The intersection curves of these ellipsoids with the **s**-z plane are shown in Figure 5.3 for a positive uniaxial crystal ($n_e > n_o$). The exterior curve $n_e(\theta)$ is a plot of Eq. (5.4-2). Plots such as Figure 5.3 are very useful since they convey at a glance the two indices. The orientation of the displacement vectors is also shown. $\mathbf{D}_e(\theta)$ is in the **s**-z plane, whereas that of the ordinary ray is at right angles to this plane.

5.5 NORMAL MODE EXPANSION OF THE ELECTROMAGNETIC FIELD IN A RESONATOR

Maxwell's equations and the current continuity equation in the MKS system of units are

$$\begin{aligned}
\nabla \times \mathbf{E} &= \frac{-\partial \mathbf{B}}{\partial t} \\
\nabla \times \mathbf{H} &= \mathbf{i} + \frac{\partial \mathbf{D}}{\partial t} \\
\nabla \cdot \mathbf{D} &= \rho \\
\nabla \cdot \mathbf{B} &= 0 \\
\nabla \cdot \mathbf{i} &= -\frac{\partial \rho}{\partial t}
\end{aligned} \tag{5.5-1}$$

We will limit ourselves, for the moment, to charge-free, isotropic, and homogeneous media so that

$$\mathbf{i} = 0 \qquad \mathbf{B} = \mu \mathbf{H} \qquad \nabla \cdot \mathbf{D} = 0 \qquad \mathbf{D} = \varepsilon \mathbf{E} \tag{5.5-2}$$

where ε is the dielectric constant.

Consider the electric field $\mathbf{E}(\mathbf{r}, t)$ and magnetic field $\mathbf{H}(\mathbf{r}, t)$ inside a volume V bounded by a surface S of perfect conductivity. The tangential component of $\mathbf{E}$, $-\mathbf{n} \times \mathbf{n} \times \mathbf{E}$, and the normal component of $\mathbf{H}$, $\mathbf{n} \cdot \mathbf{H}$, must both be zero on S ($\mathbf{n}$ is the unit vector normal to S). We will expand $\mathbf{E}$ and $\mathbf{H}$ in terms of two orthogonal sets of vector fields $\mathbf{E}_a$ and $\mathbf{H}_a$, respectively. These sets, which were introduced originally by Slater (Reference 3) obey the relations

$$k_a \mathbf{E}_a = \nabla \times \mathbf{H}_a \tag{5.5-3}$$

$$k_a \mathbf{H}_a = \nabla \times \mathbf{E}_a \tag{5.5-4}$$

where k_a is to be considered, for the moment, a constant. The tangential component of $\mathbf{E}_a$ on S is zero.

$$\mathbf{n} \times \mathbf{E}_a = 0 \qquad \text{on} \qquad S \tag{5.5-5}$$

If we take the curl of both sides of (5.5-3, 4) and use the identity

$$\nabla \times \nabla \times \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$$

they become

$$\begin{aligned} \nabla^2 \mathbf{E}_a + k_a^2 \mathbf{E}_a &= 0 \\ \nabla^2 \mathbf{H}_a + k_a^2 \mathbf{H}_a &= 0 \end{aligned} \tag{5.5-6}$$

that is, the familiar wave equation.

It follows from (5.5-3), (5.5-4), and (5.5-5) that the normal component of $\mathbf{H}_a$, $\mathbf{n} \cdot \mathbf{H}_a$ is zero on S. To prove this statement, consider an arbitrary closed contour C on S surrounding a surface S'.

$$\oint_C \mathbf{E}_a \cdot d\mathbf{l} = \oint_C (-\mathbf{n} \times \mathbf{n} \times \mathbf{E}_a) \cdot d\mathbf{l} + \oint_C (\mathbf{n} \cdot \mathbf{E}_a)\mathbf{n} \cdot d\mathbf{l} = 0 \tag{5.5-7}$$

where $\mathbf{E}_a$ is expressed as the vector sum of its tangential $(-\mathbf{n} \times \mathbf{n} \times \mathbf{E}_a)$ and normal $(\mathbf{n} \cdot \mathbf{E}_a)\mathbf{n}$ components. The first term on the right side of (5.5-7) is zero because of (5.5-5) whereas the second one is zero since $\mathbf{n}$ is perpendicular to $d\mathbf{l}$. Using Stokes' theorem on the left side of (5.5-7) gives

$$\oint_C \mathbf{E}_a \cdot d\mathbf{l} = \int_{S'} (\nabla \times \mathbf{E}_a) \cdot \mathbf{n}\, da = k_a \int_{S'} (\mathbf{H}_a \cdot \mathbf{n})\, da = 0$$

and since C is arbitrary, it follows that

$$\mathbf{H}_a \cdot \mathbf{n} = 0 \qquad \text{on} \qquad S \tag{5.5-8}$$

We will next prove that the functions $\mathbf{E}_a$ and $\mathbf{H}_a$ are orthogonal in the sense

$$\begin{aligned} \int_V \mathbf{E}_a \cdot \mathbf{E}_b\, dv &= 0 \qquad a \neq b \\ \int_V \mathbf{H}_a \cdot \mathbf{H}_b\, dv &= 0 \qquad a \neq b \end{aligned} \tag{5.5-9}$$

To prove the first of Eqs. (5.5-9), apply the vector identity $\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot \nabla \times \mathbf{A} - \mathbf{A} \cdot \nabla \times \mathbf{B}$ to $(\mathbf{E}_b \times \nabla \times \mathbf{E}_a)$, then to $(\mathbf{E}_a \times \nabla \times \mathbf{E}_b)$ and subtract. The result is

$$\begin{aligned} &\nabla \cdot (\mathbf{E}_b \times \nabla \times \mathbf{E}_a) - \nabla \cdot (\mathbf{E}_a \times \nabla \times \mathbf{E}_b) \\ &\quad = \nabla \times \mathbf{E}_a \cdot \nabla \times \mathbf{E}_b - \mathbf{E}_b \cdot \nabla \times \nabla \times \mathbf{E}_a - \nabla \times \mathbf{E}_b \cdot \nabla \times \mathbf{E}_a + \mathbf{E}_a \cdot \nabla \times \nabla \times \mathbf{E}_b \end{aligned}$$

From (5.5-4), we have $\nabla \times \mathbf{E}_a = k_a \mathbf{H}_a$ and $\nabla \times \nabla \times \mathbf{E}_a = k_a^2 \mathbf{E}_a$, which substituted in the last equation gives

$$k_a \nabla \cdot (\mathbf{E}_b \times \mathbf{H}_a) - k_b \nabla \cdot (\mathbf{E}_a \times \mathbf{H}_b) = (k_b^2 - k_a^2)\mathbf{E}_a \cdot \mathbf{E}_b$$

that, after applying Gauss's theorem (see Section 5.1), becomes

$$\int_S [k_a \mathbf{n} \cdot (\mathbf{E}_b \times \mathbf{H}_a) - k_b \mathbf{n} \cdot (\mathbf{E}_a \times \mathbf{H}_b)]\, da = (k_b^2 - k_a^2) \int_V \mathbf{E}_a \cdot \mathbf{E}_b\, dv$$

The left side of the last equality can be shown, with the aid of the identity $\mathbf{A} \cdot \mathbf{B} \times \mathbf{C} = \mathbf{C} \cdot \mathbf{A} \times \mathbf{B}$ and (5.5-5), to be zero, so that for $k_b \neq k_a$, the first of Eqs. (5.5-9) is proved. If $k_b = k_a$, that is, when $\mathbf{E}_b$ and $\mathbf{E}_a$ are members of a degenerate set, it is possible to construct linear superpositions of the degenerate functions so that orthogonality is preserved. The proof of the orthogonality of the $\mathbf{H}_a$ functions follows along identical lines.

We are free to choose the magnitude of $\mathbf{H}_a$ and $\mathbf{E}_a$ so that they are normalized according to

$$\begin{aligned} \int_V \mathbf{H}_a \cdot \mathbf{H}_b \, dv &= \delta_{a,b} \\ \int_V \mathbf{E}_a \cdot \mathbf{E}_b \, dv &= \delta_{a,b} \end{aligned} \tag{5.5-10}$$

This choice will be used throughout this text.

The total resonator fields of $\mathbf{E}(\mathbf{r}, t)$ and $\mathbf{H}(\mathbf{r}, t)$ can be expanded as

$$\begin{aligned} \mathbf{E}(\mathbf{r}, t) &= -\sum_a \frac{1}{\sqrt{\varepsilon}} p_a(t)\mathbf{E}_a(\mathbf{r}) \\ \mathbf{H}(\mathbf{r}, t) &= \sum_a \frac{1}{\sqrt{\mu}} \omega_a q_a(t)\mathbf{H}_a(\mathbf{r}) \end{aligned} \tag{5.5-11}$$

where $\omega_a = k_a/\sqrt{\mu\varepsilon}$. Substituting (5.5-11) in the first of Maxwell equations (5.5-1) and using (5.5-3) result in

$$p_a = \dot{q}_a \tag{5.5-12}$$

and in a similar fashion, from the second of (5.5-1), we have

$$\omega_a^2 q_a = -\dot{p}_a \tag{5.5-13}$$

Eliminating q_a gives

$$\ddot{p}_a + \omega_a^2 p_a = 0 \tag{5.5-14}$$

This identifies $k_a(\mu\varepsilon)^{-1/2} = \omega_a$ as the radian oscillation frequency of the ath mode.

5.6 THE QUANTIZATION OF THE RADIATION FIELD

In this section we will show that the electromagnetic field inside a resonator can be considered, formally, as an ensemble of *independent* harmonic oscillators. The formalism is related closely to that used in Chapter 4, to quantize the spectrum of lattice vibrations. To bring out the similarity (dot) multiply the first of Eqs. (5.5-11) by $\mathbf{E}_b$ and integrate over the resonator volume. The result, after using (5.5-10), is

$$p_b(t) = -\sqrt{\varepsilon} \int_V \mathbf{E}(\mathbf{r}, t) \cdot \mathbf{E}_b(\mathbf{r}) \, dv \tag{5.6-1}$$

This equation is analogous to the second of Eqs. (4.5-2). It follows from (5.6-1) and from the relation

$$q_b(t) = \frac{\sqrt{\mu}}{\omega_b} \int_V \mathbf{H}(\mathbf{r}, t) \cdot \mathbf{H}_b(\mathbf{r}) \, dv \tag{5.6-2}$$

that the state of the classical electromagnetic field can be specified by $\mathbf{H}(\mathbf{r}, t)$ and $\mathbf{E}(\mathbf{r}, t)$ or, alternatively, by the dynamical variables $p_a(t)$ and $q_a(t)$. The total energy (Hamiltonian) is

$$\mathcal{H} = \tfrac{1}{2} \int_V (\mu \mathbf{H} \cdot \mathbf{H} + \varepsilon \mathbf{E} \cdot \mathbf{E}) \, dv \tag{5.6-3}$$

Substituting for **H** and **E** their expansion (5.5-11) gives

$$\mathcal{H} = \sum_a \tfrac{1}{2}(p_a^2 + \omega_a^2 q_a^2) \tag{5.6-4}$$

which has the basic form of a sum of harmonic oscillator Hamiltonians as in (2.2-1). The dynamical variables p_a and q_a constitute canonically conjugate variables. This can be seen by considering Hamilton's equations of motion relating $\dot{p}_a$ to q_a and $\dot{q}_a$ to p_a.

$$\begin{aligned} \dot{p}_a &= -\frac{\partial \mathcal{H}}{\partial q_a} = -\omega_a^2 q_a \\ \dot{q}_a &= \frac{\partial \mathcal{H}}{\partial p_a} = p_a \end{aligned} \tag{5.6-5}$$

These are identical with Eqs. (5.5-12) and (5.5-13) obtained from Maxwell's equations.

The quantization of the electromagnetic radiation is achieved by considering p_a and q_a as *formally equivalent* to the momentum and coordinate operators of a quantum mechanical harmonic oscillator, thus taking the commutator relations connecting the dynamical variables as

$$\begin{aligned} [p_a, p_b] &= [q_a, q_b] = 0 \\ [q_a, p_b] &= i\hbar\, \delta_{a,b} \end{aligned} \tag{5.6-6}$$

In a manner analogous to that used in Chapter 2 [see (2.2-25)], we define the creation operator $a_l^\dagger$ and the annihilation operator a_l by

$$\begin{aligned} a_l^\dagger(t) &= \left(\frac{1}{2\hbar\omega_l}\right)^{1/2} [\omega_l q_l(t) - i p_l(t)] \\ a_l(t) &= \left(\frac{1}{2\hbar\omega_l}\right)^{1/2} [\omega_l q_l(t) + i p_l(t)] \end{aligned} \tag{5.6-7}$$

The commutator relations are found directly from (5.6-6) to be

$$\begin{aligned} [a_l, a_m] &= [a_l^\dagger, a_m^\dagger] = 0 \\ [a_l, a_m^\dagger] &= \delta_{l,m} \end{aligned} \tag{5.6-8}$$

Solving (5.6-7) for p_l and q_l gives

$$\begin{aligned} p_l(t) &= i\left(\frac{\hbar\omega_l}{2}\right)^{1/2} [a_l^\dagger(t) - a_l(t)] \\ q_l(t) &= \left(\frac{\hbar}{2\omega_l}\right)^{1/2} [a_l^\dagger(t) + a_l(t)] \end{aligned} \tag{5.6-9}$$

We can express the Hamiltonian in terms of the operators $a_l^\dagger$ and a_l by substituting (5.6-9) in (5.6-4) and replacing $a_l a_l^\dagger$ by $a_l^\dagger a_l + 1$ [according to (5.6-8)]. The result is

$$\mathcal{H} = \sum_l \hbar\omega_l (a_l^\dagger a_l + \tfrac{1}{2}) \tag{5.6-10}$$

which is the same as Eq. (4.5-12) obtained for the case of the lattice vibrations. The formal analogy between the operators $a_l^\dagger$, a_l and their counterparts in the case of the harmonic oscillators shows that, quantum mechanically, a stationary state of the total radiation field can be characterized by an eigenfunction Φ, which is a product of the eigenfunctions of the individual Hamiltonians $\hbar\omega_l(a_l^\dagger a_l + \frac{1}{2})$

$$\Phi = u_{n_1}u_{n_2}\cdots = \prod_{l=1}^{\infty} u_{n_l} \tag{5.6-11}$$

where

$$\begin{aligned} a_l^\dagger u_{n_l} &= \sqrt{n_l + 1}\; u_{n_l+1} \\ a_l u_{n_l} &= \sqrt{n_l}\; u_{n_l-1} \\ a_l^\dagger a_l u_{n_l} &= n_l u_{n_l} \end{aligned} \tag{5.6-12}$$

The expectation value of the operator $a_l^\dagger a_l$ is

$$\langle\Phi|\, a_l^\dagger a_l \,|\Phi\rangle = \langle n_l|\, a_l^\dagger a_l \,|n_l\rangle = n_l \tag{5.6-13}$$

and is equal to the number of quanta n_l in the lth mode of the resonator.

Plane-Wave Quantization

The discussion just concluded uses a generalized resonator of unspecified shape. We will find it useful to consider the form of the field operators in the case of a plane-wave resonator. Although such a resonator, which requires an infinite cross-sectional area does not exist, most optical resonators that employ curved mirrors as reflectors involve nearly plane-wave propagation.

To be specific, consider the lth mode of a resonator of length L along the axis and mode volume V. Let the electric and magnetic field vectors point along the y and x directions, respectively. Equations (5.5-3, 4, 9) are satisfied by

$$\begin{aligned} \mathbf{E}_l(\mathbf{r}, t) &= -\,i\hat{\mathbf{x}}\left(\frac{\hbar\omega_l}{V\varepsilon}\right)^{1/2}[a_l^\dagger(t) - a_l(t)]\sin k_l z \\ \mathbf{H}_l(\mathbf{r}, t) &= \hat{\mathbf{y}}\left(\frac{\hbar\omega_l}{V\mu}\right)^{1/2}[a_l^\dagger(t) + a_l(t)]\cos k_l z \end{aligned} \tag{5.6-14}$$

where V is the mode volume. In the case of plane wave modes, the total field can be written as a summation over all modes $\mathbf{k}$

$$\begin{aligned} \mathbf{E} &= \sum_{\mathbf{k},\lambda} -\,i\,\hat{\mathbf{e}}_{\mathbf{k},\lambda}\sqrt{\frac{\hbar\omega_k}{2\varepsilon V}}\left(a_{\mathbf{k},\lambda}^\dagger\, e^{-i\mathbf{k}\cdot\mathbf{r}} - a_{\mathbf{k},\lambda}e^{i\mathbf{k}\cdot\mathbf{r}}\right) \\ \mathbf{H} &= \sum_{\mathbf{k},\lambda} -\,i\,\frac{\mathbf{k}}{k}\times\hat{\mathbf{e}}_{\mathbf{k},\lambda}\sqrt{\frac{\hbar\omega_k}{2\mu V}}\left(a_{\mathbf{k},\lambda}^\dagger\, e^{-i\mathbf{k}\cdot\mathbf{r}} + a_{\mathbf{k},\lambda}e^{i\mathbf{k}\cdot\mathbf{r}}\right) \end{aligned} \tag{5.6-15}$$

E and **H** can be derived using (5.1-22, 23) from a quantized vector potential

$$\mathbf{A} = \sum_{\mathbf{k},\lambda} \hat{\mathbf{e}}_{\mathbf{k},\lambda} \sqrt{\frac{\hbar}{2\varepsilon V\omega_k}} \left(a^{\dagger}_{\mathbf{k},\lambda}\, e^{-i\mathbf{k}\cdot\mathbf{r}} + a_{\mathbf{k},\lambda} e^{i\mathbf{k}\cdot\mathbf{r}}\right) \tag{5.6-16}$$

5.7 MODE DENSITY AND BLACKBODY RADIATION

The number of electromagnetic modes with resonant frequencies between ν and $\nu + d\nu$ is, in general, a function of the specific form of the electromagnetic enclosure. If, however, the typical dimensions of the enclosure are large compared to the wavelength of the radiation under consideration, this is no longer true and the mode density can be calculated by using an arbitrarily shaped enclosure. Limiting ourselves to this case, we pick as the resonator a box whose sides are equal to L. The propagation characteristics of a mode are then described by $\exp(i\mathbf{k}\cdot\mathbf{r})$. Using the boundary condition that the field be periodic in L imposes the following condition on the Cartesian components of **k**:

$$k_x = \frac{2\pi l}{L} \qquad k_y = \frac{2\pi m}{L} \qquad k_z = \frac{2\pi n}{L} \tag{5.7-1}$$

where l, m, and n are integers.

From Maxwell's equations, we have

$$\left(\frac{\varepsilon}{\varepsilon_0}\right)\frac{\omega^2}{c^2} = k^2 = (k_x^2 + k_y^2 + k_z^2)$$

From (5.7-1), it follows that with each mode we can associate a volume $dk_x\, dk_y\, dk_z = (2\pi/L)^3$ in **k** space. The number of modes N_k whose wave vectors have magnitudes between 0 and k is derived simply by dividing the total volume in **k** space $(\frac{4}{3})\pi k^3$ by the volume per mode $(2\pi/L)^3$ and by multiplying the result by 2, since for each wave vector **k** there are two possible directions of polarization, each of which corresponds to an independent solution of Maxwell equations. The result is

$$N_k = \frac{k^3L^3}{3\pi^2}$$

or, if we use $k = 2\pi\nu n/c$, where $n = (\varepsilon/\varepsilon_0)^{1/2}$ is the index of refraction[2]

$$\frac{N_\nu}{V} = \frac{8\pi\nu^3 n^3}{3c^3} \tag{5.7-2}$$

for the total number of modes per unit volume between $\nu = 0$ and ν. The mode density per unit frequency $p(\nu)$ is given by

$$p(\nu) = \frac{1}{V}\frac{dN_\nu}{d\nu} = \frac{8\pi\nu^2 n^3}{c^3} \tag{5.7-3}$$

[2] Not to be confused with the integer n in (5.7-1).

If the enclosure is in thermal equilibrium at a temperature T, the average energy per mode $\bar{E}$ is derived by a calculation identical to that employed in the case of the lattice modes, which led to (4.6-3). We can thus write directly

$$\bar{E} = \frac{h\nu}{2} + \frac{h\nu}{e^{h\nu/kT} - 1} \tag{5.7-4}$$

The thermal radiation (blackbody) energy density per unit frequency width is thus

$$\rho(\nu) = p(\nu)\bar{E} = \frac{8\pi h n^3 \nu^3}{c^3}\left(\frac{1}{2} + \frac{1}{e^{h\nu/kT} - 1}\right)$$

If we integrate $\rho(\nu)$ over all frequencies, we get, because of the presence of the $\frac{1}{2}$ inside the brackets, an infinite result. This situation is not acceptable and does not agree with experiments. This "paradox" may be resolved if we recognize that the term $\frac{1}{2}$ represents the zero-field energy $h\nu/2$ of the radiation oscillators. This is the lowest energy that an oscillator may possess and is thus not available for energy exchange or, equivalently, measurement. From the thermodynamic point of view, therefore, we can write

$$\rho(\nu)\,d\nu = \frac{8\pi h n^3 \nu^3\,d\nu}{c^3}\left(\frac{1}{e^{h\nu/kT} - 1}\right) \tag{5.7-5}$$

for the blackbody radiation density.

5.8 THE COHERENT STATE

The electric field of a single field mode is characterized according to (5.5-11) by the operator $p(t)$ and a field Hamiltonian

$$H = \tfrac{1}{2}(p^2 + \omega^2 q^2) \tag{5.8-1}$$

Since p and q are canonically conjugate "momenta," they are related as in (1.1-25) by

$$q \rightarrow i\hbar \frac{\partial}{\partial p} \tag{5.8-2}$$

so that the Hamiltonian in the p representation becomes

$$H = \tfrac{1}{2}\left(p^2 - \omega^2\hbar^2 \frac{\partial^2}{\partial p^2}\right) \tag{5.8-3}$$

If we define

$$\xi \equiv \alpha p = \frac{p}{\sqrt{\hbar\omega}}, \qquad \Lambda = \frac{2E_n}{\hbar\omega} \tag{5.8-4}$$

the Schrödinger equation $Hu_n = E_n u_n$ assumes the form

$$\frac{\partial^2 u_n}{\partial \xi^2} + [\Lambda_n - \xi^2]u_n = 0 \tag{5.8-5}$$