HOMEWORK – Week 3

Topics: eigenvalues and eigenvectors; diagonalizable matrices.

1. Show that $v = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is an eigenvector of the matrix $A = \begin{pmatrix} 1 & 1 \\ 3 & 3 \end{pmatrix}$. What is the corresponding eigenvalue?

2. Find the eigenvalues and the eigenvectors of the following matrices; for each eigenvalue, indicate its algebraic and geometric multiplicity:

$$\text{(a) } A = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix};\ \text{(b) } B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix};\ \text{(c) } C = \begin{pmatrix} 2 & 1 & -1 \\ 1 & 2 & 1 \\ 1 & 1 & 0 \end{pmatrix};\ \text{(d) } D = \begin{pmatrix} 2 & 0 & 0 & 4 \\ 0 & 3 & 0 & 5 \\ 0 & 5 & -2 & 5 \\ 4 & 0 & 0 & 2 \end{pmatrix}.$$

   For each matrix, check that the sum of the eigenvalues equals the trace of the matrix and that the product of the eigenvalues equals the determinant of the matrix.
   Specify whether the matrices are diagonalizable; calculate the power matrix A^8.

3. Diagonalize the following matrix A and calculate the power matrix A^k for any integer $k \geq 1$:

$$A = \begin{pmatrix} 1 & -1 \\ 2 & 4 \end{pmatrix}$$

4. Given the matrix:

$$A = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 2 & 1 & 4 \end{pmatrix}$$

   a. find the eigenvalues of A;

   b. indicate three linearly independent eigenvectors of A;

   c. indicate whether A is diagonalizable and find a diagonal matrix D and a non singular matrix P such that $A = P \cdot D \cdot P^{-1}$.

5. Given the matrix:

$$A = \begin{pmatrix} 3 & 2 & 1 \\ -3 & -2 & h+1 \\ 6 & 4 & 2 \end{pmatrix}$$

   a. find the value of the parameter h for which A has an eigenvalue equal to 3;

   b. for this value of h, prove that A is diagonalizable and indicate a diagonal matrix D and a non–singular matrix P such that $A = P \cdot D \cdot P^{-1}$.

6. Given the matrix:

$$A = \begin{pmatrix} 1 & 0 & 3 \\ 2 & k & 2 \\ 3 & 0 & 1 \end{pmatrix}$$

a. find its eigenvalues and their algebraic multiplicity;

b. find for which values k the matrix is diagonalizable.

7. Suppose A is a 2×2 matrix with distinct eigenvalues λ and μ.
   Show that $tr(A) = \lambda + \mu$ and $det(A) = \lambda \cdot \mu$.

8. Let A be an $n \times n$ square matrix. Prove that A and its transpose A^t have the same eigenvalues.