

# An example of potential games: network congestion games

Serena Fugaro

UNIVERSITÀ DEGLI STUDI DI NAPOLI " FEDERICO II"  
PhD in Matematica e Applicazioni

March 16th, 2018



# Agenda

- Introduction on potential games
- Definition of congestion games (CG)
- Complexity of finding Nash Equilibria for CG
- Example

# Potential games

A 2-players game  $\Gamma = \langle 2; X, Y; f, g \rangle$  is called **team game** if  $f = g$  and **symmetric game** if  $X = Y$  and  $f(x, y) = g(y, x)$ .

# Potential games

A 2-players game  $\Gamma = \langle 2; X, Y; f, g \rangle$  is called **team game** if  $f = g$  and **symmetric game** if  $X = Y$  and  $f(x, y) = g(y, x)$ .

## Definition

A 2-players game  $\Gamma = \langle 2; X, Y; f, g \rangle$  is called **exact potential game** if there exists a function  $P: X \times Y \mapsto \mathbb{R}$  such that:

$$\begin{aligned}
 f(x_1, y) - f(x_2, y) &= P(x_1, y) - P(x_2, y) & \forall x_1, x_2 \in X & \quad \forall y \in Y \\
 g(x, y_1) - g(x, y_2) &= P(x, y_1) - P(x, y_2) & \forall y_1, y_2 \in Y & \quad \forall x \in X
 \end{aligned}$$

# Potential games

A 2-players game  $\Gamma = \langle 2; X, Y; f, g \rangle$  is called **team game** if  $f = g$  and **symmetric game** if  $X = Y$  and  $f(x, y) = g(y, x)$ .

## Definition

A 2-players game  $\Gamma = \langle 2; X, Y; f, g \rangle$  is called **exact potential game** if there exists a function  $P: X \times Y \mapsto \mathbb{R}$  such that:

$$\begin{aligned} f(x_1, y) - f(x_2, y) &= P(x_1, y) - P(x_2, y) \quad \forall x_1, x_2 \in X \quad \forall y \in Y \\ g(x, y_1) - g(x, y_2) &= P(x, y_1) - P(x, y_2) \quad \forall y_1, y_2 \in Y \quad \forall x \in X \end{aligned}$$

Given a potential game  $\Gamma$ ,

$$\begin{aligned} (x^*, y^*) \in NE(\Gamma) &\iff \begin{cases} f(x^*, y^*) - f(x, y^*) \geq 0 & \forall x \in X \\ g(x^*, y^*) - g(x^*, y) \geq 0 & \forall y \in Y \end{cases} \\ &\iff \begin{cases} P(x^*, y^*) - P(x, y^*) \geq 0 & \forall x \in X \\ P(x^*, y^*) - P(x^*, y) \geq 0 & \forall y \in Y \end{cases} \end{aligned}$$

### Proposition

Given a potential game  $\Gamma$  and the **coordination game** associated to it  $\Gamma_P = \langle 2; X, Y; P, P \rangle$ , then  $NE(\Gamma) = NE(\Gamma_P)$ .

### Proposition

Given a potential game  $\Gamma$  we have  $\operatorname{argmax}_{X \times Y} P = P_{\max} \subseteq NE(\Gamma)$ .

### Theorem (Existence of Nash Equilibria)

If  $X$  and  $Y$  are compact sets and  $P$  is an upper semicontinuous function then  $\Gamma$  admits a Nash equilibrium.

## Congestion games (Rosenthal 1973)

A **congestion game** is a tuple  $(\mathcal{N}, \mathcal{R}, (\Sigma_i)_{i \in \mathcal{N}}, (d_r)_{r \in \mathcal{R}})$  where

- $\mathcal{N} = \{1, 2, \dots, n\}$  denotes the set of players
- $\mathcal{R} = \{1, 2, \dots, m\}$  is the set of resources
- $\Sigma_i$  is an arbitrary collection of subsets of  $\mathcal{R}$  and represents the strategy space for player  $i$
- $d_r: \mathbb{N} \mapsto \mathbb{Z}$  is the so called **delay function** i.e. a cost function associated with resource  $r$

## Congestion games (Rosenthal 1973)

A **congestion game** is a tuple  $(\mathcal{N}, \mathcal{R}, (\Sigma_i)_{i \in \mathcal{N}}, (d_r)_{r \in \mathcal{R}})$  where

- $\mathcal{N} = \{1, 2, \dots, n\}$  denotes the set of players
- $\mathcal{R} = \{1, 2, \dots, m\}$  is the set of resources
- $\Sigma_i$  is an arbitrary collection of subsets of  $\mathcal{R}$  and represents the strategy space for player  $i$
- $d_r: \mathbb{N} \mapsto \mathbb{Z}$  is the so called **delay function** i.e. a cost function associated with resource  $r$

Given a state of the game  $S = (S_1, S_2, \dots, S_n)$

- ▶ the **congestion**  $n_r(S)$  on resource  $r$  is  $|\{i \in \mathcal{N} : r \in S_i\}|$
- ▶ the **individual cost** of player  $i$  is  $\delta_i(S) = \sum_{r \in S_i} d_r(n_r(S))$
- ▶ an **improvement step** of player  $i$  is a change of its strategy from  $S_i$  to  $S'_i$  such that  $\delta_i(S_{-i}, S'_i) < \delta_i(S_{-i}, S_i)$

## Theorem (Rosenthal (1973))

Every CG has at least one pure-strategy Nash equilibrium.

**Proof [Hints]:** minimize  $\sum_{r \in \mathcal{R}} \sum_{i=0}^{n_r(S)} d_r(i)$

subject to:  $\sum_{r \in S_i} x_{r_i} = 1 \quad i = 1, \dots, n$

$$n_r(S) - \sum_{i=1}^n \sum_{r \in r_i} x_{r_i} = 0 \quad r = 1, \dots, m$$

$$x_{r_i} \in \{0, 1\} \quad i = 1, \dots, n \quad \square$$

## Theorem (Rosenthal (1973))

Every CG has at least one pure-strategy Nash equilibrium.

**Proof [Hints]:** minimize  $\sum_{r \in \mathcal{R}} \sum_{i=0}^{n_r(S)} d_r(i)$

subject to:  $\sum_{r \in S_i} x_{r_i} = 1 \quad i = 1, \dots, n$

$$n_r(S) - \sum_{i=1}^n \sum_{r \in r_i} x_{r_i} = 0 \quad r = 1, \dots, m$$

$$x_{r_i} \in \{0, 1\} \quad i = 1, \dots, n \quad \square$$

Potential function:  $\Phi(S) = \sum_{r \in \mathcal{R}} \sum_{i=0}^{n_r(S)} d_r(i)$

## Proposition (Finite improvement property)

For every congestion game, every sequence of improvement steps is finite.

# Network Congestion Games

In a **network congestion game** we are given

- a directed graph  $(V, E)$  where  $V = \{1, 2, \dots, n\}$  is the set of *nodes* and  $E = \{(i, j) : i, j \in V\}$  is the set of *edges*
- for each player, a source node and a destination node
- set of resources  $\mathcal{R} = E$
- set of strategies  $\Sigma_i = \{\text{paths from } i\text{'s source to } i\text{'s destination}\}$

The **delay** of an edge depends on the number of players using that edge.

## Aim

Every player looks for a **minimum delay path** connecting its source with its destination.

**Symmetric variant:** same source and same destination for all the players

# Complexity of finding Nash equilibria

A **local search problem**  $\Pi$  is given by the set of its instances  $\mathcal{I}_\Pi$ . For every instance  $I \in \mathcal{I}_\Pi$  we are given:

- a finite set of feasible solutions  $\mathcal{F}(I)$
- an objective function  $c: \mathcal{F}(I) \mapsto \mathbb{Z}$
- a neighbourhood  $\mathcal{N}(S, I) \subseteq \mathcal{F}(I)$  for every feasible solution  $S$

The *transition graph*: node  $v(S) \forall S \in \mathcal{F}(I)$  edge  $(v(S_1), v(S_2))$  if  $c(S_2)$  is strictly better than  $c(S_1)$ . The sinks are the local optima.

LS problem  $\Pi$  is in the class **PLS (Polynomially Local Search)** if there exist three polynomial time algorithms:

- A which computes  $\forall I \in \mathcal{I}_\Pi$  an  $S \in \mathcal{F}(I)$
- B which computes  $\forall I \in \mathcal{I}_\Pi, \forall S \in \mathcal{F}(I)$  the objective value  $c(S)$
- C which determines  $\forall I \in \mathcal{I}_\Pi, \forall S \in \mathcal{F}(I)$  whether  $S$  is locally optimal or not and finds a better solution in  $\mathcal{N}(S, I)$  in the latter case.

A **reduction** from a problem  $\Pi_1 \in PLS$  to a problem  $\Pi_2 \in PLS$  is two polynomial algorithms:

- A which maps every instance  $I \in \mathcal{I}_{\Pi_1}$  to an instance  $A(I)$  of  $\Pi_2$
- B which maps every local optimum of  $A(I)$  to a local optimum of  $I$ .

A local search problem  $\Pi$  from PLS is **PLS-complete** if every problem in PLS is PLS-reducible to  $\Pi$ .

A **reduction** from a problem  $\Pi_1 \in PLS$  to a problem  $\Pi_2 \in PLS$  is two polynomial algorithms:

- A which maps every instance  $I \in \mathcal{I}_{\Pi_1}$  to an instance  $A(I)$  of  $\Pi_2$
- B which maps every local optimum of  $A(I)$  to a local optimum of  $I$ .

A local search problem  $\Pi$  from PLS is **PLS-complete** if every problem in PLS is PLS-reducible to  $\Pi$ .

In a congestion game:

- ▶  $\mathcal{F}(I)$  is the set of the states of the game  $\Sigma = \times_{i=1}^n \Sigma_i$
- ▶ the objective function is Rosenthal's potential
- ▶  $\mathcal{N}(S, I)$  consists of those states that deviate from  $S$  only in one's player strategy
- ▶ sequences of improvement steps correspond to paths in the transition graph whose sinks are  $NE(CG)$ .

A **reduction** from a problem  $\Pi_1 \in PLS$  to a problem  $\Pi_2 \in PLS$  is two polynomial algorithms:

- A which maps every instance  $I \in \mathcal{I}_{\Pi_1}$  to an instance  $A(I)$  of  $\Pi_2$
- B which maps every local optimum of  $A(I)$  to a local optimum of  $I$ .

A local search problem  $\Pi$  from PLS is **PLS-complete** if every problem in PLS is PLS-reducible to  $\Pi$ .

In a congestion game:

- ▶  $\mathcal{F}(I)$  is the set of the states of the game  $\Sigma = \times_{i=1}^n \Sigma_i$
- ▶ the objective function is Rosenthal's potential
- ▶  $\mathcal{N}(S, I)$  consists of those states that deviate from  $S$  only in one's player strategy
- ▶ sequences of improvement steps correspond to paths in the transition graph whose sinks are  $NE(CG)$ .

Heuristic procedure based on **local search paradigm**: the number of states is exponential in the number of players  $\implies$  the procedure may fail!

## Theorem

*Computing Nash equilibria in congestion games is PLS-complete.*

**Proof [Hints]:** Reduction from MAX-CUT problem. □

## Theorem

*Computing Nash equilibria in congestion games is PLS-complete.*

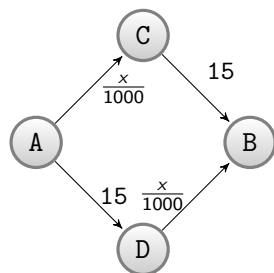
**Proof [Hints]:** Reduction from MAX-CUT problem. □

## Theorem

*There is a polynomial time algorithm for computing Nash equilibria in symmetric network congestion games with non-decreasing delay functions.*

**Proof [Hints]:** Reduction to min-cost flow problem. □

# Braess' paradox (1968)

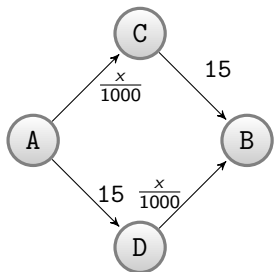


12000 players

Cost of path  $ACB$ :  $\frac{x}{1000} + 15$

Cost of path  $ADB$ :  $15 + \frac{x}{1000}$

# Braess' paradox (1968)



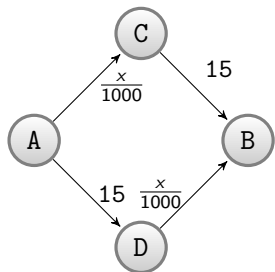
12000 players

Cost of path  $ACB$ :  $\frac{x}{1000} + 15$

Cost of path  $ADB$ :  $15 + \frac{x}{1000}$

$x$  players choose route  $ACB$  and  $12000 - x$  players choose route  $ADB$

## Braess' paradox (1968)



12000 players

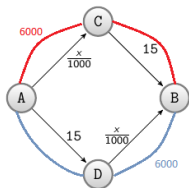
Cost of path  $ACB$ :  $\frac{x}{1000} + 15$

Cost of path  $ADB$ :  $15 + \frac{x}{1000}$

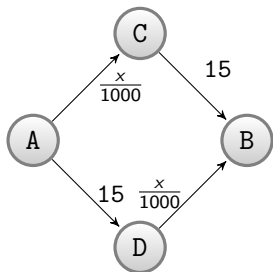
$x$  players choose route  $ACB$  and  $12000 - x$  players choose route  $ADB$

At the equilibrium no one wants to deviate i.e.

$$\frac{x}{1000} + 15 = \frac{12000-x}{1000} + 15 \implies x = 6000$$



# Braess' paradox (1968)



12000 players

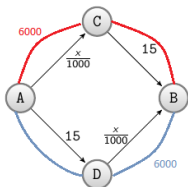
Cost of path  $ACB$ :  $\frac{x}{1000} + 15$

Cost of path  $ADB$ :  $15 + \frac{x}{1000}$

$x$  players choose route  $ACB$  and  $12000 - x$  players choose route  $ADB$

At the equilibrium no one wants to deviate i.e.

$$\frac{x}{1000} + 15 = \frac{12000-x}{1000} + 15 \implies x = 6000$$

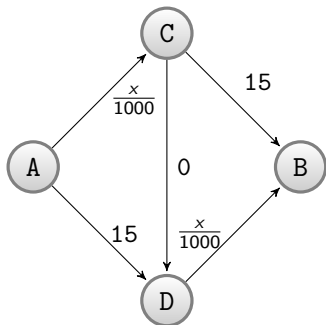


If the number of players is even, a state is a Nash equilibrium for the network congestion game if and only if the players equally divide among the two possible paths.

12000 players

Cost of path  $AC$ :  $\frac{x}{1000} < 15 \quad \forall x \leq 12000$

Cost of path  $DB$ :  $\frac{x}{1000} < 15 \quad \forall x \leq 12000$

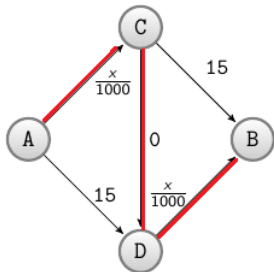
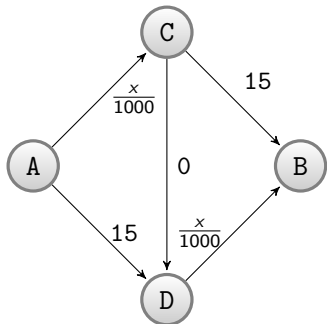


12000 players

Cost of path  $AC$ :  $\frac{x}{1000} < 15 \quad \forall x \leq 12000$

Cost of path  $DB$ :  $\frac{x}{1000} < 15 \quad \forall x \leq 12000$

Each player has an incentive to cross the freeway  $CD$



Total cost of  $ACDB$ :  $\frac{12000}{1000} + 0 + \frac{12000}{1000} > \frac{6000}{1000} + 15 \implies \text{PARADOX!}$

# Bibliography



Constantinos Daskalakis, Paul W Goldberg, and Christos H Papadimitriou. “The complexity of computing a Nash equilibrium”. In: *SIAM Journal on Computing* 39.1 (2009), pp. 195–259.



Robert W Rosenthal. “A class of games possessing pure-strategy Nash equilibria”. In: *International Journal of Game Theory* 2.1 (1973), pp. 65–67.



Tim Roughgarden. “CS364A: Algorithmic Game Theory Lecture# 19: Pure Nash Equilibria and PLS-Completeness”. In: (2013).



Berthold Vöcking and R Aachen. “Congestion Games: Optimization in Competition.” In: *ACiD*. 2006, pp. 9–20.

THANKS FOR THE ATTENTION