Mathematics for economic analysis – Academic Year 2016/2017

HOMEWORK – Week 2

Topics: basis and dimension of a vector space; null space and row space of a matrix; recap exercises on vector spaces.

Warning: $< S >$ and $\text{Span}\, S$ are equivalent notations for the span of set S.

1. Consider the following vectors in $\mathbb{R}^4$:

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}; \quad v_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 2 \end{pmatrix}; \quad v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

Determine:

a. whether they are linearly independent;

b. whether they form a basis of $\mathbb{R}^4$.

Then, find the dimension and a basis for the vector subspace generated by these three vectors.

2. Prove that the following vectors:

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

form a basis of $\mathbb{R}^3$ and find the components of the vectors $e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, $w = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$ with respect to such basis.

3. In the vector space $\mathbb{R}^2$, consider the following vectors $u = (1, 2)$ e $v = (a, 1)$, where a is a real number. Determine for which values of the parameter a they form a basis of $\mathbb{R}^2$. For such values of a, find the components of the vector $w = (3, 2)$ with respect to such basis.

4. Write down the standard basis of the set $M(2, 2)$ of all matrices 2×2. Consider then the following matrices:

$$A = \begin{pmatrix} 0 & 0 \\ k & 0 \end{pmatrix}; \quad B = \begin{pmatrix} 1 & k \\ -2 & 0 \end{pmatrix}; \quad C = \begin{pmatrix} k & 1 \\ -1 & 1 \end{pmatrix}.$$

For which values of the parameter k are they linearly independent? For these values of k, do they form a basis of the vector space $M(2, 2)$?

5. True/false (provide reasons; if false, find examples that demonstrate it):

a. four vectors in $\mathbb{R}^6$ are always linearly independent;

b. six vectors in $\mathbb{R}^4$ are always linearly independent;

c. the vectors $(1, \frac{1}{2})$ and $(2, 1)$ are linearly dependent;

d. the vectors $(1, \frac{1}{2})$, $(2, 1)$ and $(3, 0)$ are linearly dependent;

e. each vector space has a unique basis.

6. Determine which of the following subsets are vector subspaces and find a basis for them:

$$\begin{aligned} S_1 &= \{(x, y) \in \mathbb{R}^2 : y + x = 0\}; \\ S_2 &= \{(x, y) \in \mathbb{R}^2 : 3x = y + 2\}. \end{aligned}$$

7. Let $u = \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}$ and $v = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$.

   a. Find a vector w_1, different from u and v, such that $\langle u, v, w_1 \rangle = \langle u, v \rangle$.

   b. Find a vector w_2 so that $\langle u, v, w_2 \rangle \neq \langle u, v \rangle$.

8. Suppose that S is a subset of $\mathbb{R}^n$. Prove that the zero vector is an element of $\langle S \rangle$.

9. Let $S = \left\{ \begin{pmatrix} x \\ 2x + 3z \\ z \end{pmatrix} : x, z \in \mathbb{R} \right\}$.

   Prove that:

$$S = \left\langle \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} \right\rangle.$$

   Find a basis for S and explain why it cannot be a basis for $\mathbb{R}^3$. Indicate the dimension of S.

10. Consider the following two vectors:

$$u = (1, 0, 0, 1) \text{ and } v = (0, 1, 0, 1).$$

   a. Determine for which values of the parameter t, the vector $w = (2, t, 0, 1)$ belongs to the subspace generated by u e v.

   b. Find the dimension of the subspace generated by u, v and w (depending on the parameter t).

11. Consider the matrix:

$$A = \begin{pmatrix} 1 & -2 & 0 & 0 & 3 \\ 2 & -5 & -3 & -2 & 6 \\ 0 & 5 & 15 & 10 & 0 \\ 2 & 6 & 18 & 8 & 6 \end{pmatrix}$$

   a. Find a basis of the row space of A and indicate its dimension.

   b. Compute the rank of A.

12. Consider the matrix:

$$A = \begin{pmatrix} -1 & 0 & -1 & 2 \\ 2 & 0 & 2 & 0 \\ 1 & 0 & 1 & -1 \end{pmatrix}$$

a. Find a basis of the row space of A.

b. Find a basis of the null space of A and compute the nullity of A.

c. Check that the sum of the nullity and the rank of A equals the number of columns of A.