Strains, strain rate, stresses

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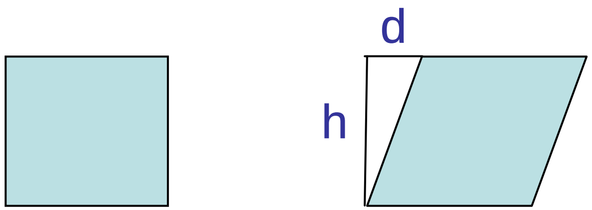
Broad definitions

- The strain measures the change in shape of a small material element. Ratio of lengths, e.g. before and after the deformation, is used. Hence strains are nondimensional quantities.
- The strain rate is the strain achieved per unit time, and is particularly important in fluid materials. Dimension is inverse time, typically s^{-1}. The strain rate is related to the velocity gradient of the flow.
- The stress is related to the force per unit area that a small material element exerts by contact on its surroundings. A simple example is the pressure in a stagnant liquid, where such a force is the same in all directions (isotropy). More complex is the situation arising when the material is deformed, and the stress becomes anisotropic. Typical unit is Pa (Pascal = Newton/m^2).

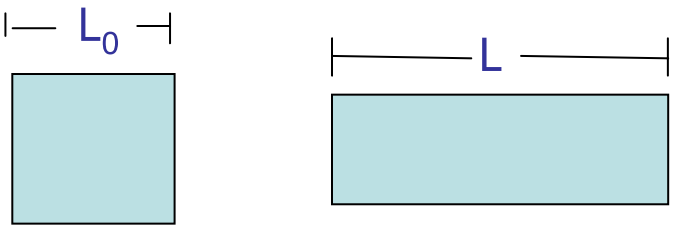
Typical deformations

Shear deformation



Shear strain = γ = d/h

Elongational deformation



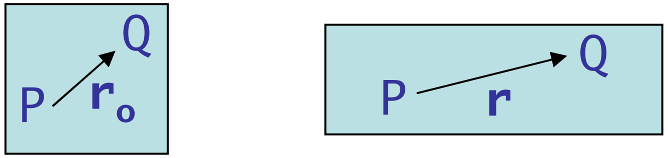
Stretch ratio = $\lambda = L/L_0$

Hencky strain = $\varepsilon = \ln(L/L_0)$

We are particularly interested in volume-preserving deformations

Deformation tensor

If we deform in an arbitrary way a small volume of material, we may consider the relationship between vector $\mathbf{r_o}$ connecting any two material points P and Q before, and vector **r** linking the same material points after, the deformation.



Because of the smallness of the material element, the relationship between $\mathbf{r_o}$ and **r** is linear. A general linear relationship between vectors is described by a tensor. We write:

$$\mathbf{r} = \mathbf{E} \cdot \mathbf{r_o} \qquad (1)$$

where **E** is the deformation tensor, fully describing how the undeformed element (to the left in the figure) changes its shape (to the right). It is important to note that linearity arises from the smallness of the element, not of the deformation. Arbitrarily large deformations can be considered, as are often encountered in rubber and rubber-like materials.

Matrix of deformation tensor

- Let us use a Cartesian coordinate system. If x_o, y_o, z_o are coordinates of a material point of the body before deformation, and x, y, z those of the same point after deformation, the three following functions:

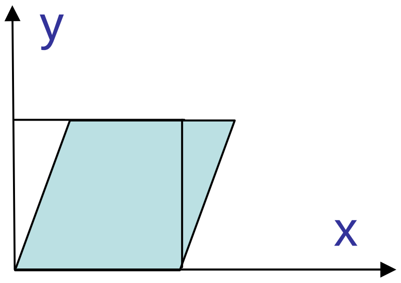
  $$x(x_o,y_o,z_o),\quad y(x_o,y_o,z_o),\quad z(x_o,y_o,z_o)$$

  fully describe body deformation.
- At any point in the body, we may consider the 9 partial derivatives of those functions, that we organize in the following matrix 3 x 3:

$$\|E\| = \begin{vmatrix} \partial x/\partial x_0 & \partial x/\partial y_0 & \partial x/\partial z_0 \\ \partial y/\partial x_0 & \partial y/\partial y_0 & \partial y/\partial z_0 \\ \partial z/\partial x_0 & \partial z/\partial y_0 & \partial z/\partial z_0 \end{vmatrix} \qquad (2)$$

- This is the matrix of tensor **E** defined in (1), also called “deformation gradient”.
- The determinant of the matrix in (2) gives the volume ratio due to deformation. Hence, if volume preserving deformations are considered, the determinant must be unity.

Example 1: shear deformation



- This deformation is described by the following (linear) functions:

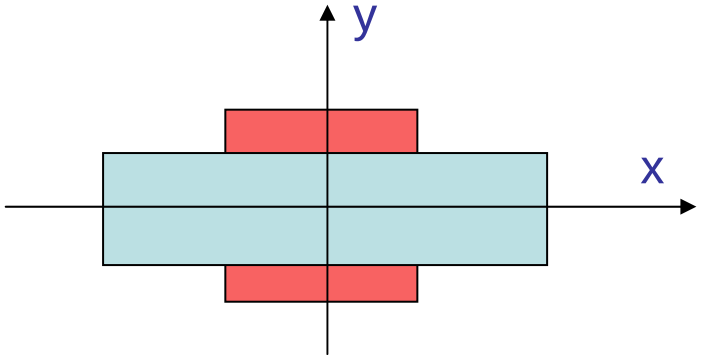
$$x = x_0 + \gamma\, y_0$$
$$y = y_0 \qquad (3)$$
$$z = z_0$$

generating the matrix:

$$\|E\| = \begin{vmatrix} 1 & \gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \qquad (4)$$

- Notice that the determinant is unity.

Example 2: Uniaxial elongation



- Linear functions for this deformation are:

$$x = \lambda x_0$$

$$y = y_0 / \sqrt{\lambda}$$

$$z = z_0 / \sqrt{\lambda}$$

where the stretch ratio λ is larger than unity. The deformation matrix is:

$$\|E\| = \begin{vmatrix} \lambda & 0 & 0 \\ 0 & 1/\sqrt{\lambda} & 0 \\ 0 & 0 & 1/\sqrt{\lambda} \end{vmatrix} \qquad (5)$$

- Also here the determinant is unity.

Elongational deformations in general

- All elongational deformations are described by a symmetric tensor **E**. Then, by a suitable choice of Cartesian coordinates, the matrix of **E** has the canonical form:

$$\|E\| = \begin{vmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{vmatrix} \qquad (6)$$

  where $\lambda_1\lambda_2\lambda_3 = 1$ for volume preservation. Hence, some of the λ's will be larger, and others smaller, than unity.

- If two of the λ's are equal, the deformation is called uniaxial, like the uniaxial stretch of the previous example. Similarly, one can have a uniaxial compression. In both cases, there is an axis of symmetry.

- If one λ is unity, the deformation is called planar, since in one direction there is no deformation.

- In the general case, the deformation is called biaxial, because one has to assign the values of λ in two directions, the third being determined by volume conservation.

More on Eqs. (1) and (2)

- In terms of Cartesian components, Eq. (1) defining tensor **E** can be interpreted as follows. Imagine that the 3 components of vector $\mathbf{r_o}$ are written as a column matrix:

$$\|r_o\| = \begin{vmatrix} r_{ox} \\ r_{oy} \\ r_{oz} \end{vmatrix}$$

  Multiplication row by column of the matrix in Eq. (2) times the matrix of vector $\mathbf{r_o}$ then gives the matrix of vector **r**.

- If vector $\mathbf{r_o}$ is taken from the origin, and hence its components coincide with the coordinates x_o, y_o, z_o of the vector tip, one can readily verify that multiplication row by column of, for example, the shear deformation matrix of Eq. (4) times the matrix of $\mathbf{r_o}$ indeed gives the components of vector **r** as reported in Eq. (3).
- For future reference, it is worth noting that linear operators on vectors (i.e. tensors) can be combined to operate in series, i.e. one after the other. For example, two consecutive deformations $\mathbf{E_1}$ and $\mathbf{E_2}$ combine in the single deformation **E** given by

$$\mathbf{E} = \mathbf{E_2} \cdot \mathbf{E_1}$$

  where the matrix multiplication of $\mathbf{E_2}$ to $\mathbf{E_1}$ is rows by columns.

Rotations

- Eq. (1) defining tensor **E** also includes particular "deformations" which are in fact rigid rotations of the element.
- However, we can easily recognize a tensor representing a rigid rotation.
- Indeed, suppose tensor **R** is a rigid rotation. We then consider its transpose **R**$^{\mathbf{T}}$. (The transpose of a tensor has a matrix where rows and columns have been interchanged.)
- Next we calculate the product **R** • **R**$^{\mathbf{T}}$. If, and only if, **R** is orthogonal, then such a product generates the unit tensor. Tensors representing rigid rotations are orthogonal.
- For example, let us consider tensor **R** representing a rigid rotation by an angle φ around the z axis. Its matrix is:

$$\|R\| = \begin{vmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

- The matrix of the transpose is:

$$\|R^T\| = \begin{vmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

- One can readily verify that multiplying these two matrices rows by columns gives the unit matrix.

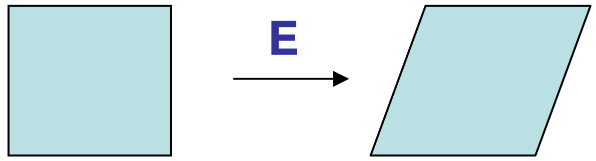
Why considering rigid rotations

- Generally deformations include both deformations proper, called pure deformations, and rigid rotations.
- Tensors representing pure deformations are symmetric, i.e. $\mathbf{E} = \mathbf{E}^T$ for them.
- Elongational deformations previously encountered are pure deformations, and it is immediately verified that the matrix of **E** is indeed symmetric.
- Conversely, for a shear deformation **E** is not symmetric, which implies that the deformation is not "pure", and also incorporates a rotation.
- The fact that, generally, deformations also include rotations is unfortunate because we expect that the stress arising in materials is due to pure deformations, not to rigid rotations. In other words, we cannot expect to link the stress to tensor **E** as such, because **E** generally contains a rotation as well.
- Fortunately, there exists a theorem stating that any tensor **E** can be split in a single way into the product of a symmetric positive-definite tensor (called **U**) to an orthogonal one **R**, i.e.:

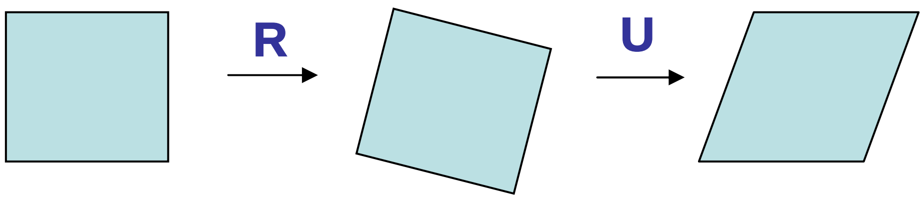
$$\mathbf{E} = \mathbf{U} \cdot \mathbf{R} \tag{7}$$

E is split into rotation times pure deformation

The meaning of Eq. (7) is illustrated, for example, by a shear deformation **E**



which can be split into a rigid rotation **R** followed by a pure deformation **U**:



- As said previously, the pure deformation **U** is a symmetric positive-definite tensor, positive-definite meaning that all 3 principal values (or eigen-values) of the tensor are positive numbers (here, the 3 principal stretch ratios).

A simple way to eliminate rotation

- As said before, we want to link stress to pure deformation. Hence we need to eliminate the rigid rotation **R** from tensor **E**.
- A simple way to obtain such a result is to operate the product:

$$\mathbf{B} = \mathbf{E} \cdot \mathbf{E}^T \qquad (8)$$

- Indeed, in view of Eq. (7), and since the transpose of a product is made by transposing the factors in reverse order, we obtain:

$$\mathbf{B} = (\mathbf{U} \cdot \mathbf{R}) \cdot (\mathbf{U} \cdot \mathbf{R})^T = \mathbf{U} \cdot \mathbf{R} \cdot \mathbf{R}^T \cdot \mathbf{U}^T = \mathbf{U} \cdot \mathbf{U}^T = \mathbf{U} \cdot \mathbf{U} = \mathbf{U}^2$$

- In this calculation we have used the property that an orthogonal tensor **R** times its transpose gives the unit tensor (disappearing from the product). We have also used the symmetry of the pure deformation tensor **U**, whereby the product with the transpose is equivalent to taking the square.
- Quadratic measures of the deformation are named after Cauchy, Green and Finger. We call **B** the Finger tensor.

Finger tensor for the shear deformation

- By way of example, let us calculate tensor **B** for a shear deformation.

- The matrix of the deformation tensor **E** is given by Eq. (4), hence in view of Eq. (8) the matrix of **B** is obtained as:

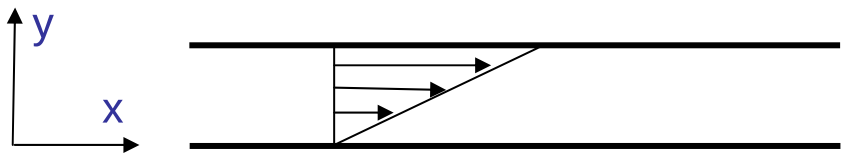
$$\|B\| = \begin{vmatrix} 1 & \gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \begin{vmatrix} 1 & 0 & 0 \\ \gamma & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1+\gamma^2 & \gamma & 0 \\ \gamma & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \qquad \textbf{(9)}$$

- Notice that **B** (differently from **E**) is symmetric, as it should, since it represents a pure deformation. Obviously, the determinant of **B** remains unity.

- Finger tensor **B** appears very frequently in modeling the rheology of polymeric systems.

Strain rate

- We now move on from strain to strain rate, a quantity which is particularly useful in flowing systems.
- A flow process is just a continuous deformation. Hence the examples of possible deformations previously considered (shear, uniaxial elongation, etc.) are readily transformed into the corresponding flows.
- Flows are usually represented by velocity fields. Thus, for example, a simple shear flow between parallel plates is depicted as:



- The 3 components of the velocity vector **v** are in this case:

$$v_x = (V/h)\, y \qquad v_y = v_z = 0$$

where V is the speed of the moving plate, and h the gap size.

- The ratio V/h is the shear rate, usually indicated with the symbol $\dot{\gamma}$ because it is in fact the time derivative of the shear strain γ.

Velocity gradient

- Since the velocity is a vector, the velocity gradient at any point of a velocity field is the tensor **L**, the 9 Cartesian components of which are (3 by 3) the derivatives of v_x, v_y, v_z, with respect to x, y, z.
- For example, for the simple shear flow just considered, since the only nonzero derivative is $\partial v_x/\partial v_y$, the matrix of **L** is:

$$\|L\| = \begin{vmatrix} 0 & \dot{\gamma} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix} \qquad (10)$$

- As for all tensors, also **L** can be seen as a linear operator transforming vectors into vectors. Specifically, the velocity gradient at a point P associates to the vector **r** reaching from P to a neighboring point Q the velocity difference $\mathbf{v}_Q$-$\mathbf{v}_P$ at Q and P, respectively:

$$\mathbf{v_Q} - \mathbf{v_P} = \mathbf{L} \bullet \mathbf{r} \qquad (11)$$

- The physical dimensions of **L** are those of inverse time.

Strain rate and vorticity

- The velocity gradient tensor is generally non-symmetric, as can be seen in the example of Eq. (10). This is due to the fact that, during flow, a fluid element is, in the general case, simultaneously strained and rotated.
- We need to distinguish between the rates at which these two effects occur, which is accomplished by splitting **L** into the following sum:

$$\mathbf{L} = (\mathbf{L} + \mathbf{L}^T)/2 + (\mathbf{L} - \mathbf{L}^T)/2$$

- Here, the first term is symmetric, representing the strain rate tensor **D**:

$$\mathbf{D} = (\mathbf{L} + \mathbf{L}^T)/2 \qquad (12)$$

- The second term is anti-symmetric, giving the rotation rate or vorticity tensor **W**:

$$\mathbf{W} = (\mathbf{L} - \mathbf{L}^T)/2 \qquad (13)$$

(Anti-symmetry means that $\mathbf{W}^T = -\mathbf{W}$.)

- In conclusion we write:

$$\mathbf{L} = \mathbf{D} + \mathbf{W} \qquad (14)$$

having split the velocity gradient into a strain rate and a vorticity.

- For a shear flow, both the strain rate and the vorticity matrices contain the shear rate divided by 2.

Elongational flows

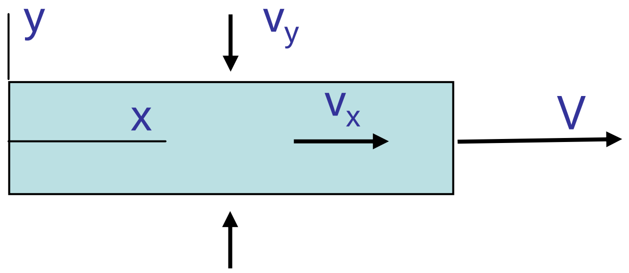
- Elongational flows are characterized by the fact that the vorticity is zero. Hence the velocity gradient **L** coincides in this case with the strain rate **D**, and is a symmetric tensor.
- If a tensor is symmetric, it is always possible to orient the Cartesian coordinates in such a way that the matrix of the tensor is diagonal, meaning that all non-diagonal terms of the matrix are zero. The coordinate axes then coincide with the principal directions (or eigen-directions) of the tensor.
- The velocity gradient for a general elongational flow can then be written as:

$$\|L\| = \begin{vmatrix} \partial v_x/\partial x & 0 & 0 \\ 0 & \partial v_y/\partial y & 0 \\ 0 & 0 & \partial v_z/\partial z \end{vmatrix} = \begin{vmatrix} \dot{\varepsilon}_x & 0 & 0 \\ 0 & \dot{\varepsilon}_y & 0 \\ 0 & 0 & \dot{\varepsilon}_z \end{vmatrix} \qquad (15)$$

- The 3 terms on the diagonal are the principal strain rates. Their sum must be zero for volume conservation (also called "incompressibility"). The sum of the diagonal elements is called the trace of the tensor, indicated as tr**L**.
- As shown in the next example, the principal strain rates are the time derivatives of the corresponding Hencky strains.

Uniaxial elongational flow

- Let us consider a uniaxial stretching flow in the x direction. Velocity field is:



$$v_x = \dot{\varepsilon}\, x$$
$$v_y = -\frac{\dot{\varepsilon}}{2} y$$
$$v_z = -\frac{\dot{\varepsilon}}{2} z$$

- The stretch rate in the x direction, called $\dot{\varepsilon}$, can be related to the velocity V at the end of the element, and to the element length L, as V/L. Hence:

$$\dot{\varepsilon} = \frac{V}{L} = \frac{1}{L}\frac{dL}{dt} = \frac{d \ln L}{dt} = \frac{d \ln\left(L/L_0\right)}{dt} = \frac{d\varepsilon}{dt}$$

showing that a principal strain rate is the time derivative of the corresponding Hencky strain (previously defined, see the second slide).

Different elongational flows

- Because their sum must be zero, some of the principal strain rates are positive and others negative.
- The case of one positive and two negative ones (equal to one another) corresponds to filament stretching, like in fiber spinning. In the previous slide we have called this case uniaxial elongational flow.
- The case of two positive ones (and one negative of course) is encountered in film blowing, where the film extends in the machine direction as well as transversally. It is called biaxial elongational flow.
- Finally, we might have that one principal stretch rate is zero, and the other two are then opposite in sign and equal in magnitude. This case is encountered in film casting, where because the film adheres to the chill roll the width of the cast film is essentially fixed, so that the plastic film is extended only in the machine direction, correspondingly reducing its thickness. This case is called planar elongational flow.
- One may have noted the similarity with the discussion following Eq. (6). Indeed there exists a general relationship between the velocity gradient **L** and the deformation tensor **E**, which is considered next.

Relationship between E and L

- Let us imagine that, starting from some initial configuration at time t=0, we deform a material element progressively in time. We can describe such a process through the tensor function **E**(t) which, at any time t, relates the current configuration of the element to the initial one, used as reference.
- On the other hand, a continuous deformation is equivalent to a flow, and hence we can equally well describe the same process through the velocity gradient **L**(t), itself a function of time in the general case.
- The relationship between the two descriptions can be obtained as follows. We start from Eq. (1), stating that **E** transforms vector $\mathbf{r}_o$ connecting two material points (P and Q, say) in the reference configuration to vector **r** connecting the same points in the current configuration:

$$\mathbf{r}(t) = \mathbf{E}(t) \bullet \mathbf{r}_o \qquad (16)$$

- Time differentiating this equation, since $d\mathbf{r}/dt = \mathbf{v}_Q - \mathbf{v}_P$, we get:

$$\mathbf{v}_Q - \mathbf{v}_P = d\mathbf{E}/dt \bullet \mathbf{r}_o$$

- In the current configuration, the velocity difference obeys Eq. (11). Hence:

$$\mathbf{L} \bullet \mathbf{r} = d\mathbf{E}/dt \bullet \mathbf{r}_o$$

- We now use again Eq. (16) to replace **r** by $\mathbf{E} \bullet \mathbf{r}_o$, finally obtaining:

$$\mathbf{L} \bullet \mathbf{E} = d\mathbf{E}/dt \qquad \text{or, equivalently,} \qquad \mathbf{L} = d\mathbf{E}/dt \bullet \mathbf{E}^{-1} \qquad (17)$$

Stress tensor

- The force per unit area exerted by a material element on the surroundings (or the other way around) generally depends on how the contact surface is oriented.
- Let us characterize the contact surface by both its extent (the area A) and its orientation (the unit vector **n** normal to the surface). The vector A**n** then fully describes the contact surface.
- Indicating with **f** the force exchanged through the surface A**n**, the relationship between these vectors is (provided the element is sufficiently small) a linear one. Hence:

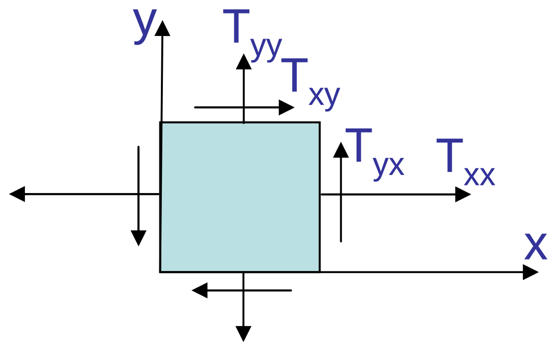
$$\mathbf{f} = \mathbf{T} \bullet A\mathbf{n}$$

  where **T** is the stress tensor (dimensions of force/square length).
- To complete the definition we need to specify that, choosing to orient **n**, say, outwards from the element, then **f** is taken to be the force exerted by the surroundings upon the element. With such a choice, tractions come out as positive stresses while compressions are negative, which is the convention usually adopted in rheology. (Fluid dynamics uses the opposite one.)

Components of T, called “stresses”

- The Cartesian components of T are obtained as follows. We take the small material element to be a cube with edges parallel to the coordinate axes. Then we consider the 3 force vectors acting on the three faces of the cube normal to x, y, z, and divide them by the face area. For each of these forces per unit area, we consider the 3 components. The total of 9 stresses thus obtained is then organized in the matrix:

$$\|T\| = \begin{vmatrix} T_{xx} & T_{xy} & T_{xz} \\ T_{yx} & T_{yy} & T_{yz} \\ T_{zx} & T_{zy} & T_{zz} \end{vmatrix}$$



- As shown in the figure, the second index in each stress refers to the face of the cube, the first to the specific component on that face. However, the order of the indices is not important in most cases because, with the rare exception of materials with internal torques, torque balance imposes that $T_{ij} = T_{ji}$, i.e., the stress tensor is symmetric.
- Stresses with equal indices (along the matrix main diagonal) are normal stresses. Those with different indices are called tangential or shear stresses.

Pressure and normal stresses

- In a stagnant fluid, the stress tensor is isotropic, i.e., all shear stresses are zero, and all normal stresses are the same:

$$\|T\| = \begin{vmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{vmatrix} = -p \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \qquad (18)$$

- The minus sign in front of the pressure p is due to the sign convention. In tensor form, the above equation is written as:

$$\mathbf{T} = -\,p\,\mathbf{I} \qquad (18')$$

  where **I** is the unity (or identity) tensor.
- Of course, the static pressure p has nothing to do with rheology.
- Due to flow or deformation, **T** becomes anisotropic, and normal stresses generally become different. In view of incompressibility, rheology only determines such normal stress differences (like $T_{xx}-T_{yy}$, say). The absolute level of the normal stresses is not determined locally, i.e., by the rheology of the element. Rather, it ultimately depends on conditions at the boundary of the whole body of matter.

Stress tensor in shear

- Because of the symmetry of a shear flow or deformation, the matrix of the stress tensor reduces to

$$\|T\| = \begin{vmatrix} T_{xx} & T_{xy} & 0 \\ T_{yx} & T_{yy} & 0 \\ 0 & 0 & T_{zz} \end{vmatrix} \qquad (19)$$

  where it is understood that the x-axis is along the shear direction, and the y-axis is normal to the shearing surfaces.
- The relevant stresses in shear are therefore only 3; precisely:

  $\sigma = T_{xy} = T_{yx} =$ shear stress
  $N_1 = T_{xx} - T_{yy} =$ first (or primary) normal stress difference
  $N_2 = T_{yy} - T_{zz} =$ second normal stress difference
- Symmetry also requires that σ is an odd function of the shear strain or of the shear rate, while N_1 and N_2 are even functions.
- In a steady shear flow, the ratio of σ to the shear rate is the viscosity η. The ratio of N_1 or N_2 to the square of the shear rate is called first or second normal stress coefficient, ψ_1 or ψ_2.

Stress tensor in elongation

- We have seen previously that in elongation both the deformation tensor **E** and the velocity gradient **L** are symmetric tensors, and hence by a suitable choice of coordinates (along the principal directions of those tensors) their matrices can be made diagonal.
- For symmetry reasons, the stress tensor **T** will have the same principal directions as **E** or **L**. With that coordinate system, the matrix of **T** then is:

$$\|T\| = \begin{vmatrix} T_{xx} & 0 & 0 \\ 0 & T_{yy} & 0 \\ 0 & 0 & T_{zz} \end{vmatrix} \qquad (20)$$

- Hence, relevant stresses in elongation are (at most) 2, namely:

  $N_1 = T_{xx} - T_{yy}$ = first normal stress difference

  $N_2 = T_{yy} - T_{zz}$ = second normal stress difference
- In the uniaxial case, symmetry further reduces the stress state to a single nonzero normal stress difference.
- In uniaxial elongational flow, the ratio of that normal stress difference to the stretching rate is called Trouton viscosity.

Constitutive equations

- Constitutive equations of interest to rheology somehow link the stress tensor **T** to strain and/or strain rate.
- Because of the incompressibility constraint, it is understood that normal stresses are determined only as differences. In other words, **T** is obtained from the constitutive equation only to within an arbitrary isotropic tensor (i.e., one like that in Eq. 17).
- For an amorphous elastic solid, **T** must be a function only of a pure deformation tensor, like the Finger tensor **B**, measured from the stress-free equilibrium state used as reference.
- For the ideal rubber, it can be shown that such a function reduces to a simple proportionality: **T** = G **B**. The factor G is called shear modulus. Indeed, from Eqs. (9) and (19) one finds for the shear stress the simple proportionality $\sigma = G\,\gamma$. Yet the same equations also give $N_1 = G\,\gamma^2$ (which has important consequences at large deformations), and $N_2 = 0$. Actual rubber does not depart too much from these predictions.
- For a viscous liquid, **T** must be a function only of the strain rate tensor **D**. For many liquids (called Newtonian), this function simply is **T** = 2η**D**. For Newtonian liquids, the Trouton viscosity is 3 times the shear viscosity η.
- The viscoelastic materials will be considered starting from the 2nd lecture.