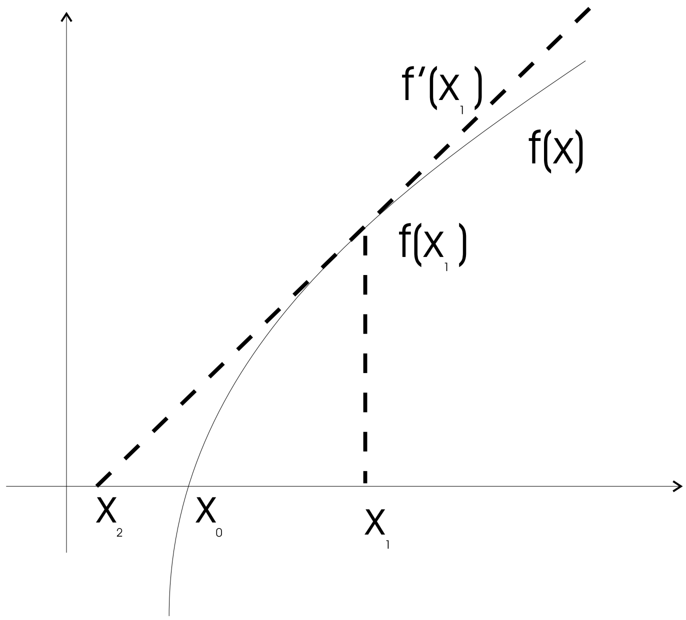

Figure 8.2: Newton's method

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \tag{8.45}$$

so that the iterative solution becomes

$$x_{j+1} = x_j - \frac{f(x_j)}{f'(x_j)} \quad ; \quad j = 1, 2, 3, \cdots . \tag{8.46}$$

The starting value x_1 we can find by sampling $f(x)$ to determine when the function changes sign. In the present case we obtain by differentiating the expression for $f(x)$

$$f'(x) = \frac{\alpha}{\cosh^2(\alpha x)} + \frac{1}{x^2}. \tag{8.47}$$

Numerically determined values for n:

$$\begin{aligned} &T = 12 \text{ s} \quad d = 100 \text{ m} \quad \rightarrow n = 1.00721 \\ &T = 12 \text{ s} \quad d = 25 \text{ m} \quad \rightarrow n = 1.3547. \end{aligned} \tag{8.48}$$

8.7 The wavelength in water of infinite depth

Find an expression for the wavelength λ_0 in deep water for a harmonic plane surface wave with period T. Determine λ_0 when $T = 10$ s and $T = 15$ s.

Solution:

Using $k_0 = \frac{2\pi}{\lambda_0}$ and (8.38), we get

$$\frac{2\pi}{\lambda_0} = \frac{(2\pi)^2}{T^2}\frac{1}{g} \Rightarrow \underline{\lambda_0 = \frac{gT^2}{2\pi}}. \tag{8.49}$$

$$T = 10\text{s} \quad \Rightarrow \quad \lambda_0 = \frac{1}{6.28} \cdot 9.81 \frac{\text{m}}{\text{s}^2} \cdot 100\text{s}^2 = 156\text{m} \tag{8.50}$$

$$T = 15\text{s} \quad \Rightarrow \quad \lambda_0 = \frac{1}{6.28} \cdot 9.81 \frac{\text{m}}{\text{s}^2} \cdot 225\text{s}^2 = 351.3\text{m}. \tag{8.51}$$

8.8 The wavelength in water of finite depth

Determine the wavelength λ for the wave in water of constant depth d expressed in terms of λ_0, d, and the refractive index n. Compute λ for $T = 12$ s and $d = 100$ m and for $T = 12$ s and $d = 25$ m.

Solution:

$$\lambda = \frac{\lambda_0}{n} = \frac{\lambda_0}{\coth(\frac{2\pi n d}{\lambda_0})}. \tag{8.52}$$

For $T = 12$ s and $d = 100$ m, we have that $n = 1.00721$, so that

$$\lambda = \frac{\lambda_0}{n} = \frac{gT^2}{2\pi n} = \frac{9.81 \cdot (12)^2}{6.28 \cdot 1.00721} = \frac{224.83}{1.00721} = 223.22 \text{ m} \tag{8.53}$$

and for $T = 12$ s and $d = 25$ m, we have $n = 1.3547$, and hence

$$\lambda = \frac{\lambda_0}{n} = \frac{gT^2}{2\pi n} = \frac{9.81 \cdot (12)^2}{6.28 \cdot 1.3547} = \frac{224.83}{1.3547} = 165.96 \text{ m}. \tag{8.54}$$

8.9 Refraction of waves that propagate towards a beach

A plane wave with a period $T = 12$ s propagates from infinitely deep water towards an area with a constant finite depth of $d = 25$ m. The angle of incidence θ^i (see Fig. 8.3) is 30°. Use Snell′s law for water waves ($n_1 \sin\theta^i = n_2 \sin\theta^t$) to determine the angle of refraction θ^t.

Solution:

For $T = 12$ s and $d = 25$ m, we have $n = 1.3547$, and hence

$$\sin\theta^t = \frac{\sin\theta^i}{n_2} = \frac{\sin 30°}{1.3547} = 0.369 \Rightarrow \theta^t = 21.66°. \tag{8.55}$$

This change of direction explains why a wave that travels towards a beach – irrespective of its direction of incidence – changes its direction such that the wave crest finally becomes parallel to the beach.

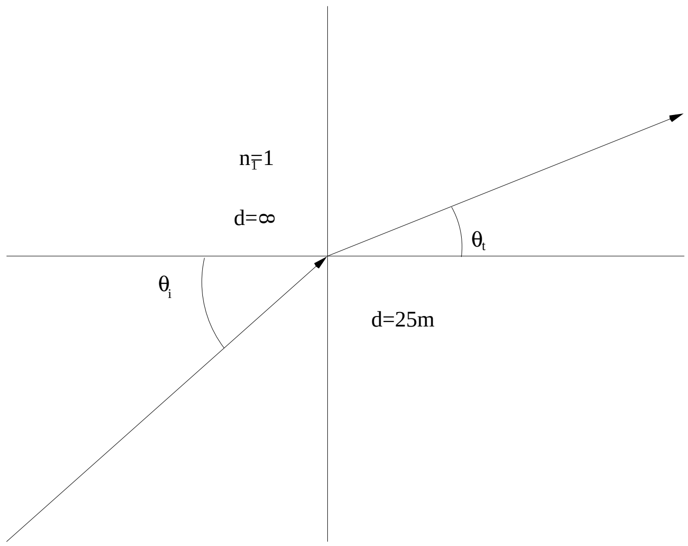

Figure 8.3: Vertical view.

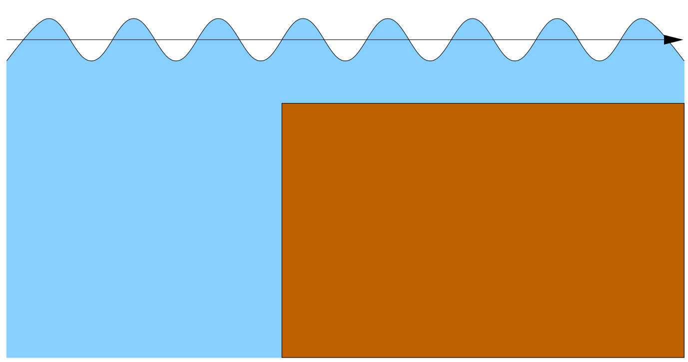

Figure 8.4: Horisontal view.

Chapter 9

Fresnel's formulas

9.1 Rewriting of Fresnel's formulas

Use Snell's law to show that the Fresnel formulas for reflection and transmission

$$T^{TM} = \frac{2n_1 \cos\theta^i}{n_2\cos\theta^i + n_1\cos\theta^t} \quad ; \quad R^{TM} = \frac{n_2\cos\theta^i - n_1\cos\theta^t}{n_2\cos\theta^i + n_1\cos\theta^t}$$
$$T^{TE} = \frac{2n_1 \cos\theta^i}{n_1\cos\theta^i + n_2\cos\theta^t} \quad ; \quad R^{TE} = \frac{n_1\cos\theta^i - n_2\cos\theta^t}{n_1\cos\theta^i + n_2\cos\theta^t} \tag{9.1}$$

can be expressed as follows:

$$T^{TM} = \frac{2\sin\theta^t\cos\theta^i}{\sin(\theta^i+\theta^t)\cos(\theta^i-\theta^t)} \quad ; \quad R^{TM} = \frac{\tan(\theta^i-\theta^t)}{\tan(\theta i+\theta^t)}$$
$$T^{TE} = \frac{2\sin\theta^t\cos\theta^i}{\sin(\theta^i+\theta^t)} \quad ; \quad R^{TE} = -\frac{\sin(\theta^i-\theta^t)}{\sin(\theta^i+\theta^t)}. \tag{9.2}$$

Solution:

We start by considering the expression for T^{TM}, and we use Snell's law $n_1 \sin\theta^i = n_2 \sin\theta^t$ to rewrite it as follows

$$\begin{aligned}
T^{TM} &= \frac{2n_1\cos\theta^i}{n_2\cos\theta^i + n_1\cos\theta^t} \\
&= \frac{2n_1\sin\theta^i\cos\theta^i}{n_2\sin\theta^i\cos\theta^i + n_1\sin\theta^i\cos\theta^t} \\
&= \frac{2n_2\sin\theta^t\cos\theta^i}{n_2\sin\theta^i\cos\theta^i + n_2\sin\theta^t\cos\theta^t} \\
&= \frac{2\sin\theta^t\cos\theta^i}{\sin\theta^i\cos\theta^i + \sin\theta^t\cos\theta^t}.
\end{aligned} \tag{9.3}$$

Next, we show that $\sin x\cos x \pm \sin y\cos y = \sin(x\pm y)\cos(x\mp y)$:

$$\begin{aligned}\sin(x \pm y)\cos(x \mp y) &= (\sin x \cos y \pm \cos x \sin y)(\cos x \cos y \pm \sin x \sin y) \\ &= \sin x \cos x(\cos^2 y + \sin^2 y) \pm \sin y \cos y(sin^2 x + \cos^2 x) \\ &= \sin x \cos x \pm \sin y \cos y.\end{aligned} \tag{9.4}$$

Using (9.4), we may rewrite (9.2) as follows

$$T^{TM} = \frac{2\sin\theta^t \cos\theta^i}{\sin(\theta^i + \theta^t)\cos(\theta^i - \theta^t)}. \tag{9.5}$$

For the refelction coefficient for TM waves we have

$$\begin{aligned}R^{TM} &= \frac{n_2\cos\theta^i - n_1\cos\theta^t}{n_2\cos\theta^i + n_1\cos\theta^t} = \frac{\frac{n_2}{n_1}\cos\theta^i - \cos\theta^t}{\frac{n_2}{n_1}\cos\theta^i + \cos\theta^t} \\ &= \frac{\frac{\sin\theta^i}{\sin\theta^t}\cos\theta^i - \cos\theta^t}{\frac{\sin\theta^i}{\sin\theta^t}\cos\theta^i + \cos\theta^t} = \frac{\sin\theta^i\cos\theta^i - \sin\theta^t\cos\theta^t}{\sin\theta^i\cos\theta^i + \sin\theta^t\cos\theta^t} \\ &= \frac{\sin(\theta^i - \theta^t)\cos(\theta^i + \theta^t)}{\sin(\theta^i + \theta^t)\cos(\theta^i - \theta^t)}\end{aligned} \tag{9.6}$$

where we have used (9.4) in the final step. Thus, we have

$$R^{TM} = \frac{\tan(\theta^i - \theta^t)}{\tan(\theta^i + \theta^t)}. \tag{9.7}$$

For the transmission coefficient for TE waves we have

$$\begin{aligned}T^{TE} &= \frac{2n_1\cos\theta^i}{n_1\cos\theta^i + n_2\cos\theta^t} = \frac{2\cos\theta^i}{\cos\theta^i + \frac{\sin\theta^i}{\sin\theta^t}\cos\theta^t} \\ &= \frac{2\sin\theta^t\cos\theta^i}{\sin\theta^t\cos\theta^i + \cos\theta^t\sin\theta^i} \\ \Rightarrow T^{TE} &= \frac{2\sin\theta^t\cos\theta^i}{\sin(\theta^i + \theta^t)}.\end{aligned} \tag{9.8}$$

For the reflection coefficient for TE waves we have

$$\begin{aligned}R^{TE} &= \frac{n_1\cos\theta^i - n_2\cos\theta^t}{n_1\cos\theta^i + n_2\cos\theta^t} = \frac{\cos\theta^i - \frac{\sin\theta^i}{\sin\theta^t}\cos\theta^t}{\cos\theta^i + \frac{\sin\theta^i}{\sin\theta^t}\cos\theta^t} \\ &= \frac{\sin\theta^t\cos\theta^i - \sin\theta^i\cos\theta^t}{\sin\theta^t\cos\theta^i + \sin\theta^i cos\theta^t} \\ \Rightarrow R^{TE} &= -\frac{\sin(\theta^i - \theta^t)}{\sin(\theta^i + \theta^t)}.\end{aligned} \tag{9.9}$$

9.2 The sign of the reflection and transmission coefficients

Provided that θ^i and θ^t are real angles, determine under what circumstances T^{TM}, R^{TM}, T^{TE}, and R^{TE} are positive and negative. What physical interpretation do we associate with negative values?

<u>Solution</u>:

We assume θ^i og θ^t are real agles and that $0 \leq \theta^i < \frac{\pi}{2}$; $0 \leq \theta^t \leq \frac{\pi}{2}$. Then $0 \leq \theta^i + \theta^t \leq \pi$ and hence $\sin(\theta^i + \theta^t) > 0$. Further, we have that $\cos(\theta^i - \theta^t) = \cos\theta^i \cos\theta^t + \sin\theta^i \sin\theta^t > 0$, and therefore T^{TM} og T^{TE} are always positive. To analyse the the reflection coefficients we first assume that

$$\begin{array}{rccl} n_2 > n_1 \quad \text{or} & \theta^i > \theta^t & \Rightarrow & R^{TE} < 0 \\ \text{If} \quad \theta^i + \theta^t < \dfrac{\pi}{2} & \text{we have} & : & R^{TM} > 0 \\ \text{and if} \quad \theta^i + \theta^t > \dfrac{\pi}{2} & & : & R^{TM} < 0. \end{array} \tag{9.10}$$

Next, we assume that

$$\begin{array}{rccl} n_2 < n_1 \quad \text{or} & \theta^i < \theta^t & \Rightarrow & R^{TE} > 0 \\ \text{If} \quad \theta^i + \theta^t < \dfrac{\pi}{2} & \text{we have} & : & R^{TM} < 0 \\ \text{and if} \quad \theta^i + \theta^t > \dfrac{\pi}{2} & & : & R^{TM} > 0. \end{array} \tag{9.11}$$

A negative value for a reflection coefficient implies that there is a phase change of π between the incident and the reflected wave.

Chapter 10

Reflectivity and transmissivity

10.1 Energy conservation for TE and TM components

From the formulas for the reflectivities and the transmissivities, given by

$$\mathcal{T}^{TM} = \frac{\sin 2\theta^i \sin 2\theta^t}{\sin^2(\theta^i + \theta^t)\cos^2(\theta^i - \theta^t)} \quad ; \quad \mathcal{R}^{TM} = \frac{\tan^2(\theta^i - \theta^t)}{\tan^2(\theta^i + \theta^t)}$$
$$\mathcal{T}^{TE} = \frac{\sin 2\theta^t \sin 2\theta^i}{\sin^2(\theta^i + \theta^t)} \quad ; \quad \mathcal{R}^{TE} = -\frac{\sin^2(\theta^i - \theta^t)}{\sin^2(\theta^i + \theta^t)} \tag{10.1}$$

show that

$$\mathcal{T}^{TM} + \mathcal{R}^{TM} = 1 \quad ; \quad \mathcal{T}^{TE} + \mathcal{R}^{TE} = 1. \tag{10.2}$$

Solution:

$$\begin{aligned}
\mathcal{R}^{TM} &= \frac{\tan^2(\theta^i - \theta^t)}{\tan^2(\theta^i + \theta^t)} = \frac{\sin^2(\theta^i - \theta^t)\cos^2(\theta^i + \theta^t)}{\sin^2(\theta^i + \theta^t)\cos^2(\theta^i - \theta^t)} \\
\mathcal{T}^{TM} &= \frac{\sin 2\theta^i \sin 2\theta^t}{\sin^2(\theta^i + \theta^t)\cos^2(\theta^i - \theta^t)} \\
\Rightarrow \mathcal{R}^{TM} + \mathcal{T}^{TM} &= \frac{\sin 2\theta^i \sin 2\theta^t + \sin^2(\theta^i - \theta^t)\cos^2(\theta^i + \theta^t)}{\sin^2(\theta^i + \theta^t)\cos^2(\theta^i - \theta^t)}.
\end{aligned} \tag{10.3}$$

Thus, we have

$$\mathcal{R}^{TM} + \mathcal{T}^{TM} = 1 + \frac{N}{D} \tag{10.4}$$

where

$$\begin{aligned}
N &= \sin 2\theta^i \sin 2\theta^t + A - D \\
A &= \sin^2(\theta^i - \theta^t)\cos^2(\theta^i + \theta^t) \\
D &= \sin^2(\theta^i + \theta^t)\cos^2(\theta^i - \theta^t).
\end{aligned} \tag{10.5}$$

Next, we use the formula $\sin x \cos x \pm \sin y \cos y = \sin(x \pm y)\cos(x \mp y)$ to obtain

$$\begin{aligned} A &= \sin^2(\theta^i - \theta^t)\cos^2(\theta^i + \theta^t) = (\sin\theta^i \cos\theta^i - \sin\theta^t \cos\theta^t)^2 \\ D &= \sin^2(\theta^i + \theta^t)\cos^2(\theta^i - \theta^t) = (\sin\theta^i \cos\theta^i + \sin\theta^t \cos\theta^t)^2 \\ \Rightarrow A - D &= -4\sin\theta^i \cos\theta^i \sin\theta^t \cos\theta^t = -\sin 2\theta^i \sin 2\theta^t \end{aligned} \tag{10.6}$$

which shows that $N = 0$, and hence that

$$\mathcal{T}^{TM} + \mathcal{R}^{TM} = 1 \tag{10.7}$$

which was to be proven. For TE polarisation we have

$$\begin{aligned} \mathcal{R}^{TE} &= \frac{\sin^2(\theta^i - \theta^t)}{\sin^2(\theta^i + \theta^t)} \\ \mathcal{T}^{TE} &= \frac{\sin 2\theta^t \sin 2\theta^i}{\sin^2(\theta^i + \theta^t)} \end{aligned} \tag{10.8}$$

$$\mathcal{R}^{TE} + \mathcal{T}^{TE} = 1 + \frac{N}{D} \tag{10.9}$$

where in this case

$$\begin{aligned} N &= \sin 2\theta^i \sin 2\theta^t + A - D \\ A &= \sin^2(\theta^i - \theta^t) = (\sin\theta^i \cos\theta^t - \sin\theta^t \cos\theta^i)^2 \\ D &= \sin^2(\theta^i + \theta^t) = (\sin\theta^i \cos\theta^t + \sin\theta^t \cos\theta^i)^2 \\ \Rightarrow A - D &= -4\sin\theta^i \cos\theta^t \sin\theta^t \cos\theta^i = -\sin 2\theta^i \sin 2\theta^t. \end{aligned} \tag{10.10}$$

This shows that $N = 0$, and hence that

$$\mathcal{T}^{TE} + \mathcal{R}^{TE} = 1 \tag{10.11}$$

which was to be proven.

10.2 Energy conservation

Show that

$$\mathcal{T} + \mathcal{R} = 1 \tag{10.12}$$

where

$$\begin{aligned} \mathcal{R} &= \mathcal{R}^{TM}\cos^2\alpha^i + \mathcal{R}^{TE}\sin^2\alpha^i \\ \mathcal{T} &= \mathcal{T}^{TM}\cos^2\alpha^i + \mathcal{T}^{TE}\sin^2\alpha^i. \end{aligned} \tag{10.13}$$

Solution:

It follows from (10.13) that

$$\mathcal{R} + \mathcal{T} = (\mathcal{R}^{TM} + \mathcal{T}^{TM}) \cos^2 \alpha^i + (\mathcal{R}^{TE} + \mathcal{T}^{TE}) \sin^2 \alpha^i \tag{10.14}$$

but since

$$\mathcal{T}^{TM} + \mathcal{R}^{TM} = 1 \tag{10.15}$$

and

$$\mathcal{T}^{TE} + \mathcal{R}^{TE} = 1 \tag{10.16}$$

we get

$$\mathcal{R} + \mathcal{T} = \cos^2 \alpha^i + \sin^2 \alpha^i = 1. \tag{10.17}$$

Chapter 11

Total reflection 1

Consider a harmonic, plane wave that is incident upon a plane interface under an angle of incidence θ^i that is greater than the critical angle θ^{ic}, so that we have total reflection. The components of the electric field are then given by (see section 9.3.5 of [**?**]):

$$\begin{aligned} E_x^t &= T^{TM} E^{TMi} \cos\theta^t \quad e^{i(k_x x - \omega t)} e^{-|k_{z2}| z} \\ E_y^t &= T^{TE} E^{TEi} \quad e^{i(k_x x - \omega t)} e^{-|k_{z2}| z} \\ E_z^t &= -\tfrac{\sin\theta^i}{n} T^{TM} E^{TMi} \quad e^{i(k_x x - \omega t)} e^{-|k_{z2}| z} \end{aligned} \tag{11.1}$$

where

$$\begin{aligned} k_x = k_1 \sin\theta^i = \frac{1}{n}\frac{\omega}{v_2}\sin\theta^i \quad &; \quad n = \frac{n_2}{n_1} < 1 \\ k_{z2} = k_2 \cos\theta^t \quad &; \quad \cos\theta^t = \frac{i}{n}\sqrt{\sin^2\theta^i - n^2}. \end{aligned} \tag{11.2}$$

11.1 Transmitted magnetic field

Show from Maxwell's equation (with $\mu_1 = \mu_2 = 1$)

$$\bigtriangledown \times \mathbf{E}^t = -\frac{1}{c}\,\dot{\mathbf{H}}^t \tag{11.3}$$

that the transmitted magnetic field $\mathbf{H}^t$ has the following components:

$$\begin{aligned} H_x^t &= -n_2 \cos\theta^t T^{TE} E^{TEi} \quad e^{i(k_x x - \omega t)} e^{-|k_{z2}| z} \\ H_y^t &= n_2 T^{TM} E^{TMi} \quad e^{i(k_x x - \omega t)} e^{-|k_{z2}| z} \\ H_z^t &= n_1 \sin\theta^i T^{TE} E^{TEi} \quad e^{i(k_x x - \omega t)} e^{-|k_{z2}| z}. \end{aligned} \tag{11.4}$$

Solution: First we determine $\mathbf{H}^t$ from Maxwell's equation

$$\bigtriangledown \times \mathbf{E}^t = -\frac{1}{c}\,\dot{\mathbf{H}}^t \tag{11.5}$$

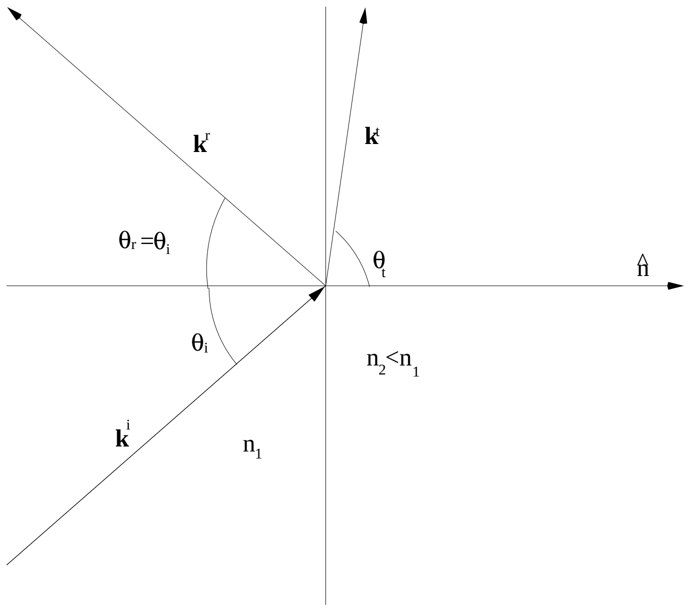

Figure 11.1: Refraction near the critical angle of total reflection.

which with a time variation of $e^{-i\omega t}$ gives:

$$\mathbf{H}^t = \frac{1}{ik_0} \bigtriangledown \times \mathbf{E}^t \quad ; \quad k_0 = \frac{\omega}{c}. \tag{11.6}$$

Since

$$\mathbf{E}^t = \mathbf{E}_0^t e^{i(\mathbf{k}^t \cdot \mathbf{r} - \omega t)} \tag{11.7}$$

we get

$$\bigtriangledown \times \mathbf{E}^t = i\mathbf{k}^t \times \mathbf{E}^t \tag{11.8}$$

so that

$$\mathbf{H}^t = \frac{1}{k_0} \mathbf{k}^t \times \mathbf{E}^t. \tag{11.9}$$

For the z component of $\mathbf{H}^t$ we have:

$$H_z^t = \frac{1}{k_0}(k_x E_y^t - k_y E_x^t) = n_1 \sin\theta^i E_y^t \tag{11.10}$$

since $k_y = 0$ and $k_x = k_1 \sin\theta^i = n_1 k_0 \sin\theta^i$. Thus, we have

$$H_z^t = n_1 \sin\theta^i T^{TE} E^{TEi} e^{i(k_x x - \omega t)} e^{-|k_{z2}|z}. \tag{11.11}$$

For the y component we have:

$$H_y^t = \frac{1}{k_0}(k_{z2} E_x^t - k_x E_z^t). \tag{11.12}$$

Since $\mathbf{k} \cdot \mathbf{E} = 0$, and hence

$$E_z^t = \frac{-k_x}{k_{z2}} E_x^t \tag{11.13}$$

we get

$$\begin{aligned} H_y^t &= \frac{1}{k_0}\left(k_{z2}E_x^t - k_x E_z^t\right) = \frac{1}{k_0}\left(k_{z2}E_x^t + \frac{{k_x}^2}{k_{z2}}E_x^t\right) = \frac{1}{k_0 k_{z2}}\left(k_{z2}^2 + {k_x}^2\right)E_x^t = \frac{{k_2}^2}{k_0 k_{z2}}E_x^t \\ &= \frac{k_2^2}{k_2 \cos\theta^t k_0} T^{TM} E^{TMi} \cos\theta^t e^{i(k_x x - \omega t)} e^{-|k_{z2}|z} = n_2 T^{TM} E^{TMi} e^{i(k_x x - \omega t)} e^{-|k_{z2}|z}. \end{aligned} \tag{11.14}$$

For the x component we have

$$H_x^t = \frac{1}{k_0}(k_y E_z^t - k_{z2} E_y^t) = -\frac{k_{z2}}{k_0} E_y^t \tag{11.15}$$

since $k_y = 0$. Thus, we get

$$H_x^t = -n_2 \cos\theta^t E_y^t = -n_2 \cos\theta^t T^{TE} E^{TEi} e^{i(k_x x - \omega t)} e^{-|k_{z2}|z}. \tag{11.16}$$

11.2 Transmitted Poynting vector

Show that the Poynting vector of the transmitted field

$$\mathbf{S}^t = \frac{c}{4\pi} \mathbf{E}^t \times \mathbf{H}^t \tag{11.17}$$

has the following components:

$$\begin{aligned} S_x^t = \frac{c}{16\pi} n_1 \sin\theta^i e^{-2A} \quad & \{(E^{TEi})^2 \left[(T^{TE})^2 e^{2i\phi} + ((T^{TE})^*)^2 e^{-2i\phi} + 2|T^{TE}|^2\right] \\ & + (E^{TMi})^2 \left[(T^{TM})^2 e^{2i\phi} + ((T^{TM})^*)^2 e^{-2i\phi} + 2|T^{TM}|^2\right]\} \end{aligned} \tag{11.18}$$

$$S_y^t = \frac{c}{4\pi} n_1 \sin\theta^i e^{-2A} E^{TMi} E^{TEi} \, \Im\left\{T^{TM}[T^{TE}]^*\right\} \tag{11.19}$$

$$S_z^t = -\frac{c}{8\pi} n_2 |\cos\theta^t| e^{-2A} \left\{(E^{TMi})^2 \, \Im\left\{\left[T^{TE} e^{i\phi}\right]^2\right\} + (E^{TEi})^2 \, \Im\left\{\left[T^{TM} e^{i\phi}\right]^2\right\}\right\} \tag{11.20}$$

where

$$A = |k_{z2}|z \quad ; \quad \phi = k_x x - \omega t. \tag{11.21}$$

Solution: Since the Poynting vector is given as the vectorial product of **E** and **B**, we must use real quantities. Thus, we express the electric field components as follows

$$\begin{aligned} E_x^t = & \quad E^{TMi} e^{-A} & \frac{1}{2}\left[T^{TM} \cos\theta^t e^{i\phi} + (T^{TM} \cos\theta^t)^* e^{-i\phi}\right] \\ E_y^t = & \quad E^{TEi} e^{-A} & \frac{1}{2}\left[T^{TE} e^{i\phi} + (T^{TE})^* e^{-i\phi}\right] \\ E_z^t = & -\frac{\sin\theta^i}{n} E^{TMi} e^{-A} & \frac{1}{2}\left[T^{TM} e^{i\phi} + (T^{TM})^* e^{-i\phi}\right]. \end{aligned} \tag{11.22}$$

Similarly, we express the magnetic field components as:

$$\begin{aligned} H_x^t = & \quad -n_2 E^{TEi} e^{-A} & \frac{1}{2}\left[T^{TE}\cos\theta^t e^{i\phi} + (T^{TE}\cos\theta^t)^* e^{-i\phi}\right] \\ H_y^t = & \quad n_2 E^{TMi} e^{-A} & \frac{1}{2}\left[T^{TM} e^{i\phi} + (T^{TM})^* e^{-i\phi}\right] \\ H_z^t = & \quad n_1 \sin\theta^i E^{TEi} e^{-A} & \frac{1}{2}\left[T^{TE} e^{i\phi} + (T^{TE})^* e^{-i\phi}\right] \end{aligned} \tag{11.23}$$

where

$$A = |k_{z2}|z \quad ; \quad \phi = k_x x - \omega t. \tag{11.24}$$

Since $\cos\theta^t$ is purely imaginary, $(\cos\theta^t)^* = -\cos\theta^t$, so that we have

$$\begin{aligned} E_x^t = & \quad E^{TMi} e^{-A}\cos\theta^t & \frac{1}{2}\left[T^{TM} e^{i\phi} - (T^{TM})^* e^{-i\phi}\right] \\ E_y^t = & \quad E^{TEi} e^{-A} & \frac{1}{2}\left[T^{TE} e^{i\phi} + (T^{TE})^* e^{-i\phi}\right] \\ E_z^t = & \quad -\tfrac{\sin\theta^i}{n} E^{TMi} e^{-A} & \frac{1}{2}\left[T^{TM} e^{i\phi} + (T^{TM})^* e^{-i\phi}\right] \end{aligned} \tag{11.25}$$

and

$$\begin{aligned} H_x^t = & \quad -n_2 E^{TEi} e^{-A}\cos\theta^t & \frac{1}{2}\left[T^{TE} e^{i\phi} - (T^{TE})^* e^{-i\phi}\right] \\ H_y^t = & \quad n_2 E^{TMi} e^{-A} & \frac{1}{2}\left[T^{TM} e^{i\phi} + (T^{TM})^* e^{-i\phi}\right] \\ H_z^t = & \quad n_1 \sin\theta_i E^{TEi} e^{-A} & \frac{1}{2}\left[T^{TE} e^{i\phi} + (T^{TE})^* e^{-i\phi}\right]. \end{aligned} \tag{11.26}$$

The components S_x^t, S_y^t, and S_z^t of the Poynting vector are given by:

$$\begin{aligned} S_x^t &= \frac{c}{4\pi}[E_y^t H_z^t - E_z^t H_y^t] \\ S_y^t &= \frac{c}{4\pi}[E_z^t H_x^t - E_x^t H_z^t] \\ S_z^t &= \frac{c}{4\pi}[E_x^t H_y^t - E_y^t H_x^t]. \end{aligned} \tag{11.27}$$

Substitution from (11.25) and (11.26) into (11.27) gives

$$\begin{aligned} S_x^t = \frac{c}{16\pi} e^{-2A} n_1 \sin\theta^i \quad & \{(E^{TEi})^2 \left[T^{TE} e^{i\phi} + (T^{TE})^* e^{-i\phi}\right]^2 \\ & +(E^{TMi})^2 \left[T^{TM} e^{i\phi} + (T^{TM})^* e^{-i\phi}\right]^2\} \end{aligned} \tag{11.28}$$

$$\begin{aligned} S_y^t = \frac{c}{16\pi} e^{-2A} n_1 \sin\theta^i \cos\theta^t E^{TEi} E^{TMi} \quad & \{\left[T^{TM} e^{i\phi} + (T^{TM})^* e^{-i\phi}\right]\left[T^{TE} e^{i\phi} - (T^{TE})^* e^{-i\phi}\right] \\ & -\left[T^{TM} e^{i\phi} - (T^{TM})^* e^{-i\phi}\right]\left[T^{TE} e^{i\phi} + (T^{TE})^* e^{-i\phi}\right]\} \end{aligned} \tag{11.29}$$

$$\begin{aligned} S_z^t = \frac{c}{16\pi} e^{-2A} n_2 \cos\theta^t \quad & \{(E^{TMi})^2 \left[T^{TM} e^{i\phi} - (T^{TM})^* e^{-i\phi}\right]\left[T^{TM} e^{i\phi} + (T^{TM})^* e^{-i\phi}\right] \\ & -(E^{TEi})^2 \left[T^{TE} e^{i\phi} + (T^{TE})^* e^{-i\phi}\right]\left[T^{TE} e^{i\phi} - (T^{TE})^* e^{-i\phi}\right]\}. \end{aligned} \tag{11.30}$$

Carrying out the multiplications, we get

$$
\begin{aligned}
S_x^t = \quad & \tfrac{cn_1 \sin\theta^i}{16\pi} e^{-2A} & \{(E^{TEi})^2 \left[(T^{TE})^2 e^{2i\phi} + ((T^{TE})^*)^2 e^{-2i\phi} + 2|T^{TE}|^2\right] \\
& & +(E^{TMi})^2 \left[(T^{TM})^2 e^{2i\phi} + ((T^{TM})^*)^2 e^{-2i\phi} + 2|T^{TM}|^2\right]\} \\
S_y^t = \quad & \tfrac{cn_1 \sin\theta^i \cos\theta^t}{8\pi} e^{-2A} & E^{TMi} E^{TEi} \left[(T^{TM})^* T^{TE} - T^{TM} (T^{TE})^*\right] \\
S_z^t = \quad & \tfrac{cn_2 \cos\theta^t}{16\pi} e^{-2A} & \{(E^{TMi})^2 \left[(T^{TM})^2 e^{2i\phi} - ((T^{TM})^*)^2 e^{-2i\phi}\right] \\
& & +(E^{TEi})^2 \left[(T^{TE})^2 e^{2i\phi} - ((T^{TE})^*)^2 e^{-2i\phi}\right]\}.
\end{aligned}
\tag{11.31}
$$

This expression for S_x^t is equal to the one given in (11.18). To show that the expressions for S_y^t and S_z^t given above are equal to those given in (11.19) and (11.20), respectively, we note that $\cos\theta^t = i|\cos\theta^t|$ and use the relation $i(z - z^*) = -\Im z$, which is valid for any arbitrary number z.

11.3 Time average of the Poynting vector

Show that the time-average of the z component of the Poynting vector vanishes, i.e.

$$
< S_z^t >= 0, \tag{11.32}
$$

and that the time-averages of the x and y components are given by:

$$
\begin{aligned}
< S_x^t > \quad &= \quad \frac{cn_1 \sin\theta^i}{8\pi} e^{-2A} \left\{|T^{TE}|^2 (E^{TEi})^2 + |T^{TM}|^2 (E^{TEi})^2\right\} \\
< S_y^t > \quad &= \quad -\frac{cn_1 \sin\theta^i}{4\pi} e^{-2A} |\cos\theta^t| E^{TMi} E^{TEi}\, \Im\left\{(T^{TM})^* T^{TE}\right\}.
\end{aligned}
\tag{11.33}
$$

What is the physical explanation of this result? (Time averaging implies integration over an interval T' that is much larger than the period $T = \frac{2\pi}{\omega}$, i.e. $< \mathbf{S}^t >= \frac{1}{2T'} \int_{-T'}^{T'} \mathbf{S}^t dt$, where $T' >> T$.)

Solution: Because

$$
\frac{1}{2T'} \int_{-T'}^{T'} e^{\pm 2i\omega t} dt = \frac{\sin(2\omega T')}{4\pi} \left(\frac{T}{T'}\right) \tag{11.34}
$$

terms in (11.31) that include $e^{\pm 2i\phi}$ will disappear on time averaging over an interval such that $T' >> T$. Thus, we have

$$
\begin{aligned}
< S_z^t > \quad &= \quad 0 \\
< S_x^t > \quad &= \quad \frac{cn_1 \sin\theta^i}{8\pi} e^{-2A} ((E^{TEi})^2 |T^{TE}|^2 + (E^{TMi})^2 |T^{TM}|^2) \\
< S_y^t > \quad &= \quad -\frac{cn_1 \sin\theta^i |\cos\theta^t|}{4\pi} e^{-2A} E^{TMi} E^{TEi} \Im\{(T^{TM})^* T^{TE}\}.
\end{aligned}
\tag{11.35}
$$

This means that the time average of the energy flux is zero in the z direction. The energy propagates in directions parallel to the interface and in the plane of incidence as long as the polarisation is either normal to the plane of incidence ($E^{TMi} = 0$) or parallel to the plane of incidence ($E^{TEi} = 0$). But in general we have $< S_y^t >\neq 0$.

Chapter 12

Fresnel's rhomb

Fig. 12.1 shows a Fresnel's rhomb, which can be used to produce circularly polarised light from linearly polarised light or vice versa. The required phase difference of $\delta = 90°$ can be obtained through two successive total reflections, each introducing a phase difference of 45°. For a single total reflection the phase difference δ is given by the fomula

$$\tan\frac{\delta}{2} = \frac{\cos\theta^i\sqrt{\sin^2\theta^i - n^2}}{\sin^2\theta^i} \tag{12.1}$$

where θ^i is the angle of incidence and $n = \frac{n_2}{n_1} < 1$.

12.1 Solution for $\sin\theta^i$.

Solve (12.1) with respect to $\sin\theta^i$, and show that

$$\sin\theta^i = \left(\frac{n^2+1 \pm \sqrt{(n^2+1)^2 - 4n^2(1+\tan^2\frac{\delta}{2})}}{2(\tan^2\frac{\delta}{2}+1)}\right)^{\frac{1}{2}}. \tag{12.2}$$

Solution: From (12.1 we get

$$\begin{aligned}\sin^4\theta^i\tan^2\frac{\delta}{2} &= (1-\sin^2\theta^i)(\sin^2\theta^i - n^2)\\ \Rightarrow 0 &= \sin^4\theta^i(\tan^2\frac{\delta}{2}+1) - (n^2+1)\sin^2\theta^i + n^2\end{aligned}$$

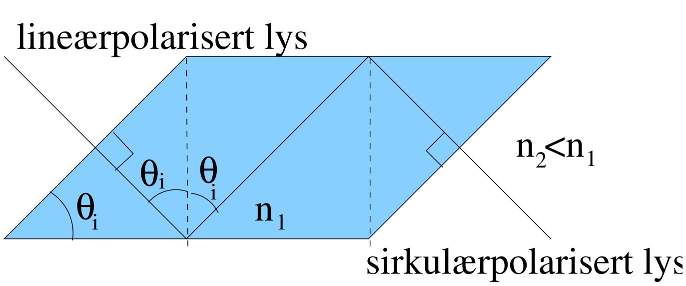

Figure 12.1: Fresnel's rhomb

$$\Rightarrow \sin^2\theta^i = \frac{n^2+1\pm\sqrt{(n^2+1)^2-4n^2(1+\tan^2\frac{\delta}{2})}}{2(\tan^2\frac{\delta}{2}+1)}$$
$$\Rightarrow \sin\theta^i = \left(\frac{n^2+1\pm\sqrt{(n^2+1)^2-4n^2(1+\tan^2\frac{\delta}{2})}}{2(\tan^2\frac{\delta}{2}+1)}\right)^{\frac{1}{2}}. \tag{12.3}$$

12.2 Phase difference of $45°$, $n = 1/1.52 = 0.6579$

For $n_{21} = \frac{1}{n} = 1.52$ determine those angles of incidence which give a phase difference of 45°.

Solution: When $n_{12} = \frac{1}{n} = 1.52$ and $\frac{\delta}{2} = 22.5°$, we get by substitution into (12.3):

$$\sin\theta^{i+} = 0.81371 \quad\Rightarrow\quad \theta^{i+} = 55.45752°$$
$$\sin\theta^{i-} = 0.73790 \quad\Rightarrow\quad \theta^{i-} = 47.55312°. \tag{12.4}$$

12.3 Phase difference of $45°$, $n = 1/1.49 = 0.6711$

Repeat the task in exercise 12.2 for $n_{21} = 1.49$, and explain the result.

Solution: For $n_{21} = 1.49$ the expression inside the square root in (12.3) becomes negative. This means that in order to obtain a phase difference of 45° in one single total reflection, n_{21} must be larger than 1.49.

12.4 Maximum phase difference

Show from (12.1) that δ has a maximum value δ_m for $\theta^i = \theta^{im}$ given by

$$\sin^2\theta^{im} = \frac{2n^2}{1+n^2} \tag{12.5}$$

and that δ_m is given by:

$$\tan\frac{\delta_m}{2} = \frac{1-n^2}{2n}. \tag{12.6}$$

Solution: From (12.1) we see that $\delta = 0$ when $\theta^i = \theta^{ic}$ (critical angle of incidence) and $\theta^i = \pi/2$ (grazing incidence). Between tese two angles there is an angle of incidence $\theta^i = \theta^{im}$ which gives a maximum phase difference of $\delta = \delta_m$, where δ_m is determined by

$$\left.\frac{d\delta}{d\theta^i}\right|_{\theta^i=\theta^{im}} = 0. \tag{12.7}$$

Since $\delta = 2\arctan[\tan(\frac{\delta}{2})]$, we have from (12.7)

$$\frac{d\delta}{d\theta_i} = \frac{2}{1+\tan^2\frac{\delta}{2}}\frac{d}{d\theta_i}(\tan\frac{\delta}{2}) = 0. \tag{12.8}$$

This means that

$$\frac{d}{d\theta^i}(\tan\frac{\delta}{2}) = 0 \quad \Rightarrow \quad \frac{d}{d\theta^i}\left[\frac{\cos\theta^i\sqrt{\sin^2\theta^i - n^2}}{\sin^2\theta^i}\right] = 0$$
$$\Rightarrow \quad \frac{\sin^2\theta^i[-\sin\theta^i\sqrt{(...)} + \cos\theta^i\frac{\sin\theta^i\cos\theta^i}{\sqrt{(...)}}] - \cos\theta^i\sqrt{(...)}2\sin\theta^i\cos\theta^i}{\sin^4\theta^i} = 0. \tag{12.9}$$

We multiply the last result by $\sqrt{\sin^2\theta^i - n^2}\sin^3\theta^i$ to obtain

$$\begin{aligned}
-\sin^2\theta^i(\sin^2\theta^i - n^2) + \sin^2\theta^i\cos^2\theta^i - 2\cos^2\theta^i(\sin^2\theta^i - n^2) &= 0 \\
&\Downarrow \\
-(\sin^2\theta^i + \cos^2\theta^i)(\sin^2\theta^i - n^2) + \sin^2\theta^i\cos^2\theta^i - \cos^2\theta^i(\sin^2\theta^i - n^2) &= 0 \\
&\Downarrow \\
-\sin^2\theta^i + n^2 + n^2(1 - \sin^2\theta^i) &= 0 \\
&\Downarrow \\
2n^2 - (1 + n^2)\sin^2\theta^i = 0
\end{aligned} \tag{12.10}$$

so that $\frac{d\delta}{d\theta^i}\Big|_{\theta^i=\theta^{im}} = 0$ gives:

$$\begin{aligned}
\sin^2\theta^{im} &= \frac{2n^2}{1+n^2} \\
\cos^2\theta^{im} = 1 - \frac{2n^2}{1+n^2} &= \frac{1-n^2}{1+n^2}.
\end{aligned} \tag{12.11}$$

By substitution of this result into (12.1), we get

$$\tan\frac{\delta_m}{2} = \frac{\sqrt{1-n^2}}{\sqrt{1+n^2}}\frac{\sqrt{\frac{2n^2}{1+n^2} - n^2}}{\frac{2n^2}{1+n^2}} = \frac{1-n^2}{2n}. \tag{12.12}$$

12.5 Phase differences of 45° and 90°

What minimum value of $n_{12} = \frac{1}{n}$ is required to obtain a phase difference of

1. 90° or
2. 45°?

Solution: From Exercise (12.4) we have that $\tan\frac{\delta}{2} \leq \tan\frac{\delta_m}{2} = \frac{1-n^2}{2n}$, from wich it follows that:

$$\begin{aligned}
n^2 + 2n\tan\frac{\delta}{2} - 1 &\leq 0 \\
&\Downarrow \\
(n^2 + \tan\frac{\delta}{2})^2 &\leq 1 + \tan^2\frac{\delta}{2}
\end{aligned}$$

$$\begin{aligned} & \Downarrow \\ n & \leq \sqrt{1+\tan^2\frac{\delta}{2}}-\tan\frac{\delta}{2} \\ & \Downarrow \\ n_{12}=\frac{1}{n} & \geq \frac{1}{\sqrt{1+\tan^2\frac{\delta}{2}}-\tan\frac{\delta}{2}}. \end{aligned} \tag{12.13}$$

1. $\delta = 90^\circ \Rightarrow \frac{\delta}{2} = 45^\circ$ and $\tan\frac{\delta}{2} = 1$ so that

$$n_{21} \geq \frac{1}{\sqrt{2}-1} = \sqrt{2}+1 = 2.4142. \tag{12.14}$$

2. $\delta = 45^\circ \Rightarrow \frac{\delta}{2} = 22.5^\circ$ and $\tan\frac{\delta}{2} = \frac{1-\cos\delta}{\sin\delta} = \frac{1-\frac{1}{2}\sqrt{2}}{\frac{1}{2}\sqrt{2}} = \sqrt{2}-1$, so that

$$n_{21} \geq \frac{1}{\sqrt{1+(\sqrt{2}-1)^2}-(\sqrt{2}-1)} = \frac{1}{1-\sqrt{2}+\sqrt{2(2-\sqrt{2})}} = 1.4966. \tag{12.15}$$

This result agrees with our finding in Exercise 12.3 showing that the expression inside the square root in (12.1) becomes negative for $n_{12} = 1.49$.

Chapter 13

Total reflection 2

Consider reflection and refraction of a plane wave at a plane interface between two media, and let $n_2 \leq n_1$, as illustrated in Fig. 13.1. From Snell's law

$$n_1 \sin\theta^i = n_2 \sin\theta^t \tag{13.1}$$

or

$$\sin\theta^t = \frac{\sin\theta^i}{n} \quad ; \quad n = \frac{n_2}{n_1} \leq 1 \tag{13.2}$$

it follows that when $\theta^i > \theta^{ic}$, where $\sin\theta^{ic} = n$, then $\sin\theta^t > 1$. Thus, θ^t must be a complex number. Show that $\theta^t = \alpha + i\beta$, where

$$\begin{aligned} \alpha &= \frac{\pi}{2} \\ \beta &= \ln\left[\frac{\sin\theta^i}{n} - \sqrt{\left(\frac{\sin\theta^i}{n}\right)^2 - 1}\right]. \end{aligned} \tag{13.3}$$

Solution: We start with

$$\cos\theta^t = \sqrt{1-\sin^2\theta^t} = \sqrt{1-\frac{\sin^2\theta^i}{n^2}}. \tag{13.4}$$

Since $n < \sin\theta^i$ when $\theta^i > \theta^{ic}$, we have

$$\cos\theta^t = iA \quad ; \quad A = \sqrt{\left(\frac{\sin\theta^i}{n}\right)^2 - 1}. \tag{13.5}$$

Further, we have $\cos\theta^t = \cos(\alpha + i\beta) = \cos\alpha\cos(i\beta) - \sin\alpha\sin(i\beta)$, and since

$$\begin{aligned} \cos(i\beta) &= \frac{e^{i(i\beta)} + e^{-i(i\beta)}}{2} = \frac{e^{-\beta} + e^{\beta}}{2} = \cosh(\beta) \\ \sin(i\beta) &= \frac{e^{i(i\beta)} - e^{-i(i\beta)}}{2i} = i\frac{e^{-\beta} - e^{\beta}}{2} = i\sinh(\beta) \end{aligned} \tag{13.6}$$

we get

$$iA = \cos\alpha\cosh\beta - i\sin\alpha\sinh\beta \tag{13.7}$$

and hence

$$\cos\alpha\cosh\beta = 0 \tag{13.8}$$

$$-\sin\alpha\sinh\beta = A. \tag{13.9}$$

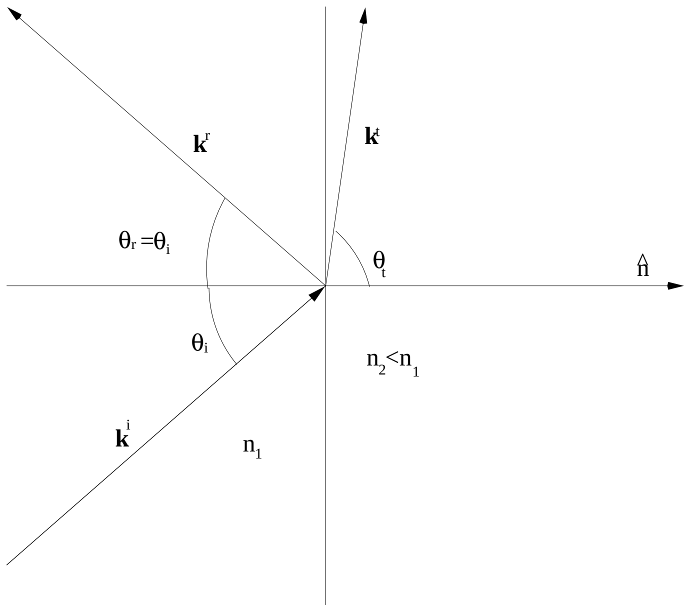

Figure 13.1: Refraction close to the critical angle.

Since $\cosh\beta \neq 0$, we have from (13.8) that $\cos\alpha = 0$, or that $\alpha = \frac{\pi}{2}$, implying that (13.9) gives:

$$\begin{aligned}
\sinh\beta = \frac{1}{2}(e^{\beta} - e^{-\beta}) &= -A \\
&\Downarrow \\
e^{2\beta} + 2Ae^{\beta} - 1 &= 0 \\
&\Downarrow \\
e^{\beta} = \frac{-2A \pm \sqrt{4A^2+4}}{2} &= -A \pm \sqrt{A^2+1}.
\end{aligned} \tag{13.10}$$

Since $e^{\beta} > 0$, the lower sign must be disregarded, so that we have

$$\begin{aligned}
e^{\beta} &= \sqrt{A^2+1} - A \\
&\Downarrow \\
\beta &= \ln\left[\sqrt{A^2+1} - A\right] \\
&\Downarrow \\
\beta &= \ln\left[\frac{\sin\theta_i}{n} - \sqrt{\left(\frac{\sin\theta_i}{n}\right)^2 - 1}\right].
\end{aligned} \tag{13.11}$$

Part II

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Chapter 14

Reflection and refraction of a plane acoustical wave

Consider two media that are separated by a plane interface and that have densities ρ_1 and ρ_2 and sound velocities v_1 and v_2. In linear acoustics the sound pressure $p(\mathbf{r},t)$ is the solution of the wave equation

$$(\bigtriangledown^2 - \frac{1}{v^2}\frac{\partial^2}{\partial t^2})p(\mathbf{r},t) = 0. \tag{14.1}$$

Therefore, a plane time-harmonic acoustical wave can be expressed as follows

$$p(\mathbf{r},t) = \mathrm{Re}\{p(\mathbf{r})e^{-i\omega\mathbf{t}}\} \tag{14.2}$$

where

$$p(\mathbf{r}) = p_0 e^{i\mathbf{k}\cdot\mathbf{r}}. \tag{14.3}$$

Here the amplitude p_0 is a constant. A plane harmonic pressure wave is incident upon a plane interface as illustrated in Fig. 14.1. The incident wave $p^i(\mathbf{r})$ is given by

$$p^i(\mathbf{r}) = p_0^i e^{i\mathbf{k}^i\cdot\mathbf{r}} \quad ; \quad \mathbf{k}^i = k_x^i\hat{\mathbf{e}}_x + k_z^i\hat{\mathbf{e}}_z. \tag{14.4}$$

The incident wave gives rise to a reflected plane wave and a transmitted plane wave of the same frequency, i.e.

$$\begin{aligned}
p^r(\mathbf{r},t) &= \mathrm{Re}\{p^r(\mathbf{r})e^{-i\omega\mathbf{t}}\} \\
p^r(\mathbf{r}) &= p_0^r e^{i\mathbf{k}^r\cdot\mathbf{r}} \quad ; \quad \mathbf{k}^r = k_x^r\hat{\mathbf{e}}_x + k_y^r\hat{\mathbf{e}}_y + k_z^r\hat{\mathbf{e}}_z \\
p^t(\mathbf{r},t) &= \mathrm{Re}\{p^t(\mathbf{r})e^{-i\omega\mathbf{t}}\} \\
p^t(\mathbf{r}) &= p_0^t e^{i\mathbf{k}^t\cdot\mathbf{r}} \quad ; \quad \mathbf{k}^t = k_x^t\hat{\mathbf{e}}_x + k_y^t\hat{\mathbf{e}}_y + k_z^t\hat{\mathbf{e}}_z.
\end{aligned} \tag{14.5}$$

The particle velocity $\mathbf{v}$ is given by:

$$\mathbf{v} = \mathrm{Re}\{\mathbf{v}^q(\mathbf{r})e^{-i\omega\mathbf{t}}\} \tag{14.6}$$

where

$$\mathbf{v}^q(\mathbf{r}) = \frac{1}{i\omega\rho^q}\bigtriangledown p^q \quad ; \quad (q = i,r,t) \tag{14.7}$$

with $\rho^i = \rho^r = \rho_1$ and $\rho^t = \rho_2$.

The boundary conditions that must be satisfied at the interface $z = 0$, are that the pressure p and the component of the particle velocity normal to the interface, i.e. $\mathbf{v}\cdot\hat{\mathbf{e}}_z$, both must be continuous across the interface, i.e.

$$\begin{aligned}
\left[p^i + p^r - p^t\right]_{z=0} &= 0 \\
\left[(\mathbf{v}^i + \mathbf{v}^r - \mathbf{v}^t)\cdot\hat{\mathbf{e}}_z\right]_{z=0} &= 0.
\end{aligned} \tag{14.8}$$

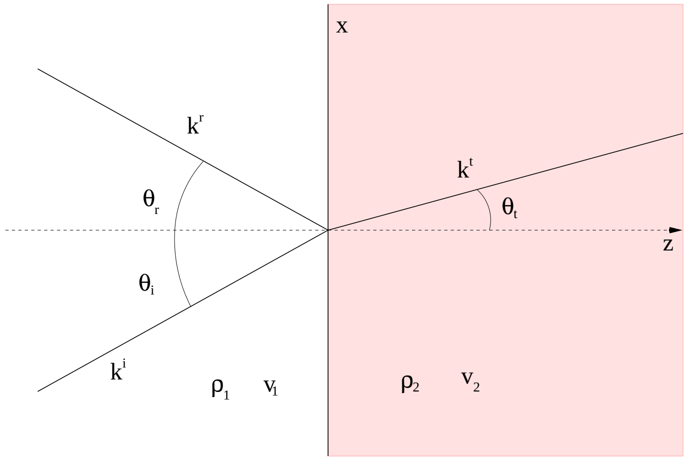

Figure 14.1: Incident, reflected, and transmitted wave.

14.1 Snell's law and the reflection law

Derive Snell's law and the reflection law.

Solution: For the boundary conditions in (14.8) to be valid the phase of the different terms must be equal, i.e.

$$\left[i\mathbf{k}^i \cdot \mathbf{r}\right]_{z=0} = \left[i\mathbf{k}^r \cdot \mathbf{r}\right]_{z=0} = \left[i\mathbf{k}^t \cdot \mathbf{r}\right]_{z=0} \tag{14.9}$$

or

$$k_x^i x\hat{\mathbf{e}}_x = k_x^r x\hat{\mathbf{e}}_x + k_y^r y\hat{\mathbf{e}}_y = k_x^t x\hat{\mathbf{e}}_x + k_y^t y\hat{\mathbf{e}}_y. \tag{14.10}$$

Because the condition in (14.10) must be satisfied for all values of x and y, we must have that $k_y^r = k_y^t = 0$, and hence that $k_x^i = k_x^r = k_x^t$. Thus, $\mathbf{k}^i$, $\mathbf{k}^r$, and $\mathbf{k}^t$ must lie in the same plane (the plane of incidence), and the projection of $\mathbf{k}^q$ onto the interface must be the same for $q = i$, $q = r$, and $q = t$.

Reflection law:

$$k_x^i = k_x^r \Rightarrow \theta^i = \theta^r. \tag{14.11}$$

Snell's law:

$$k_x^i = k_x^t \Rightarrow k_1 \sin\theta_i = k_2 \sin\theta_t \Rightarrow n_1 \sin\theta_i = n_2 \sin\theta_t. \tag{14.12}$$

14.2 Reflection and transmission coefficients

Determine the reflection coefficient

$$R = \frac{p_0^r}{p_0^i} \tag{14.13}$$

and the transmission coefficient

$$T = \frac{p_0^t}{p_0^i}. \tag{14.14}$$

Solution: The wave vectors are given by:

$$\begin{aligned}
\mathbf{k}^i &= k_x^i\hat{\mathbf{e}}_x + k_z^i\hat{\mathbf{e}}_z = k_x^i\hat{\mathbf{e}}_x + k_{z1}\hat{\mathbf{e}}_z \\
\mathbf{k}^r &= k_x^r\hat{\mathbf{e}}_x + k_z^r\hat{\mathbf{e}}_z = k_x^r\hat{\mathbf{e}}_x - k_{z1}\hat{\mathbf{e}}_z \\
\mathbf{k}^t &= k_x^t\hat{\mathbf{e}}_x + k_z^t\hat{\mathbf{e}}_z = k_x^t\hat{\mathbf{e}}_x + k_{z2}\hat{\mathbf{e}}_z.
\end{aligned} \tag{14.15}$$

By use of the result from Exercise 14.1 we have

$$\left[p^i + p^r - p^t\right]_{z=0} = 0 \quad \Rightarrow \quad p_0^i + p_0^r - p_0^t = 0 \tag{14.16}$$

$$\left[(\mathbf{v}^i + \mathbf{v}^r - \mathbf{v}^t) \cdot \hat{\mathbf{e}}_z\right]_{z=0} = 0 \quad \Rightarrow \quad \frac{1}{i\omega}\left[\frac{k_{z1}}{\rho_1}(p_0^i - p_0^r) - \frac{k_{z2}}{\rho_2}p_0^t\right] = 0. \tag{14.17}$$

or

$$\begin{aligned} 1 + R &= T \\ 1 - R &= \frac{k_{z2}}{k_{z1}}\frac{\rho_1}{\rho_2}T \end{aligned} \tag{14.18}$$

which have the solutions

$$T = \frac{2\rho_2 k_{z1}}{\rho_2 k_{z1} + \rho_1 k_{z2}} \tag{14.19}$$

$$R = \frac{\rho_2 k_{z1} - \rho_1 k_{z2}}{\rho_2 k_{z1} + \rho_1 k_{z2}}. \tag{14.20}$$

14.3 Comparison with electromagnetic waves

Compare the results with the reflection and transmission coefficients for a plane electromagnetic wave.

Solution: The reflection and transmission coefficients for an electromagnetic TE wave are given by:

$$T^{TE} = \frac{2\mu_2 k_{z1}}{\mu_2 k_{z1} + \mu_1 k_{z2}} \tag{14.21}$$

$$R^{TM} = \frac{\mu_2 k_{z1} - \mu_1 k_{z2}}{\mu_2 k_{z1} + \mu_1 k_{z2}}. \tag{14.22}$$

Thus, permeability μ plays precisely the same role here as the density ρ does in the acoustical case. The reflection and transmission coefficients for acoustical waves are the same as for electromagnetic TE waves when we replace the permeability μ in the latter coefficients with the density ρ.

Chapter 15

Fourier representation of a real function

Given the Fourier transform pair

$$\begin{aligned}\hat{f}(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\omega)e^{-i\omega t}d\omega \\ f(\omega) &= \int_{-\infty}^{\infty} \hat{f}(t)e^{i\omega t}dt \end{aligned} \tag{15.1}$$

show that when $\hat{f}(t)$ is real, we can express $\hat{f}(t)$ as follows

$$\hat{f}(t) = 2Re\left\{\frac{1}{2\pi}\int_{0}^{\infty} f(\omega)e^{-i\omega t}d\omega\right\}. \tag{15.2}$$

(Hint: Start by showing $f^*(\omega) = f(-\omega)$.)

Solution: Since $\hat{f}(t)$ is real, it follows that $\hat{f}(t) = \hat{f}(t)^*$, and hence

$$\begin{aligned}f^*(\omega) &= \int_{-\infty}^{\infty} \hat{f}(t)e^{-i\omega t}dt \\ f(-\omega) &= \int_{-\infty}^{\infty} \hat{f}(t)e^{-i\omega t}dt = f^*(\omega). \end{aligned} \tag{15.3}$$

In addition we have

$$\hat{f}(t) = \frac{1}{2\pi}\left\{\int_{-\infty}^{0} f(\omega)e^{-i\omega t}d\omega + \int_{0}^{\infty} f(\omega)e^{-i\omega t}d\omega\right\}. \tag{15.4}$$

In the first integral we let $\omega \to -\omega$, and thus we obtain

$$\int_{-\infty}^{0} f(\omega)e^{-i\omega t}d\omega = -\int_{\infty}^{0} f(-\omega)e^{i\omega t}d\omega = \int_{0}^{\infty} f^*(\omega)e^{i\omega t}d\omega = \left[\int_{0}^{\infty} f(\omega)e^{-i\omega t}d\omega\right]^*. \tag{15.5}$$

Thus, we have

$$\hat{f}(t) = \frac{1}{2\pi}\left\{\left[\int_{0}^{\infty} f(\omega)e^{-i\omega t}d\omega\right]^* + \int_{0}^{\infty} f(\omega)e^{-i\omega t}d\omega\right\} = 2Re\left\{\frac{1}{2\pi}\int_{0}^{\infty} f(\omega)e^{-i\omega t}d\omega\right\} \tag{15.6}$$

since $z + z^* = 2Re\{z\}$ holds for any arbitrary complex number.

Chapter 16

Convolution theorem, autocorrelation theorem, and Parseval's theorem

Let $G(k_x, k_y)$ and $g(x, y)$, and also $H(k_x, k_y)$ and $h(x, y)$ be Fourier transform pairs so that

$$\begin{aligned} A(k_x, k_y) &= \int\int_{-\infty}^{\infty} a(x, y) e^{-i(k_x x + k_y y)} dx dy \\ a(x, y) &= \left(\frac{1}{2\pi}\right)^2 \int\int_{-\infty}^{\infty} A(k_x, k_y) e^{i(k_x x + k_y y)} dk_x dk_y \end{aligned} \tag{16.1}$$

if we let a and A stand for either g and G or h and H.

16.1 Convolution

Prove the convolution theorem

$$\left(\frac{1}{2\pi}\right)^2 \int\int_{-\infty}^{\infty} G(k_x, k_y) H(k_x, k_y) e^{i(k_x x + k_y y)} dk_x dk_y = \int\int_{-\infty}^{\infty} g(x', y') h(x - x', y - y') dx' dy'. \tag{16.2}$$

Solution: We denote functions of (x, y) by lower case letters and their Fourirer transforms, which are functions of (k_x, k_y) by capital letters. Thus, the Fourier transform of a function $a(x, y)$, denoted by $A(k_x, k_y)$, is defined as follows

$$A(k_x, k_y) = \int\int_{-\infty}^{\infty} a(x, y) e^{-i(k_x x + k_y y)} dx dy = \mathcal{F}\{a(x, y)\}. \tag{16.3}$$

The inverse Fourier transform is given by

$$a(x, y) = \left(\frac{1}{2\pi}\right)^2 \int\int_{-\infty}^{\infty} A(k_x, k_y) e^{i(k_x x + k_y y)} dk_x dk_y = \mathcal{F}^{-1}\{A(k_x, k_y)\}. \tag{16.4}$$

The integral

$$f(x, y) = \int\int_{-\infty}^{\infty} a(x', y') b(x \pm x', y \pm y') dx' dy' \tag{16.5}$$

is a correlation or a convolution integral depending on whether we use the upper or lower sign on the right-hand side of (16.5). By expressing $b(x \pm x', y \pm y')$ in (16.5) by means of its Fourier transform, we find

$$f(x,y) = \int\int_{-\infty}^{\infty} a(x',y') \left\{ \left(\frac{1}{2\pi}\right)^2 \int\int_{-\infty}^{\infty} B(k_x,k_y) e^{i(k_x(x\pm x')+k_y(y\pm y'))} dk_x dk_y \right\} dx'dy'. \quad (16.6)$$

We exchange the order of the integrations to obtain

$$f(x,y) = \left(\frac{1}{2\pi}\right)^2 \int\int_{-\infty}^{\infty} B(k_x,k_y) \left\{ \int\int_{-\infty}^{\infty} a(x',y') e^{-i((\mp k_x)x'+(\mp k_y)y')} dx'dy' \right\} e^{i(k_x x+k_y y)} dk_x dk_y \quad (16.7)$$

or

$$f(x,y) = \left(\frac{1}{2\pi}\right)^2 \int\int_{-\infty}^{\infty} B(k_x,k_y) A(\mp k_x, \mp k_y) e^{i(k_x x+k_y y)} dk_x dk_y. \quad (16.8)$$

By combining (16.5) and (16.8) with $a = g$ and $b = h$, and using the lower sign in both equations, we obtain the convolution theorem:

$$\left(\frac{1}{2\pi}\right)^2 \int\int_{-\infty}^{\infty} G(k_x,k_y) H(k_x,k_y) e^{i(k_x x+k_y y)} dk_x dk_y = \int\int_{-\infty}^{\infty} g(x',y') h(x-x', y-y') dx'dy'. \quad (16.9)$$

16.2 Autocorrelation

Prove the autocorrelation theorem

$$\left(\frac{1}{2\pi}\right)^2 \int\int_{-\infty}^{\infty} |G(k_x,k_y)|^2 e^{i(k_x x+k_y y)} dk_x dk_y = \int\int_{-\infty}^{\infty} g(x',y') g^*(x'-x, y'-y) dx'dy'. \quad (16.10)$$

Solution: By choosing $a = g$ and $b = g^*(-x,-y)$ we obtain from (16.5) and (16.8) with the lower sign:

$$\int\int_{-\infty}^{\infty} g(x',y') g^*(x'-x, y'-y) dx'dy' = \left(\frac{1}{2\pi}\right)^2 \int\int_{-\infty}^{\infty} G(k_x,k_y) B(k_x,k_y) e^{i(k_x x+k_y y)} dk_x dk_y \quad (16.11)$$

where

$$B(k_x,k_y) = \mathcal{F}\{g^*(-x,-y)\} = \int\int_{-\infty}^{\infty} g^*(-x,-y) e^{-i(k_x x+k_y y)} dx dy. \quad (16.12)$$

By the change of variables $x \to -x$ and $y \to -y$, we get:

$$B(k_x,k_y) = \int\int_{-\infty}^{\infty} g^*(x,y) e^{i(k_x x+k_y y)} dx dy = \left[\int\int_{-\infty}^{\infty} g(x,y) e^{-i(k_x x+k_y y)} dx dy\right]^* = [G(k_x,k_y)]^* \quad (16.13)$$

which upon substitution in (16.10) gives the autocorrelation theorem:

$$\left(\frac{1}{2\pi}\right)^2 \int\int_{-\infty}^{\infty} |G(k_x,k_y)|^2 e^{i(k_x x+k_y y)} dk_x dk_y = \int\int_{-\infty}^{\infty} g(x',y') g^*(x'-x, y'-y) dx'dy'. \quad (16.14)$$

16.3 Parseval's theorem

Prove Parseval's theorem

$$\left(\frac{1}{2\pi}\right)^2 \int\int_{-\infty}^{\infty} |G(k_x, k_y)|^2 dk_x dk_y = \int\int_{-\infty}^{\infty} |g(x, y)|^2 dxdy. \tag{16.15}$$

Solution: The desired result follows from the autocorrelation theorem by setting $x = y = 0$ in (16.14).

Chapter 17

Angular-spectrum representation of a spherical wave (Weyl's formula)

My work has always tried to unite the true with the beautiful and when I had to choose one or the other, I usually chose the beautiful. Hermann Weyl.

The field associated with a diverging spherical wave with centre at the origin is given by

$$u(x,y,z) = \frac{e^{ikR}}{R} \quad ; \quad R = \sqrt{x^2+y^2+z^2}. \tag{17.1}$$

In the plane $z = 0$ we have

$$u(x,y,0) = \frac{e^{ik\sqrt{x^2+y^2}}}{\sqrt{x^2+y^2}}. \tag{17.2}$$

According to the angular-spectrum representation, we have

$$u(x,y,z) = \left(\frac{1}{2\pi}\right)^2 \int\int_{-\infty}^{\infty} U(k_x,k_y) e^{i(k_x x + k_y y + k_z |z|)} dk_x dk_y \tag{17.3}$$

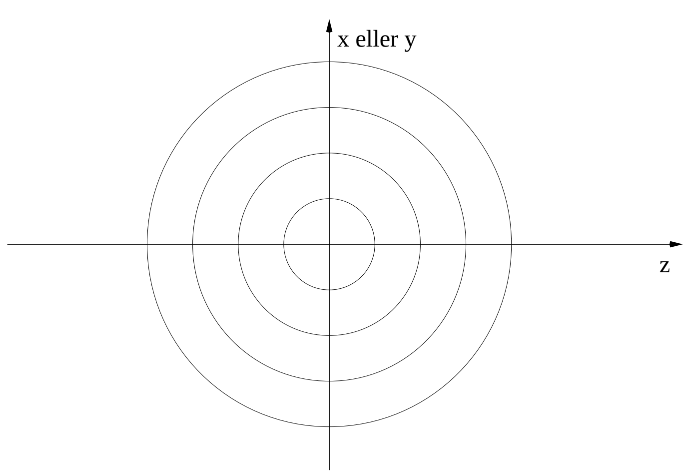

Figure 17.1: Wave fronts of a diverging spherical wave.

where the angular spectrum $U(k_x, k_y)$ is given by

$$U(k_x, k_y) = \int\int_{-\infty}^{\infty} u(x, y, 0) e^{-i(k_x x + k_y y)} dx dy \tag{17.4}$$

and where

$$k_z = \begin{cases} \sqrt{k^2 - k_x^2 - k_y^2} & \text{for} \quad k^2 > k_x^2 + k_y^2 \\ i\sqrt{k_x^2 + k_y^2 - k^2} & \text{for} \quad k^2 < k_x^2 + k_y^2. \end{cases} \tag{17.5}$$

17.1 The angular spectrum

Show that the angular spectrum of a spherical wave can be written

$$U(k_x, k_y) = 2\pi \int_0^{\infty} e^{ikt} J_0(q, t) dt \tag{17.6}$$

where

$$q^2 = k_x^2 + k_y^2. \tag{17.7}$$

Solution: Substituting (17.2) in (17.4), we have

$$U(k_x, k_y) = \int\int_{-\infty}^{\infty} \frac{e^{ik\sqrt{x^2+y^2}}}{\sqrt{x^2+y^2}} e^{-i(k_x x + k_y y)} dx dy. \tag{17.8}$$

We make the change of integration variables

$$\begin{aligned} x &= r\cos\phi \\ y &= r\sin\phi \\ dxdy &= rdrd\phi \\ \sqrt{x^2+y^2} &= r \end{aligned} \tag{17.9}$$

and let

$$\begin{aligned} k_x &= q\cos\psi \\ k_y &= q\sin\psi \\ \sqrt{k_x^2+k_y^2} &= q \end{aligned} \tag{17.10}$$

to obtain

$$\begin{aligned} U(k_x, k_y) &= \int_0^{\infty} e^{ikr} \int_0^{2\pi} e^{-iqr\cos(\phi-\psi)} d\phi dr \\ &= 2\pi \int_0^{\infty} e^{ikr} J_0(qr) dr. \end{aligned} \tag{17.11}$$

17.2 Weyl's plane-wave expansion of a spherical wave

Use the formulas

$$\int_0^{\infty} J_0(ax)\sin(bx) dx = \begin{cases} 0 & 0 < b < a \\ \frac{1}{\sqrt{b^2-a^2}} & 0 < a < b \end{cases} \tag{17.12}$$

and

$$\int_0^\infty J_0(ax)\cos(bx)dx = \begin{cases} \frac{1}{\sqrt{a^2-b^2}} & 0 < b < a \\ \infty & a = b \\ 0 & 0 < a < b \end{cases} \tag{17.13}$$

to show th at $U(k_x, k_y)$ in (17.6) can be written

$$U(k_x, k_y) = \frac{2\pi i}{k_z}. \tag{17.14}$$

Use this result to find a plane-wave expansion of a spherical wave.

Solution: We start by rewriting (17.11) as follows:

$$U(k_x, k_y) = 2\pi \left\{ \int_0^\infty J_0(qt)\cos(kt)dt + i\int_0^\infty J_0(qt)\sin(kt)dt \right\}. \tag{17.15}$$

i) Consider first the case in which $k_z^2 > 0$. Then we have $k = \sqrt{k_x^2 + k_y^2 + k_z^2} = \sqrt{q^2 + k_z^2} > q$, so that (17.12) and (17.13) give:

$$\int_0^\infty J_0(qt)\cos(kt)dt = 0 \tag{17.16}$$

$$\int_0^\infty J_0(qt)\sin(kt)dt = \frac{1}{\sqrt{k^2 - q^2}} = \frac{1}{\sqrt{k^2 - k_x^2 - k_y^2}} = \frac{1}{k_z} \tag{17.17}$$

where we have used (17.5). Substitution of these results in (17.15) gives

$$U(k_x, k_y) = \frac{2\pi i}{k_z}. \tag{17.18}$$

ii) Consider next the case in which $k_z^2 < 0$. Then $k = \sqrt{q^2 - |k_z|^2} < q$, so that (17.12) and (17.13) give [cf. (17.5)]

$$\int_0^\infty J_0(qt)\cos(kt)dt = \frac{1}{\sqrt{q^2 - k^2}} = \frac{1}{\sqrt{k_x^2 + k_y^2 - k^2}} = \frac{i}{i\sqrt{k_x^2 + k_y^2 - k^2}} = \frac{i}{k_z} \tag{17.19}$$

$$\int_0^\infty J_0(qt)\sin(kt)dt = 0 \tag{17.20}$$

which upon substitution in (17.15) give

$$U(k_x, k_y) = \frac{2\pi i}{k_z}. \tag{17.21}$$

Substitution of (17.21) in 17.3 gives

$$u(x,y,z) = \left(\frac{1}{2\pi}\right)^2 \int\int_{-\infty}^\infty \frac{2\pi i}{k_z} e^{i(k_x x + k_y y + k_z|z|)} dk_x dk_y = \frac{i}{2\pi}\int\int_{-\infty}^\infty \frac{e^{i(k_x x + k_y y + k_z|z|)}}{k_z} dk_x dk_y \tag{17.22}$$

which is Weyl's plane-wave expansion of a spherical wave $\exp\{ikR\}/R$.

Chapter 18

The Airy diffraction pattern

Show that

$$\int_0^1 J_0(vt)tdt = \frac{1}{2}\left(\frac{2J_1(v)}{v}\right). \tag{18.1}$$

Hint: The following recursion formulas apply to Bessel functions:

$$\begin{aligned} \frac{d}{dx}\left\{x^{n+1}J_{n+1}(x)\right\} &= x^{n+1}J_n(x) \\ \frac{d}{dx}\left\{x^{-n}J_n(x)\right\} &= -x^{-n}J_{n+1}(x). \end{aligned} \tag{18.2}$$

Solution: We are to show that $I = \int_0^1 J_0(vt)tdt = \frac{1}{2}\left(\frac{2J_1(v)}{v}\right)$. By the change of integration variable $x = vt$ we get:

$$I = \frac{1}{v^2}\int_0^v xJ_0(x)dx. \tag{18.3}$$

From the first of the two recursion relations with $n = 0$ it follows that: $[xJ_1(x)]' = xJ_0(x)$, so that we get:

$$I = \frac{1}{v^2}\left[xJ_1(x)\right]_0^v = \frac{J_1(v)}{v} = \frac{1}{2}\left(\frac{2J_1(v)}{v}\right). \tag{18.4}$$

Note: $\lim_{v\to 0}\frac{2J_1(v)}{v} = 1$.

Chapter 19

Integrated energy of the Airy diffraction pattern

19.1 Total differential for $\frac{J_1^2(x)}{x}$

Use the recursion formulas in Exercise 18 with $n = 0$ to show that

$$\frac{J_1^2(x)}{x} = -\frac{1}{2}\frac{d}{dx}[J_0^2(x) + J_1^2(x)]. \tag{19.1}$$

Solution: From $\frac{d}{dx}\left\{x^{n+1}J_{n+1}(x)\right\} = x^{n+1}J_n(x)$ with $n = 0$ we get:

$$\frac{d}{dx}\left\{xJ_1(x)\right\} = xJ_0(x) = J_1(x) + xJ_1'(x). \tag{19.2}$$

We multiply by $\frac{J_1}{x}$ on both sides of (19.2) to obtain

$$\frac{J_1^2(x)}{x} = -J_1(x)J_1'(x) + J_0(x)J_1(x). \tag{19.3}$$

From the recurrence relation $\frac{d}{dx}\left\{x^{-n}J_n(x)\right\} = -x^{-n}J_{n+1}(x)$ with $n = 0$ we get $J_0'(x) = -J_1(x)$, which upon substitution in (19.3) gives:

$$\begin{aligned}\frac{J_1^2(x)}{x} &= -J_1(x)J_1'(x) - J_0(x)J_0'(x) \\ &= -\frac{1}{2}\frac{d}{dx}[J_0^2(x) + J_1^2(x)].\end{aligned} \tag{19.4}$$

19.2 Encircled energy

Fraunhofer diffraction through a circular aperture gives rise to the Airy diffraction pattern, so that the intensity is given by

$$I(v) = C\left[\frac{2J_1(v)}{v}\right]^2 \tag{19.5}$$

where C is a constant and where v is a dimensionless co-ordinate given by

$$v = k\frac{a}{z_2}r. \tag{19.6}$$

Here r is the distance from the optical axis $z = 0$ to the observation point. The integrated energy $E(v_0)$ inside a circle with dimensionless radius v_0 is given by the integral of $I(v)$ over the circular area, i.e.

$$E(v_0) = \int_0^{v_0} \int_0^{2\pi} I(v) v dv d\psi = 2\pi \int_0^{v_0} I(v) v dv. \tag{19.7}$$

The ratio of $E(v_0)$ to the total energy is called the *encircled energy*, and is given by

$$L(v_0) = \frac{E(v_0)}{E(\infty)} = \frac{\int_0^{v_0} \frac{J_1^2(v)}{v} dv}{\int_0^{\infty} \frac{J_1^2(v)}{v} dv}. \tag{19.8}$$

Use the result from (19.1) and the limiting values:

$$\begin{aligned} J_0(0) &= 1 \\ J_1(0) &= J_1(\infty) = J_0(\infty) = 0 \end{aligned} \tag{19.9}$$

to show that

$$L(v_0) = 1 - J_0^2(v_0) - J_1^2(v_0). \tag{19.10}$$

Solution: Apart from a factor $2\pi C$, which is the same in the numerator and denominator in (19.8), we have

$$\begin{aligned} E(v_0) &= \int_0^{v_0} \frac{J_1^2(v)}{v} dv \\ &= -\frac{1}{2} [J_0^2(x) + J_1^2(x)]_0^{v_0} \\ &= -\frac{1}{2} \left\{ J_0^2(v_0) + J_1^2(v_0) - 1 \right\}. \end{aligned} \tag{19.11}$$

Disregarding again the factor $2\pi C$, we have $E(\infty) = \frac{1}{2}$, so that

$$L(v_0) = \frac{E(v_0)}{E(\infty)} = 1 - J_0^2(v_0) - J_1^2(v_0). \tag{19.12}$$

Chapter 20

Fresnel diffraction through an infinitely large circular aperture

From equation (11.2.7) in the lecture notes [**?**] we have for the field diffracted through a circular aperture:

$$u_I = C \int_0^1 J_0(vt) e^{i\frac{1}{2}ut^2} t dt \tag{20.1}$$

where

$$\begin{aligned} v &= k\frac{a}{z_2} r \\ u &= k\frac{a^2}{z_2} \\ C &= \frac{2\pi a^2}{i\lambda z_2} e^{i\phi} \\ \phi &= k(z_2 + \frac{r^2}{2z_2}). \end{aligned} \tag{20.2}$$

Here a is the aperture radius, and the incident field is a normally incident plane wave, as illustrated in Fig. 20.1.

20.1 Change of variable: $vt \rightarrow x$

Show that u_I can be expressed as:

$$u_I = \frac{z_2 e^{i\phi}}{ikr^2} \int_0^v J_0(x) e^{iBx^2} x dx \tag{20.3}$$

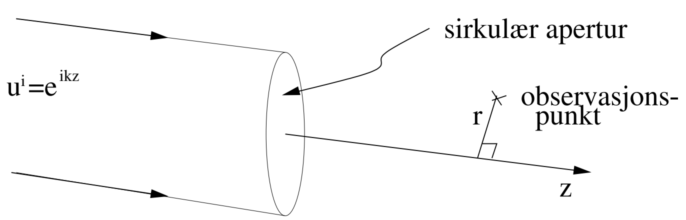

Figure 20.1: Diffraction trough a circular aperture.

where

$$B = \frac{z_2}{2kr^2}. \tag{20.4}$$

Solution: By the change of integration variable $vt = x$ we get from (20.1):

$$u_I = \frac{C}{v^2}\int_0^v J_0(x)e^{i\frac{1}{2}u\frac{x^2}{v^2}}xdx. \tag{20.5}$$

Since

$$\begin{aligned} \frac{C}{v^2} &= e^{i\phi}\frac{2\pi a^2}{i\lambda z_2}\frac{z_2^2}{k^2a^2r^2} = e^{i\phi}\frac{z_2}{ikr^2} \\ \frac{u}{v^2} &= \frac{k\frac{a^2}{z_2}}{\left(k\frac{a}{z_2}r\right)^2} = \frac{z_2}{kr^2} \end{aligned} \tag{20.6}$$

we get from (20.5)

$$u_I = \frac{z_2e^{i\phi}}{ikr^2}\int_0^v J_0(x)e^{iBx^2}xdx \quad ; \quad B = \frac{z_2}{2kr^2}. \tag{20.7}$$

20.2 Infinitely large aperture

When the aperture radius a becomes infinitely large, we get $v = k\frac{a}{z_2}r \to \infty$, so that

$$u_I = \frac{z_2e^{i\phi}}{ikr^2}\int_0^\infty J_0(x)e^{iBx^2}xdx. \tag{20.8}$$

Use integration by parts together with the results

$$\begin{aligned} \int_0^\infty \sin(ax^2)J_1(bx)dx &= \frac{1}{b}\sin\left(\frac{b^2}{4a}\right) \\ \int_0^\infty \cos(ax^2)J_1(bx)dx &= \frac{2}{b}\sin^2\left(\frac{b^2}{8a}\right) \end{aligned} \tag{20.9}$$

to show that (20.8) can be written

$$u_I = e^{ikz_2}. \tag{20.10}$$

What is the physical interpretation of this result?

Solution: From (20.8) we have

$$u_I = \frac{z_2e^{i\phi}}{ikr^2}I \quad ; \quad I = \int_0^v J_0(x)\left(e^{iBx^2}x\right)dx. \tag{20.11}$$

We let $u = J_0$, $v' = xe^{iBx^2} \Rightarrow v = \frac{e^{iBx^2}}{2iB}$, and use integration by parts $\int uv'dx = uv - \int u'vdx$, to obtain

$$I = J_0(x)\frac{e^{iBx^2}}{2iB}\Bigg|_0^\infty - \int_0^\infty J_0'(x)\frac{e^{iBx^2}}{2iB}dx. \tag{20.12}$$

Since $J_0(0) = 1$, $J_0(\infty) = 0$ and $J_0'(x) = -J_1(x)$, we get

$$I = \frac{-1}{2iB}\left\{1 - \int_0^\infty J_1(x)(\cos(Bx^2) + i\sin(Bx^2))dx\right\} \tag{20.13}$$

which by use of the formulas in (20.9) give:

$$I = \frac{i}{2B}\left\{1 - 2\sin^2\left(\frac{1}{2}\cdot\frac{1}{4B}\right) - i\sin\left(\frac{1}{4B}\right)\right\} \quad ; \quad \frac{1}{4B} = \frac{kr^2}{2z_2}. \tag{20.14}$$

Since $2\sin^2\frac{x}{2} = 1 - \cos x$, we get

$$\begin{aligned} I &= \frac{i}{2B}\left(\cos\left(\frac{1}{4B}\right) - i\sin\left(\frac{1}{4B}\right)\right) = \frac{i}{2B}\left(\cos\left(\frac{-1}{4B}\right) + i\sin\left(\frac{-1}{4B}\right)\right) \\ &= \frac{i}{2B}e^{-i\frac{1}{4B}} = \frac{2i}{4B}e^{-i\frac{1}{4B}} = 2i\frac{kr^2}{2z_2}e^{-i\frac{kr^2}{2z_2}} = \frac{ikr^2}{z_2}e^{-i\frac{kr^2}{2z_2}}. \end{aligned} \tag{20.15}$$

Substitution for I from (20.15) and for ϕ from (20.2) in (20.11) gives

$$u_I = \frac{z_2 e^{ik(z_2 + \frac{r^2}{2z_2})}}{ikr^2}\frac{ikr^2}{z_2}e^{-i\frac{kr^2}{2z_2}} = e^{ikz_2}. \tag{20.16}$$

The physical interpretation of this result is that when the aperture is infinitely large, the incident plane propagates along without obstruction of any kind.

Chapter 21

Diffraction by a half-plane

The diffracted field resulting when a plane wave is normally incident upon the edge of a half-plane, is given in terms of so-called detour parameters. The detour parameter ξ^i associated with the incident wave is defined such that $(\xi^i)^2$ is the difference between the phase measured along the diffracted ray and along the incident ray. Also, ξ^i is defined such that $\xi^i > 0$ in the shadow zone of the incident wave, whereas $\xi^i < 0$ in the lit area. From this definition and Fig. 21.1, we have

$$(\xi^i)^2 = k(s - D) \tag{21.1}$$
$$\operatorname{sgn}(\xi^i) = \operatorname{sgn}(\theta_0 - \theta). \tag{21.2}$$

21.1 Detour parameter associated with the incident wave

Show from (21.1) and (21.2) that

$$\xi^i = -\sqrt{2ks}\sin\frac{1}{2}(\theta - \theta_0). \tag{21.3}$$

Solution: From (21.1) and Fig. (21.1) it follows that

$$(\xi^i)^2 = ks(1 - \frac{D}{s}) \tag{21.4}$$

or

$$(\xi^i)^2 = ks(1 - \cos(\theta - \theta_0)) \tag{21.5}$$

which by the use of the formula $\sin\frac{x}{2} = \sqrt{\frac{1-\cos x}{2}}$ gives

$$(\xi^i)^2 = 2ks\sin^2\frac{1}{2}(\theta - \theta_0). \tag{21.6}$$

By use of (21.2) we therefore get

$$\xi^i = -\sqrt{2ks}\sin\frac{1}{2}(\theta - \theta_0). \tag{21.7}$$

21.2 Detour parameter associated with the reflected wave

The detour parameter ξ^r associated with the reflected wave is defined such that $(\xi^r)^2$ is the difference between the phase measured along the diffracted ray and along the reflected ray. Also, ξ^r is defined

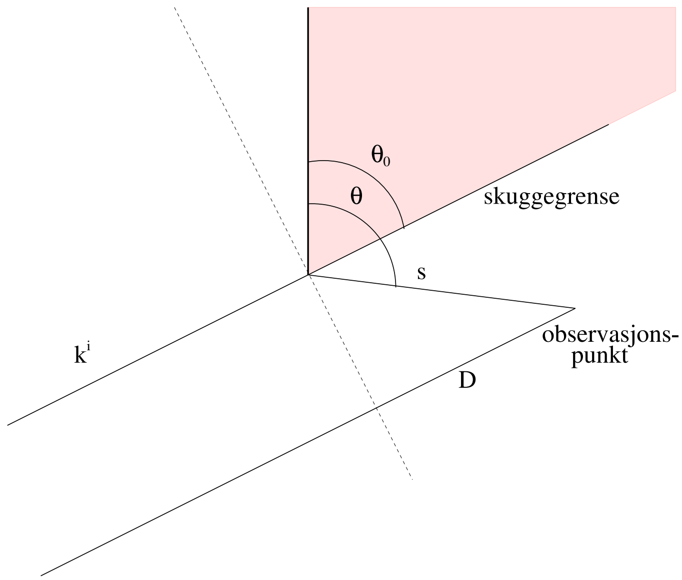

Figure 21.1: Detour parameter associated with the incident wave.

such that $\xi^r > 0$ in the shadow zone of the reflected wave, whereas $\xi^r < 0$ in the area that is lit by the reflected wave. Draw a figure and show that

$$\xi^r = \sqrt{2ks}\sin\frac{1}{2}(\theta + \theta_0). \tag{21.8}$$

Solution: From Fig. 21.2 it follows that

$$(\xi^r)^2 = k(s - D) \tag{21.9}$$

or

$$(\xi^r)^2 = ks(1 - \frac{D}{s}). \tag{21.10}$$

Also, we see from Fig. 21.2 that

$$\frac{D}{s} = cos\alpha \tag{21.11}$$

and that

$$\alpha + \beta = \theta_0. \tag{21.12}$$

Since $\beta = 2\pi - \theta$, we get

$$\begin{aligned} \alpha + 2\pi - \theta &= \theta_0 \\ &\Downarrow \\ \alpha &= \theta_0 + \theta - 2\pi \end{aligned} \tag{21.13}$$

so that

$$(\xi^r)^2 = ks(1 - \cos(\theta + \theta_0 - 2\pi)) = ks(1 - \cos(\theta + \theta_0)) = 2ks\sin^2\frac{1}{2}(\theta + \theta_0). \tag{21.14}$$

By definition $\xi^r > 0$ in the shadow zone of the reflected wave, i.e. $\xi^r > 0$ when $\theta < 2\pi - \theta_0$ (see Fig. 21.2) or when $\frac{1}{2}(\theta + \theta_0) < \pi$. Thus, we have $\text{sgn}(\xi^r) = \text{sgn}[\sin\frac{1}{2}(\theta + \theta_0)]$, and hence

$$\xi^r = \sqrt{2ks}\sin\frac{1}{2}(\theta + \theta_0). \tag{21.15}$$

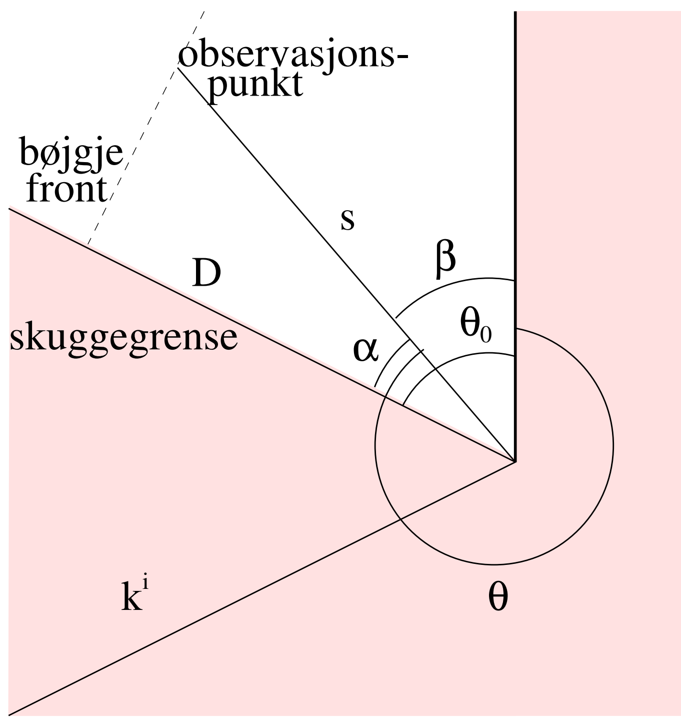

Figure 21.2: Detour parameter for the reflected wave.

Chapter 22

Diffraction through a circular aperture – axial intensity

When a plane wave is normally incident upon a circular aperture, the intensity in the diffraction pattern is given by

$$I = \left(\frac{\pi a^2}{\lambda z_2}\right)^2 \left|2\int_0^1 J_0(vt)e^{i\frac{1}{2}ut^2}tdt\right|^2. \tag{22.1}$$

Show that along the axis $v = 0$ the intensity becomes

$$I = 4\sin^2(\pi a^2/2\lambda z_2). \tag{22.2}$$

Sketch the axial intensity as a function of z_2 in units of $a^2/2\lambda$. Give a physical interpretation of this axial interference pattern and explain why the minimum intensity is zero and the maximum intensity has the value of 4.

Solution: Since $u = ka^2/z_2 = 2\pi a^2/\lambda z_2$, the given expression for the intensity distribution becomes

$$I = \left|u\int_0^1 J_0(vt)e^{i\frac{1}{2}ut^2}tdt\right|^2. \tag{22.3}$$

Along the axis $v = 0$ we have (since $J_0(0) = 1$):

$$A = u\int_0^1 J_0(vt)e^{i\frac{1}{2}ut^2}tdt = u\int_0^1 e^{i\frac{1}{2}ut^2}tdt. \tag{22.4}$$

We let $\frac{1}{2}t^2 = x$, so that $tdt = dx$ and we get:

$$\begin{aligned} A &= u\int_0^{\frac{1}{2}} e^{iux}dx = u\frac{e^{iux}}{iu}\bigg|_0^{\frac{1}{2}} = 2\frac{e^{i\frac{u}{2}}-1}{2i} \\ &= 2e^{i\frac{u}{4}}\frac{e^{i\frac{u}{4}}-e^{i\frac{-u}{4}}}{2i} = 2e^{i\frac{u}{4}}\sin\left(\frac{u}{4}\right). \end{aligned} \tag{22.5}$$

Thus, the axial intensity becomes

$$I = 4\sin^2(u/4) = 4\sin^2(\pi a^2/2\lambda z_2). \tag{22.6}$$

The axial intensity has maxima when $\pi a^2/2\lambda z_2 = (\pi/2)m$, where $m = 1, 3, 5, \cdots$. Thus, we may write

$$z_{2max}^{(m)} = \frac{a^2}{\lambda}\frac{1}{m} \quad ; \quad m = 1, 3, 5, \cdots. \tag{22.7}$$

The axial intensity has zeros when $\pi a^2/2\lambda z_2 = \pi m$, where $m = 1, 2, 3, \cdots$. Thus, we may write

$$z_{2min}^{(m)} = \frac{a^2}{\lambda}\frac{1}{m} \quad ; \quad m = 2, 4, 6, \cdots . \tag{22.8}$$

When z_2 is expressed in units of a^2/λ, the intensity has maxima for $z_2 = 1, 1/3, 1/5, 1/7, \cdots$, and minima when $z_2 = 1/2, 1/4, 1/6, 1/8, \cdots$. As z_2 increases beyond the value of a^2/λ, the intensity decays monotonically to zero, and as z_2 decreases below the value of a^2/λ, the intensity oscillates faster and faster between the values of zero and 4. For an aperture radius of $a = 1$ mm and a wavelength of $\lambda = 0.5$ μm, we have $a^2/\lambda = 2$ m.

The physical interpretation of this axial interference pattern is that at any observation point on the axis the distance from the edge to the observation point is the same for all edge-diffracted rays. Hence the edge-diffracted waves interact to reinforce one another and the resulting wave due to all edge points is of the same intensity as the incident wave. This explains why we get zero minimum intensity at axial points where the edge-diffracted wave is out of phase with the incident wave (destructive interference). The incident plane wave is of unit amplitude, and since the total edge-diffracted wave has the same amplitude as the incident wave, we get a total amplitude of 2, and hence an intensity of 4, at axial points where the edge-diffracted wave is in phase with the incident wave (constructive interference).

Chapter 23

Poisson's spot

In the diffraction problem illustrated in Fig. 23.1 a point source in the half-space $z < 0$ radiates a field u^i which is incident upon an aperture A in the plane $z = 0$. When $kR_2 >> 1$, the diffracted field is given by Rayleigh-Sommerfeld's first diffraction integral, i.e.

$$u_I = \frac{1}{i\lambda}\int\int_A u^i(x,y,0)\frac{e^{ikR_2}}{R_2}\cdot\frac{z_2}{R_2}dxdy \tag{23.1}$$

where

$$R_2 = \sqrt{(x-x_2)^2+(y-y_2)^2+{z_2}^2}. \tag{23.2}$$

For a normally incident plane wave $u^i(x,y,0) = 1$, and we get

$$u_I = \frac{1}{i\lambda}\int\int_A \frac{e^{ikR_2}}{R_2}\frac{z_2}{R_2}dxdy. \tag{23.3}$$

If u^i is due to a point source at $(x_1, y_1, -z_1)$, we get

$$u_I = \frac{1}{i\lambda}\int\int_A \frac{e^{(ikR_1+R_2)}}{R_1R_2}\frac{z_2}{R_2}dxdy \tag{23.4}$$

where

$$R_1 = \sqrt{(x-x_1)^2+(y-y_1)^2+{z_1}^2}. \tag{23.5}$$

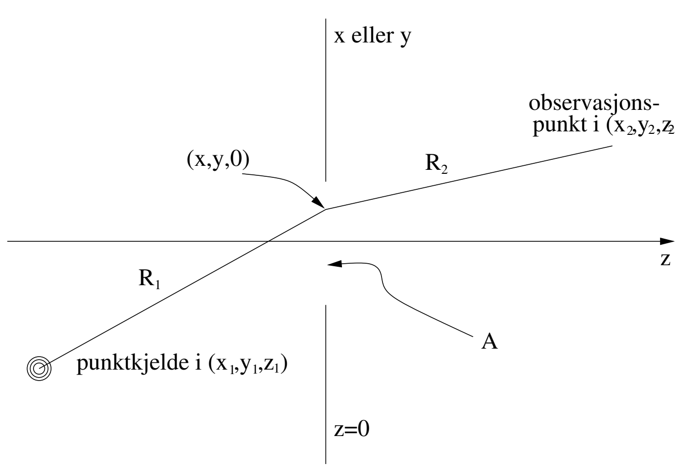

Figure 23.1: Diffracted field due to a point source.

23.1 Diffraction of a spherical wave through a circular aperture

In the lectures we have shown that when a normally incident plane wave is diffracted through a circular aperture, the first Rayleigh-Sommerfeld diffraction integral in (23.3) can be expressed as

$$u_{IA} = -iC_1 \frac{\pi a^2}{\lambda z_2} 2 \int_0^1 J_0(vt) e^{i\frac{1}{2}ut^2} t dt \tag{23.6}$$

where

$$\begin{aligned} v &= k\frac{a}{z2}r \\ u &= k\frac{a^2}{z_2} \\ C_1 &= e^{ik(z_2 + \frac{{x_2}^2 + {y_2}^2}{2z_2})}. \end{aligned} \tag{23.7}$$

Equation (23.6) applies provided that the paraxial approximation and the Fresnel approximation are satisfied. Let $x_1 = y_1 = 0$ and use the same approximations to show that (23.4) can be expressed as

$$u_{IA} = -iC_2 \frac{\pi a^2}{\lambda z_1 z_2} 2 \int_0^1 J_0(vt) e^{i\frac{1}{2}\bar{u}t^2} t dt \tag{23.8}$$

where

$$C_2 = C_1 e^{ikz_1} \quad ; \quad \bar{u} = ka^2 \left(\frac{1}{z_1} + \frac{1}{z_2} \right). \tag{23.9}$$

Solution: We use the paraxial approximation

$$\frac{z_2}{R_2} \approx 1 \quad ; \quad \frac{1}{R_1 R_2} \approx \frac{1}{z_1 z_2} \tag{23.10}$$

and the Fresnel approximation

$$R_j \approx z_j + \frac{1}{2}\frac{(x - x_j)^2 + (y - y_j)^2}{z_j} \quad ; \quad (j = 1, 2) \tag{23.11}$$

so that we get

$$R_j \approx z_j + \frac{{x_j}^2 + {y_j}^2}{2z_j} + \frac{x^2 + y^2}{2z_j} - \frac{xx_j + yy_j}{2z_j} \tag{23.12}$$

$$R_1 + R_2 \approx z_1 + z_2 + \frac{{x_1}^2 + {y_1}^2}{2z_1} + \frac{{x_2}^2 + {y_2}^2}{2z_2} + \frac{x^2 + y^2}{2z_1} + \frac{x^2 + y^2}{2z_2} - \frac{xx_1 + yy_1}{2z_1} - \frac{xx_2 + yy_2}{2z_2}. \tag{23.13}$$

We let the point source lie on the axis, so that $x_1 = y_1 = 0$, and we introduce polar co-ordinates:

$$\begin{aligned} x = \rho\cos\phi \quad &; \quad y = \rho\sin\phi \\ x_2 = r\cos\beta \quad &; \quad y_2 = r\sin\beta \end{aligned} \tag{23.14}$$

so that (23.4) gives:

$$u_I = \frac{C_2}{i\lambda z_1 z_2} \int_0^a \int_0^{2\pi} e^{-i\frac{k\rho r}{z_2}\cos(\phi - \beta)} d\phi e^{i\frac{k}{2}(\frac{1}{z_1} + \frac{1}{z_2})\rho^2} \rho d\rho \tag{23.15}$$

where

$$C_2 = C_1 e^{ikz_1}. \tag{23.16}$$

Letting $\rho = at$, we get

$$u_{IA} = -iC_1 \frac{\pi a^2}{\lambda z_1 z_2} 2 \int_0^1 J_0(vt) e^{i\frac{1}{2}\bar{u}t^2} t dt \tag{23.17}$$

where

$$C_2 = C_1 e^{ikz_1} \quad ; \quad \bar{u} = ka^2 \left(\frac{1}{z_1} + \frac{1}{z_2} \right) \tag{23.18}$$

which was to be proven.

23.2 Opaque disc

Let the incident field be a normally incident plane wave, and let the circular aperture be replaced by a circular disc. Show that the diffracted field then is given by

$$u_{IS} = e^{ikz_2} - u_{IA} \tag{23.19}$$

where u_{IA} is given by (23.6).

Solution: When we replace an aperture with an opaque disc of the same shape, we can use Babinet's principle, according to which $u_{IS} = u^i - u_{IA}$, where u_{IS} og u_{IA} are the diffracted fields due to the disc and the aperure, respectively. In the present case $u^i = e^{ikz_2}$, and therefore we obtain (23.19).

23.3 Proof of Babinet's principle

Let $\bar{A}$ be the aperture that is complementary to A, so that $\bar{A} + A$ covers the whole plane $z = 0$. Then we have

$$\begin{aligned}
u_{IA} &= -\frac{1}{2\pi}\int\int_A u^i \frac{\partial}{\partial z_2}\left(\frac{e^{ikR_2}}{R_2}\right) dxdy \\
u_{IS} &= -\frac{1}{2\pi}\int\int_{\bar{A}} u^i \frac{\partial}{\partial z_2}\left(\frac{e^{ikR_2}}{R_2}\right) dxdy \\
&= -\frac{1}{2\pi}\int\int_{-\infty}^{\infty} u^i \frac{\partial}{\partial z_2}\left(\frac{e^{ikR_2}}{R_2}\right) dxdy \\
&\quad -\left(-\frac{1}{2\pi}\int\int_A u^i \frac{\partial}{\partial z_2}\left(\frac{e^{ikR_2}}{R_2}\right) dxdy\right).
\end{aligned} \tag{23.20}$$

The first integral gives after the last equality sign in (23.20) gives u^i, because the integration is over the entire plane (i.e. no edge diffraction) and the final integral in (23.20) gives u_{IA}. Thus, we have Babinet's principle, $u_{IS} = u^i - u_{IA}$.

23.4 Axial field - incident plane wave

Show that on the axis $v = 0$ we get from (23.19)

$$u_{IS} = e^{ik\left(z_2 + \frac{r^2}{2z_2}\right)} \tag{23.21}$$

which implies that the intensity everywhere on the axis behind an opaque circular disc is the same as the intensity of the incident plane wave! This bright spot is *Poisson's spot*, named after Poisson who first predicted it on the basis of Fresnel's wave theory. Poisson used this bright spot as a proof against the wave theory. But when Arago soon afterwards carried out the experiment, he observed the bright spot on the axis.

Solution: On the axis $x_2 = y_2 = 0$ we have [see (23.6)]:

$$\begin{aligned}
-u_{IA} = +ie^{ikz_2}uJ \quad &; \quad u = \frac{2\pi}{\lambda}\frac{a^2}{z_2} \\
J = \int_0^1 e^{i\frac{1}{2}ut^2} dt \quad &; \quad \frac{1}{2}t^2 = x \Rightarrow tdt = dx
\end{aligned} \tag{23.22}$$

$$\Rightarrow \quad J = \int_0^{\frac{1}{2}} e^{iux} dx = \left. \frac{e^{iux}}{iu} \right|_0^{\frac{1}{2}} = \frac{e^{i\frac{u}{2}} - 1}{iu}$$

$$\Rightarrow \quad -u_{IA} = e^{ikz_2}(e^{i\frac{u}{2}} - 1). \tag{23.23}$$

Thus, we have from (23.19)

$$u_{IS} = e^{ikz_2} - u_{IA} = e^{ikz_2 + i\frac{u}{2}} = e^{ik(z_2 + \frac{a^2}{2z_2})}. \tag{23.24}$$

The intensity becomes

$$|u_{IS}|^2 = |u_I|^2. \tag{23.25}$$

23.5 Axial field - incident spherical wave

Repeat the derivations in Exercises 23.2 and 23.4 for an incident diverging spherical wave, so that (23.8) applies. Consider the special case in which $z_1 = z_2$ and try to simplify the result as much as possible.

Solution: According to Babinet's principle we have $u_{IS} = U^i - u_{IA}$. At the point $x_2 = y_2 = 0$ on the axis u^i is given by:

$$u^i = \frac{e^{ik(z_1+z_2)}}{z_1 + z_2} \tag{23.26}$$

and from (23.8) with $x_2 = y_2 = v = 0$ we get

$$-u_{IA} = ie^{ik(z_1+z_2)} \cdot k\frac{a^2}{z_1 z_2} \frac{e^{i\frac{\bar{u}}{2}} - 1}{i\bar{u}}. \tag{23.27}$$

Since $\bar{u} = ka^2(\frac{1}{z_1} + \frac{1}{z_2})$ [see (23.9)], we get

$$-u_{IA} = -u^i + \frac{e^{ik(z_1+z_2)+i\bar{u}/2}}{z_1 + z_2} \tag{23.28}$$

so that

$$u_{IS} = u^i - u_{IA} = \frac{\exp\left\{ik\left(z_1 + z_2\right)\left(1 + \frac{a^2}{2z_1 z_2}\right)\right\}}{z_1 + z_2}. \tag{23.29}$$

The on-axis intensity becomes

$$|u_{IS}|^2 = \left(\frac{1}{z_1 + z_2}\right)^2 = |u^i|^2. \tag{23.30}$$

Thus, on the axis behind a circular opaque disc the intensity is the same as if the disc were removed! If we let $z_1 = z_2$, we get

$$|u_{IS}|^2 = \frac{1}{4}\left(\frac{1}{z_1}\right)^2 \tag{23.31}$$

which implies that the intensity at this observation point on the axis is one quarter of the incident intensity at the center of the disc. When $z_1 >> z_2$,we get from (23.30)

$$|u_{IS}|^2 = \left(\frac{1}{z_1}\right)^2 \left(\frac{1}{1 + \frac{z_2}{z_1}}\right)^2 \approx \left(\frac{1}{z_1}\right)^2 \tag{23.32}$$

where $\left(\frac{1}{z_1}\right)^2$ is the intensity of the incident wave at the center of the disc, in accordance with the result from Exercise 23.4.

Chapter 24

Fraunhofer diffraction at oblique incidence and interference between the fields diffracted through two apertures

In the lectures we have shown that the diffracted field in the Fraunhofer zone is given by

$$u_I = \frac{C_1}{i\lambda z_2} \int \int_{\infty}^{\infty} u^i(x,y,0) t(x,y) e^{-i(k_x^0 x + k_y^0 y)} dx dy \tag{24.1}$$

where

$$C_1 = e^{ik\left(z_2 + \frac{{x_2}^2 + {y_2}^2}{2z_2}\right)} \quad ; \quad k_x^0 = \frac{kx_2}{z_2} \quad ; \quad k_y^0 = \frac{ky_2}{z_2} \tag{24.2}$$

and where u^i is the incident field and $t(x,y)$ is the transmission function of the aperture. Let us assume that $t(x,y)$ has the value 1 when (x,y) lies inside the aperture A and the value zero when (x,y) lies outside A.

24.1 Fourier representation at oblique incidence

Show that for an obliquely incident plane wave with wave vector

$$\mathbf{k}^i = {k_x}^i \hat{\mathbf{e}}_x + {k_y}^i \hat{\mathbf{e}}_y + {k_z}^i \hat{\mathbf{e}}_z \tag{24.3}$$

the diffracted field becomes

$$u_I = \frac{C_1}{i\lambda z_2} \int \int_A e^{-i(K_x x + K_y y)} dx dy \tag{24.4}$$

where

$$K_x = k_x^0 - k_x^i \quad ; \quad K_y = k_y^0 - k_y^i. \tag{24.5}$$

Solution: The obliquely incident plane is given by

$$u^i = e^{i\mathbf{k}\cdot\mathbf{r}} = e^{i({k_x}^i x + {k_y}^i y + {k_z}^i z)} \tag{24.6}$$

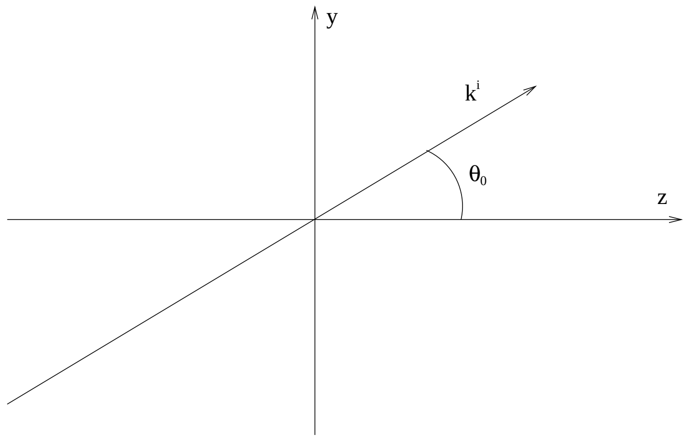

Figure 24.1: The incident plane wave makes an angle θ_0 with the z axis.

which upon substitution in (24.1) gives:

$$\begin{aligned} u_I &= \frac{C_1}{i\lambda z_2}\int\int_A e^{i(k_x{}^i x + k_y{}^i y)} e^{-i(k_x{}^0 x + k_y{}^0 x)} dxdy \\ &= \frac{C_1}{i\lambda z_2}\int\int_A e^{-i(K_x x + K_y y)} dxdy \end{aligned} \tag{24.7}$$

where

$$K_x = k_x{}^0 - k_x{}^i \quad ; \quad K_y = k_y{}^0 - k_y{}^i \tag{24.8}$$

which was to be shown.

24.2 Airy diffraction pattern at oblique incidence

Let $\mathbf{k}^i$ lie in the (y, z) plane and make an angle θ_0 with the positive z axis, and let the aperture be circular with radius a and with centre at $x = y = 0$, as shown in Fig. 24.1. Determine the diffracted field in the Fraunhofer zone. Simplify the expression as much as possible, and compare the result with that found previously for Fraunhofer diffraction through a circular aperture when the plane wave is normally incident.

Solution: From the information given in the Exercise and from (24.8) we have:

$$\begin{aligned} \mathbf{k}^i &= k\sin\theta_0 \hat{\mathbf{e}}_y + k\cos\theta_0 \hat{\mathbf{e}}_z \\ k_x{}^i &= 0 \\ K_x &= k_x{}^0 = \frac{kx_2}{z_2} \\ K_y &= k_y{}^0 - k_y{}^i = k\frac{y_2}{z_2} - k\sin\theta_0. \end{aligned} \tag{24.9}$$

We introduce polar co-ordinates

$$\begin{aligned} K_x &= K\cos\beta \\ K_y &= K\sin\beta \\ K^2 &= K_x{}^2 + K_y{}^2 = k^2\left[\left(\frac{x_2}{z_2}\right)^2 + \left(\frac{y_2}{z_2} - \sin\theta_0\right)^2\right] \\ x &= at\cos\phi \\ y &= at\sin\phi \\ dxdy &= a^2 t dt d\phi \end{aligned} \tag{24.10}$$

and get

$$K_x x + K_y y = Kat\cos(\phi - \beta) \tag{24.11}$$

which upon substitution in (24.4) gives

$$\begin{aligned} u_I &= \frac{C_1}{i\lambda z_2} 2\pi a^2 \int_0^1 \left\{ \frac{1}{2\pi} \int_0^{2\pi} e^{-i(Kat\cos(\phi-\beta))} d\phi \right\} tdt \\ &= \frac{C_1}{i\lambda z_2} 2\pi a^2 \int_0^1 J_0(Kat)tdt \\ &= \frac{C_1}{i\lambda z_2} 2\pi a^2 \int_0^1 J_0(\bar{v}t)tdt \end{aligned} \tag{24.12}$$

where

$$\bar{v} = Ka = ka\sqrt{\left(\frac{x_2}{z_2}\right)^2 + \left(\frac{y_2}{z_2} - \sin\theta_0\right)^2}. \tag{24.13}$$

Letting $\bar{v}t = x$, we get

$$I = \int_0^1 J_0(\bar{v}t)tdt = \frac{1}{\bar{v}^2}\int_0^{\bar{v}} J_0(x)xdx \tag{24.14}$$

which by use of the relation $[xJ_1(x)]' = xJ_0(x)$ gives

$$I = \frac{J_1(\bar{v})}{\bar{v}}. \tag{24.15}$$

Thus, the diffracted field becomes

$$u_I = \frac{C_1\pi a^2}{i\lambda z_2}\left(\frac{2J_1(\bar{v})}{\bar{v}}\right) \tag{24.16}$$

and the intensity distribution becomes

$$I = |u_I|^2 = \left(\frac{\pi a^2}{\lambda z_2}\right)^2 \left(\frac{2J_1(\bar{v})}{\bar{v}}\right)^2 \tag{24.17}$$

where

$$\bar{v} = Ka = ka\sqrt{\left(\frac{x_2}{z_2}\right)^2 + \left(\frac{y_2}{z_2} - \sin\theta_0\right)^2}. \tag{24.18}$$

At normal incidence $\theta_0 = 0$, we get:

$$\bar{v} = v = k\frac{a}{z_2}r \quad ; \quad r = \sqrt{{x_2}^2 + {y_2}^2} \tag{24.19}$$

in accordance with the result we found previously.

Thus, at oblique incidence upon a circular aperture, we get an Airy diffraction pattern centered at $x_2 = 0$, $y_2 = y_{20} = z_2\sin\theta_0 \approx z_2\tan\theta_0$, where the last approximation follows from the fact that θ_0 must be small for the paraxial approximation to hold. From Fig. 24.2 we see that the center of the Airy diffraction pattern is at that point where the incident wave vector $\mathbf{k}^i$ through the center of the aperture hits the observation plane.

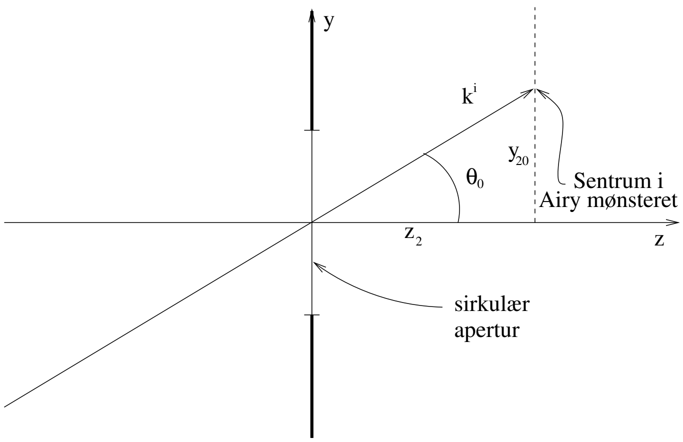

Figure 24.2: The center of the Airy diffraction pattern is at the point where $\mathbf{k}^i$ through the center of the aperture hits the observation plane.

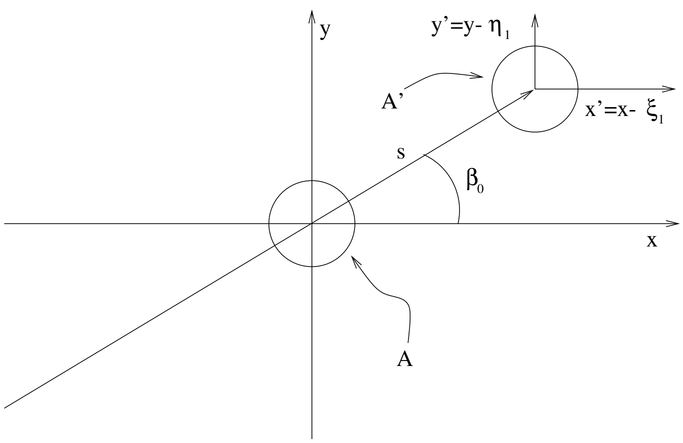

Figure 24.3: The aperture A' is displaced relative to A

24.3 Aperture displacement

Show that if we displace the aperture A in the plane $z = 0$ in (24.4) a distance s in a direction that makes an angle β_0 with the positive x axis, then the diffracted field u_I' due to the new aperture A' is given by

$$u_I' = u_I e^{-i(K_x\xi_1 + K_y\eta_1)} \tag{24.20}$$

where

$$\xi_1 = s\cos\beta_0 \quad ; \quad \eta_1 = s\sin\beta_0 \tag{24.21}$$

is the origin in a displaced co-ordinate system (x', y') that is related to A' in the same way as the original co-ordinate system (x, y) is related to A

Solution: A og A' are identical apertures, which are displaced relative to one another, as illustrated in Fig. 24.3. The diffracted field due to A' is given by:

$$u_I' = \frac{C_1}{i\lambda z_2}\int\int_{A'} e^{-i(K_x x + K_y y)}dxdy. \tag{24.22}$$

Letting $x' = x - \xi_1$ og $y' = y - \eta_1$, we get

$$u_I' = \frac{C_1}{i\lambda z_2}e^{-i(K_x\xi_1 + K_y\eta_1)}\int\int_{A'} e^{-i(K_x x' + K_y y')}dx'dy'$$

$$= \quad e^{-i(K_x\xi_1+K_y\eta_1)}\frac{C_1}{i\lambda z_2}\int\int_A e^{-i(K_x x+K_y y)}dxdy \qquad (24.23)$$

or

$$u_I' = u_I e^{-i(K_x\xi_1+K_y\eta_1)} \qquad (24.24)$$

which was to be shown.

24.4 Interference

Use the result from Exercise 24.3 to show that the diffracted intensity due to both A and A' is given by

$$I = 4I_0\cos^2\left(\frac{1}{2}\delta\right) \qquad (24.25)$$

where I_0 is the intensity due to A or A' alone, and where

$$\delta = K_x\xi_1 + K_y\eta_1. \qquad (24.26)$$

Solution: The diffracted field u_{It} due to A and A' becomes

$$u_{It} = u_I + u_I' = u_I(1+e^{-i\delta}) \qquad (24.27)$$

where $\delta = K_x\xi_1 + K_y\eta_1$ and where u_I is the diffracted field due to A alone. The intensity becomes:

$$I_t = |u_{It}|^2 = |u_I|^2(2+2\cos\delta) = 2I_0(1+\cos\delta) \qquad (24.28)$$

or

$$I_t = 4I_0\cos^2\frac{\delta}{2} \qquad (24.29)$$

where

$$I_0 = |u_I|^2. \qquad (24.30)$$

24.5 Example

Sketch the diffraction pattern given by (24.25) for the case in which we have two circular apertures with radius $a = 0.1$ mm and normal incidence. Let the other parameters be $\beta_0 = 0$, $s = 1$ mm, $\lambda = 0.5$ μm, and $z_2 = 10$ m. How large is the diameter of the Airy disc, and how many dark interference fringes are there inside the Airy disc?

Solution: For two circular apertures I_0 is the Airy diffraction pattern and $\cos^2\frac{1}{2}\delta$ gives straight interference fringes that are oriented normally to the line connecting the two apertures. This is illustrated in Fig. 24.4. The diameter of the Airy disc is given by $D = 2r_0$, where $v_0 = k\frac{a}{z_2}r_0 = 3.83$. This gives

$$D = 2.44\cdot\left(\frac{z_2}{2a}\right)\lambda = 2.44\cdot\frac{10}{2\cdot 0.1\cdot 10^{-3}}\cdot 0.5\cdot 10^{-6}\text{ m} = 6.1\text{ cm}. \qquad (24.31)$$

The factor $\cos^2\frac{1}{2}\delta$ gives a dark interference fringe when

$$\frac{1}{2}\delta = (n\pi + \frac{\pi}{2}) \quad ; \quad (n = \pm1, 2, 3, ...)$$
$$\Downarrow$$
$$\delta = n2\pi + \pi. \qquad (24.32)$$

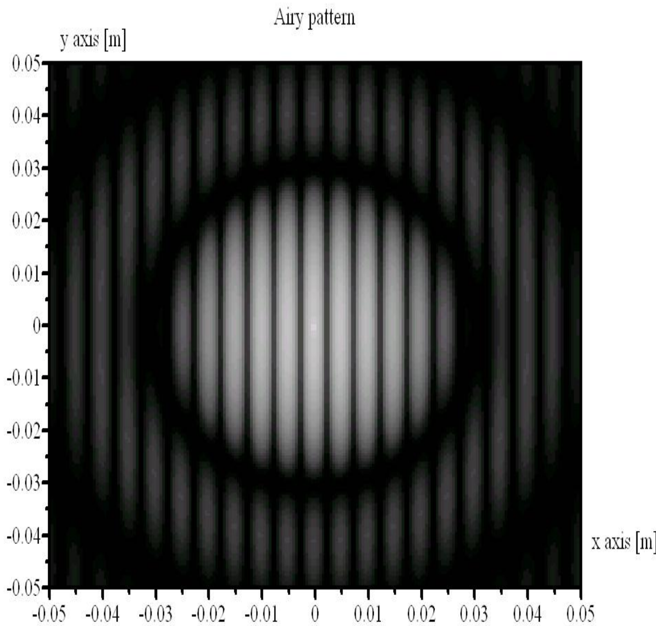

Figure 24.4: The Airy diffraction pattern due to one circular aperture with interference fringes superposed upon it due to the interaction between the fields diffracted by two circular apertures of the same size.

We have that

$$\delta = K_x\xi_1 + K_y\eta_1 = Ks\cos\beta\cos\beta_0 + Ks\sin\beta\sin\beta_0 = Ks\cos(\beta-\beta_0). \tag{24.33}$$

Normal incidence means that $\theta_0 = 0$, and $\beta_0 = 0$ means that A' lies on the x axis (see Fig 24.3). If $y_2 = 0$ then (24.9) shows at $K_y = 0$, and then it follows from (24.10) that $\beta = 0$. Substituting from the third equation in (24.10) into (24.33), we have:

$$\delta = Ks = ks\frac{x_2}{z_2} = \frac{2\pi s x_2}{\lambda z_2}. \tag{24.34}$$

Since $x_2 < 6.1$ cm, we get on substituting the given values of $s = 1$ mm, $\lambda = 0.5$ μm, and $z_2 = 10$ m into (24.34)

$$\delta < 2\pi \cdot 12.2. \tag{24.35}$$

Thus, there will be 12 dark fringes inside the Airy diffraction pattern.