Summary of relevant mathematical definitions

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1 Metric spaces

Definition 1.1 (Distance and Metric Space)**.** *Let X be a set and let*

$$d : X \times X \to \mathbb{R}^+$$

be a nonnegative *real function defined on $X \times X$, i.e. a function that maps each couple (x, y), with x and y in X, into a real number $d(x, y)$ such that $d(x, y) \geq 0$. The function d is called* ***distance*** *(or metric) on X if, for any x,y,z in X, the following three conditions hold:*

$$d(x, y) = 0 \text{ if and only if } x = y; \tag{1}$$

$$d(x, y) = d(y, x); \tag{2}$$

$$d(x, y) \leq d(x, z) + d(z, y). \tag{3}$$

If d is a distance on the set X, the couple (X, d) is called a ***metric space****.*

From now on the metric space (X, d) will be simply denoted with X as long as there are no chances of confusion.

Definition 1.2 (Ball)**.** *Let (X, d) be a metric space. For any real number $r > 0$ and for any point $x_0 \in X$, the set*

$$B_r(x_0) := \{x \in X \textit{ such that } d(x, x_0) < r\}$$

*i.e., the set of points of X whose distance from x_0 is strictly smaller than r, is called the **open ball** (or simply ball) of centre x_0 and radius r. The set*

$$\overline{B_r}(x_0) := \{x \in X \text{ such that } d(x, x_0) \le r\}$$

*is called the **closed ball** of centre x_0 and radius r.*

Definition 1.3 (Bounded set)**.** *Let X be a metric space. A subset Y of X is said to be **bounded** if there exists a ball in X (either open or closed) that contains Y.*

Definition 1.4 (Convergent sequence and Limit)**.** *Let X be a metric space and $\{x_k\}_{k\in\mathbb{N}}$ a sequence of points in X. The sequence $\{x_k\}_{k\in\mathbb{N}}$ is said to be **convergent** to the **limit** $\bar{x} \in X$ if for any real number $\varepsilon > 0$ there exist a natural number $\nu \in \mathbb{N}$ such that*

$$k > \nu \Longrightarrow d(x_k, \bar{x}) < \varepsilon.$$

Remark. It can be proved that a sequence $\{x_k\}_{k\in\mathbb{N}}$ is convergent to $\bar{x}$ in X if and only if

$$\lim_{k\in\mathbb{N}} d(x_k, \bar{x}) = 0.$$

Remark. It is not difficult to prove that each convergent sequence in a metric space admits a unique limit, so we can speak of *the* limit of a convergent sequence.

Definition 1.5 (Open set)**.** *A subset Y of a metric space X is called an **open** set if for any $x \in X$ there exists a real number $r > 0$ such that the ball of centre x and radius r is contained in Y.*

Definition 1.6 (Closed set)**.** *A subset Y of a metric space X is called a **closed** set if its complementary set (i.e. the set of points in X which are not in Y) is open.*

Remark. It can be proved that in a metric space X, a subset Y is closed if and only if the limit of any convergent sequence of points in Y still belongs to Y. It can be proved as well that any open ball is an open set and any closed ball is a closed set.

Definition 1.7 (Closure)**.** *Let X be a metric space and Y a subset of X. The **closure of** Y, which will be denoted with $\bar{Y}$, is the smallest closed set in X containing Y.*

Remark. It can be proved quite easily that the closure of a set always exists and that the a set is closed if and only if it coincides with its closure.

Definition 1.8 (Bounded sequence)**.** *A sequence $\{x_k\}_{k\in\mathbb{N}}$ in a metric space X is said to be **bounded** if there exists a bounded set Y in X such that any x_k belongs to Y.*

Remark. Any convergent sequence is bounded, while the converse is not true.

Definition 1.9 (Compact set)**.** *A subset Y of a metric space X is said to be **compact** if any sequence in Y admits a subsequence convergent to a point of Y.*

Remark. Any compact set can be proved to be bounded and closed, while the converse is not true.

Definition 1.10 (Precompact set)**.** *A subset Y of a metric space X is called **precompact** if $\bar{Y}$ is compact.*

Definition 1.11 (Cauchy sequence)**.** *A sequence $\{x_k\}_{k\in\mathbb{N}}$ in a metric space X is called a **Cauchy sequence** if for any real number $\varepsilon > 0$, there exists $\nu \in \mathbb{N}$ such that*

$$h, k > \nu \Longrightarrow d(x_h, x_k) < \varepsilon.$$

Remark. Any convergent sequence is a Cauchy sequence, but the converse is false in general, as pointed out by the following definition.

Definition 1.12 (Complete space)**.** *A metric space X is said to be **complete** if any Cauchy sequence in X is convergent.*

Definition 1.13 (Continuous function)**.** *Let (X, d_X) and (Y, d_Y) be two metric spaces and let x_0 be a point in X. A function $f : X \to Y$ is said to be **continuous in $\mathbf{x_0}$** if for any sequence $\{x_k\}_{k\in\mathbb{N}}$ in X convergent to x_0, the sequence $\{f(x_k)\}_{k\in\mathbb{N}}$ is convergent to $f(x_0)$ in Y, i.e.*

$$\lim_{k\in\mathbb{N}} d_X(x_k, x_0) = 0 \Longrightarrow \lim_{k\in\mathbb{N}} d_Y(f(x_k), f(x_0)) = 0 \tag{4}$$

*The function f is said to be **continuous** if it is continuous in every point of X.*

Remark. A function $f : X \to Y$ is continuous in $x_0 \in X$ if and only if for any real number $\varepsilon > 0$, there exists a real number $\delta > 0$ such that, for any $x \in X$,

$$d_X(x, x_0) < \delta \Longrightarrow d_Y(f(x), f(y)) < \varepsilon. \tag{5}$$

2 Linear spaces

In this section we will denote with K either the set of real numbers $\mathbb{R}$ or the set of complex numbers $\mathbb{C}$, with the usual product and sum. We will refer to K as a *field* and we will call its element *scalars.*

Definition 2.1 (Abelian group)**.** *Let A be a set and $+$ an internal operation on A, i.e. a function $+ : A \times A \to A$. For any a, b in A we will denote the element $+(a, b)$ with $a + b$. The couple $(A, +)$ is called an* ***abelian group*** *if, for any $a, b, c \in A$,*

- *$+$ is associative, i.e. $(a + b) + c = a + (b + c)$*
- *$+$ is commutative, i.e. $a + b = b + c$*

and if there exists $0 \in A$ such that, for any $a \in A$,

- *$a + 0 = a$*
- *there exists a unique element $-a \in A$ such that $a + -a = 0$*

The element 0 is called the ***neutral element*** *of A.*

Remark. The neutral element of an abelian group $(A, +)$ is unique, moreover if there exist $a, b \in A$ such that $a + b = a$, then $b = 0$

Definition 2.2 (Linear space)**.** *Let $(V, \oplus)$ be an abelian group, K a field and $* : K \times V \to V$ an external operation of K over V. The tuple $(V, \oplus, *)$ is called a* ***linear space*** *(or* ***vector space****) and its elements are called* ***vectors*** *if, for any $\alpha, \beta \in K$ and for every $\mathbf{x}, \mathbf{y}$,*

$$\alpha * (\mathbf{x} \oplus \mathbf{y}) = a * \mathbf{x} \oplus \alpha * \mathbf{y} \tag{6}$$

$$(\alpha + \beta) * \mathbf{x} = \alpha * \mathbf{x} \oplus \beta * \mathbf{x} \tag{7}$$

$$\alpha * (\beta * \mathbf{x}) = (\alpha\beta) * \mathbf{x} \tag{8}$$

$$1 * \mathbf{x} = \mathbf{x}. \tag{9}$$

From now on we will denote with $\mathbf{0}$ the neutral element of a linear space (also called its **zero vector**) and, for the sake of simplicity, with the usaual symbols of sum and product respectively its internal and external operation. The equations above thus read as follows:

$$\alpha(\mathbf{x} + \mathbf{y}) = a\mathbf{x} + \alpha\mathbf{y}$$

$$(\alpha + \beta)\mathbf{x} = \alpha\mathbf{x} + \beta\mathbf{x}$$

$$\alpha(\beta\mathbf{x}) = (\alpha\beta)\mathbf{x}$$

$$1\mathbf{x} = \mathbf{x}.$$

Remark. It is quite easy to prove that in a linear space V, the following holds for any $\mathbf{x} \in V$:

$$0 \cdot \mathbf{x} = \mathbf{0} \tag{10}$$

Definition 2.3 (Spanning set)**.** *Let $(V, +, \cdot)$ be a K-linear space and Y a subset of V. Y is called a **spanning set** (or a **set of generators**) for V if every vector* $\mathbf{x}$ *can be expressed as a finite linear combination of elements of Y, i.e. if there exist n vectors in Y and n scalars in K (respectively denoted with $\mathbf{y}_1 \dots \mathbf{y}_n$ and $\alpha_1 \dots \alpha_n$) such that*

$$\mathbf{x} = \alpha_1 \mathbf{y}_1 + \cdots + \alpha_n \mathbf{y}_n$$

Definition 2.4 (Linearly dependent and independent set)**.** *Let V be a linear space and Y a subset of Y. The set Y is said to be **linearly dependent** if the zero vector can be obtained as a nontrivial linear combination of elements of Y, i.e. if there exist n* nonzero *scalars $\alpha_1, \dots, \alpha_n$ and n* distinct *vectors $y_1, \dots, y_n$ in Y such that*

$$\mathbf{0} = \alpha_1 \mathbf{y}_1 + \cdots + \alpha_n \mathbf{y}_n.$$

*The set Y is said to be **linearly independent** if it is not linearly dependent.*

Remark. It is immediate to prove that any set containing the zero vector is linearly dependent

Definition 2.5 (Basis)**.** *A subset Y of a linear space V is said to be a **basis** of V if it is a linearly independent spanning set for V.*

Remark. It can be proved that all the bases of a linear space have the same "number" of elements (i.e. there is a 1 to 1 correspondence between any two of them) and that every linear space has a basis, so the following definition makes sense.

Definition 2.6 (Dimension)**.** *Let V be a linear space and B a basis for V. If B is finite and has n elements, we say that V has (finite) **dimension** n, otherwise V is said to have **infinite dimension**.*

3 Normed and Inner Product spaces

Definition 3.1 (Norm and normed space). *Let V be a K-linear space. A nonnegative functional $||\cdot||$, i.e. a function $||\cdot|| : V \to \mathbb{R}^+$ that maps each vector $\mathbf{x}$ into a real number $||\mathbf{x}|| \geq 0$, is called a **norm** on V if for any $\mathbf{x}, \mathbf{y} \in V$ and for any scalar λ the following hold:*

$$||\mathbf{x}|| = 0 \text{ if and only if } \mathbf{x} = \mathbf{0} \tag{11}$$

$$||\lambda \mathbf{x}|| = |\lambda| \cdot ||\mathbf{x}|| \tag{12}$$

$$||\mathbf{x} + \mathbf{y}|| \leq ||\mathbf{x}|| + ||\mathbf{y}|| \,. \tag{13}$$

*The couple $(V, ||\cdot||)$ is called a **normed space**.*

Remark. It is not difficult to prove that if $(V, ||\cdot||)$ is a normed space, the function

$$d : (\mathbf{x}, \mathbf{y}) \in V \times V \mapsto ||\mathbf{x} - \mathbf{y}|| \in \mathbb{R}$$

is a distance, (and it is called the distance *generated* by the norm $||\cdot||$); so any normed space can be naturally seen as a metric space, and the following definition makes sense.

Definition 3.2 (Banach space). *Let $(V, ||\cdot||)$ be a normed space. V is called a **Banach space** if it is complete as a metric space, i.e. if any Cauchy sequence in V converges with respect to the distance generated by the norm $||\cdot||$.*

Definition 3.3 (Inner product). *Let V be a K-linear space. A function $(\cdot, \cdot) : V \times V \to K$ is called **inner product** if, for any $\mathbf{x}, \mathbf{y}, \mathbf{z}$ in V and for any α, β in K,*

$$(\mathbf{x}, \mathbf{y}) = (\mathbf{y}, \mathbf{x})^* \tag{14}$$

$$(\mathbf{x}, \mathbf{x}) \geq 0 \text{ and } (\mathbf{x}, \mathbf{x}) = 0 \Leftrightarrow \mathbf{x} = \mathbf{0} \tag{15}$$

$$(\alpha \mathbf{x} + \beta \mathbf{y}, \mathbf{z}) = \alpha\, (\mathbf{x}, \mathbf{z}) + \beta\, (\mathbf{y}, \mathbf{z}) \tag{16}$$

where c^* denotes the complex conjugate of the complex number c.

*The couple $(V, (\cdot, \cdot))$ is called **inner product space**.*

It is worth noting that equation (14) ensures that $(\mathbf{x}, \mathbf{x})$ is a real number, so equation (15) makes sense.

Remark. If $(V, (\cdot, \cdot))$ is an inner product space, then the function

$$||\cdot|| : \mathbf{x} \in V \mapsto \sqrt{(\mathbf{x}, \mathbf{x})}$$

is a norm, so that each inner product space can be naturally seen as a normed space, and the following definition makes sense.

Definition 3.4 (Hilbert space). *Let $(V, (\cdot, \cdot))$ be an inner product space. V is called a **Hilbert space** if it is a Banach space.*

4 Operators

It is common in functional analysis to call **operator** a function between two linear spaces (especially function spaces) and **functional** a function that maps a linear space into its field K. For example a continuous function between normed spaces is often referred to as a **continuous operator** and the norm itself can be seen as a functional.

Definition 4.1 (Graph)**.** *Let $f : X \to Y$ be a function. The set*

$$\mathcal{G} = X \times f(X) \subseteq X \times Y$$

*is called **graph** of the function f.*

Definition 4.2 (Linear operator)**.** *Let V and Y be two K-linear spaces. A function $f : V \to Y$ is called a **linear transformation** or **linear operator** if, for any $\mathbf{x}, \mathbf{y}$ in V and for any scalar α,*

$$f(\mathbf{x} + \mathbf{y}) = f(\mathbf{x}) + f(\mathbf{y}) \tag{17}$$

$$f(\alpha\mathbf{x}) = \alpha f(\mathbf{x}). \tag{18}$$

Definition 4.3 (Bounded operator)**.** *Let $(V, ||\cdot||_V)$ and $(W, ||\cdot||_W)$ two normed spaces. A linear operator $f : V \to W$ is said to be **bounded operator** if there exist a real number $M > 0$ such that for any $\mathbf{v} \in V$,*

$$||f(\mathbf{v})||_W \leq M \, ||\mathbf{v}||_V \, ,$$

*the smallest of such M is called the **operator norm** of f and is denoted with $||f||$*

Remark. It is important to stress out that a linear operator is bounded *if and only if* it is continuous.

Definition 4.4 (Compact operator)**.** *Let V and W be two normed spaces. A linear operator $f : V \to W$ is called a **compact operator** if the image of any bounded set in V is precompact in W.*

References

[1] A. E. Taylor: *Introduction to Functional Analysis*, John Wiley & Sons Inc; 2Rev Ed edition (April 1980).

[2] J. R. Giles: *Introduction to the Analysis of Normed Linear Spaces*, Cambridge University Press; Australian Mathematical Society Lecture Series 13 (2000).

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