APPENDIX A

MATHEMATICAL AIDS

A. COORDINATE-FREE REPRESENTATION OF VECTORS AND TENSORS

The familiar rectangular Cartesian (x, y, z) coordinate system, used in Chapters 2, 3, 4, 5, and 8 to introduce physical concepts and conservation laws, is not always the most convenient for solution of practical problems. For general discussions of physical principles, and for the compact recording of results, a coordinate-free representation of objects such as vectors and tensors is very useful. This Appendix collects and organizes the results that are scattered throughout the book.[1]

The Algebra of Vectors and Tensors

In all that follows, a *vector* is conceived to be an object like an arrow, characterized by length and direction. Addition of vectors conforms to a parallelogram rule, shown in Fig. A1.

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[1] A more detailed mathematical treatment, which has greatly influenced the choices made here, can be found in Chapter 2 of *Theoretical Hydrodynamics*, by L. M. Milne-Thomson, Fifth Edition, MacMillan, New York, 1968.

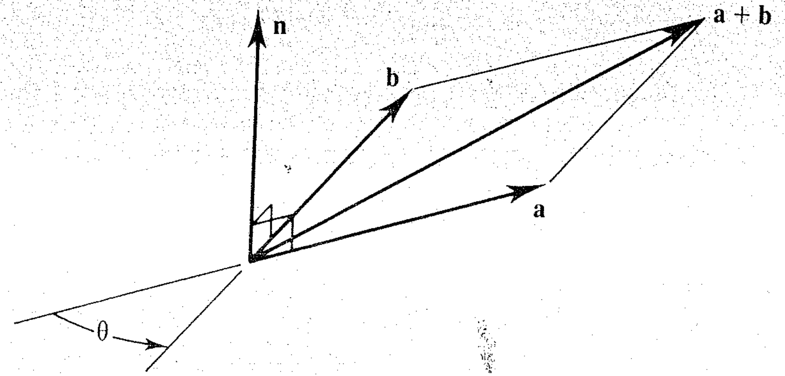

FIGURE A1

Three different results of multiplication of vectors will concern us. They are the *scalar product*

$$\mathbf{a} \cdot \mathbf{b} = ab \cos \theta \tag{A1}$$

the *vector product*

$$\mathbf{a} \times \mathbf{b} = ab \sin \theta \, \mathbf{n} \tag{A2}$$

and the *dyadic product* **a**; **b**, a second-rank tensor defined by the equation

$$\mathbf{v}(\mathbf{a}; \mathbf{b}) = (\mathbf{v} \cdot \mathbf{a})\mathbf{b} \tag{A3}$$

for any third vector **v**.

A *tensor of the second rank* or second order is a homogeneous linear algebraic operator which changes one vector into another; it may change both the length and direction of the original vector. If the original and resultant vectors are called **v** and **u** respectively, and the tensor is called A, this operation may be represented symbolically by the equation

$$\mathbf{v}\mathsf{A} = \mathbf{u} \tag{A4}$$

A matter of taste or of convention is involved in the relative placement of the vector and the tensor in the product expressions in (A3) and (A4). In this book, the vector is placed in front of the tensor. The implications of this are more clearly seen after we introduce an *orthogonal triad of unit vectors* $(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma})$ and represent an arbitrary vector as a sum of three orthogonal components, to wit

$$\mathbf{u} = u_\alpha \boldsymbol{\alpha} + u_\beta \boldsymbol{\beta} + u_\gamma \boldsymbol{\gamma} \tag{A5}$$

Because $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, and $\boldsymbol{\gamma}$ are orthogonal, we have

$$\boldsymbol{\alpha} \cdot \boldsymbol{\beta} = \boldsymbol{\beta} \cdot \boldsymbol{\gamma} = \boldsymbol{\gamma} \cdot \boldsymbol{\alpha} = 0 \tag{A6}$$

Once the direction of any two of the unit vectors is specified, that of the third is usually given by a *right-hand rule,* so that

$$\boldsymbol{\gamma} = \boldsymbol{\alpha} \times \boldsymbol{\beta}, \qquad \boldsymbol{\alpha} = \boldsymbol{\beta} \times \boldsymbol{\gamma}, \qquad \boldsymbol{\beta} = \boldsymbol{\gamma} \times \boldsymbol{\alpha} \tag{A7}$$

These unit vectors can be used to define components of a second-rank tensor, A, as follows:

$$\begin{aligned}\mathsf{A} = {} & A_{\alpha\alpha}(\boldsymbol{\alpha};\boldsymbol{\alpha}) + A_{\alpha\beta}(\boldsymbol{\alpha};\boldsymbol{\beta}) + A_{\alpha\gamma}(\boldsymbol{\alpha};\boldsymbol{\gamma}) \\ & + A_{\beta\alpha}(\boldsymbol{\beta};\boldsymbol{\alpha}) + A_{\beta\beta}(\boldsymbol{\beta};\boldsymbol{\beta}) + A_{\beta\gamma}(\boldsymbol{\beta};\boldsymbol{\gamma}) \\ & + A_{\gamma\alpha}(\boldsymbol{\gamma};\boldsymbol{\alpha}) + A_{\gamma\beta}(\boldsymbol{\gamma};\boldsymbol{\beta}) + A_{\gamma\gamma}(\boldsymbol{\gamma};\boldsymbol{\gamma})\end{aligned} \tag{A8}$$

Using (A5) and (A8), we can expand the multiplication formula $\mathbf{v}\mathsf{A} = \mathbf{u}$, to get

$$\begin{aligned}\mathbf{u} = {} & (v_\alpha A_{\alpha\alpha} + v_\beta A_{\beta\alpha} + v_\gamma A_{\gamma\alpha})\boldsymbol{\alpha} \\ & + (v_\alpha A_{\alpha\beta} + v_\beta A_{\beta\beta} + v_\gamma A_{\gamma\beta})\boldsymbol{\beta} \\ & + (v_\alpha A_{\alpha\gamma} + v_\beta A_{\beta\gamma} + v_\gamma A_{\gamma\gamma})\boldsymbol{\gamma}\end{aligned} \tag{A9}$$

Note that in each term the subscript on v is always the same as the neighboring subscript on A, while the more distant subscript on A identifies the component of the resulting vector, $\mathbf{u}$. Equation (A9) may also be written with the matrix multiplication formula

$$(v_\alpha, v_\beta, v_\gamma) = (u_\alpha, u_\beta, u_\gamma)\begin{bmatrix} A_{\alpha\alpha}A_{\alpha\beta}A_{\alpha\gamma} \\ A_{\beta\alpha}A_{\beta\beta}A_{\beta\gamma} \\ A_{\gamma\alpha}A_{\gamma\beta}A_{\gamma\gamma} \end{bmatrix}$$

The representation of a vector by a row array, rather than a column array, is consistent with the placement of $\mathbf{u}$ in front of $\mathbf{A}$ in Eq. (A4). Note that we are not narrow-minded about these conventions; in the description of the Thomas algorithm for coupled equations, in Chapter 10, we multiply a coefficient matrix times a solution vector in the other popular way, placing the vector behind the matrix, and representing it by a column, rather than a row. One convention works quite as well as the other; it is only important to know which one you have selected for a given task.

The transpose of a second-rank tensor. The tensor A is related to another tensor A^*, called its *transpose*, as follows: Let u and v be any two vectors; then

$$\mathbf{v}\cdot(\mathbf{u}\mathsf{A}^*) = \mathbf{u}\cdot(\mathbf{v}\mathsf{A}) \tag{A10}$$

If A and A^* are represented by use of the orthogonal triad of unit vectors, $(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma})$, and we pick $\mathbf{u} = \boldsymbol{\alpha}$ and $\mathbf{v} = \boldsymbol{\beta}$, we easily find from (A10) that $A^*_{\alpha\beta} = A_{\beta\alpha}$. Similarly, $A^*_{\beta\gamma} = A_{\gamma\beta}$, $A^*_{\gamma\alpha} = A_{\alpha\gamma}$, $A^*_{\alpha\alpha} = A_{\alpha\alpha}$, $A^*_{\beta\beta} = A_{\beta\beta}$ and $A^*_{\gamma\gamma} = A_{\gamma\gamma}$. The matrix array for A^* is simply obtained by interchanging the rows and the columns of that for A.

Symmetric and skew-symmetric tensors. Second-rank tensors can always be represented as the sum of a *symmetric tensor* and a *skew-symmetric tensor*. A symmetric tensor equals its transpose; a skew-symmetric tensor is the negative of its transpose. Thus, we can write

$$\mathsf{A} = \underset{\text{(symmetric)}}{(1/2)(\mathsf{A} + \mathsf{A}^*)} + \underset{\text{(skew-symmetric)}}{(1/2)(\mathsf{A} - \mathsf{A}^*)} \tag{A11}$$

Principal directions for a symmetric tensor. For any symmetric second-rank tensor S, there is a special orthogonal triad of projection vectors, say $(\boldsymbol{\alpha}', \boldsymbol{\beta}', \boldsymbol{\gamma}')$, such that all the off-diagonal components of the matrix array of the tensor vanish. Thus

$$\boldsymbol{\alpha}'\mathsf{S} = S_{\alpha'\alpha'}\boldsymbol{\alpha}', \qquad \boldsymbol{\beta}'\mathsf{S} = S_{\beta'\beta'}\boldsymbol{\beta}', \quad \text{and} \quad \boldsymbol{\gamma}'\mathsf{S} = S_{\gamma'\gamma'}\boldsymbol{\gamma}' \tag{A12}$$

We say that this special triad of unit vectors point out the *principal axes* of the symmetric tensor. We may also say that the three vectors, $S_{\alpha'\alpha'}\boldsymbol{\alpha}'$, $S_{\beta'\beta'}\boldsymbol{\beta}'$, and $S_{\gamma'\gamma'}\boldsymbol{\gamma}'$ constitute the *essence* of the symmetric tensor.

The unit tensor. A special symmetric tensor of great importance is the unit tensor I, which leaves a vector unchanged by multiplication. Thus, for any vector, $\mathbf{v}$,

$$\mathbf{v}\mathsf{I} = \mathbf{v} \tag{A13}$$

Vector associated with a skew-symmetric tensor. Suppose that B is a skew symmetric tensor. For any vector $\mathbf{v}$, we find that there is a vector $\mathbf{b}$, such that

$$\mathbf{v}\mathsf{B} = -\mathbf{v} \times \mathbf{b} \tag{A14}$$

In terms of scalar components, the relationship is

$$b_\alpha = B_{\beta\gamma}, \qquad b_\beta = B_{\gamma\alpha}, \qquad b_\gamma = B_{\alpha\beta}$$

One can say that the essence of a skew-symmetric tensor is its associated vector. Thus, if you find it easy to visualize a vector, but hard to visualize a second-rank tensor, you may be reassured to know that the tensor is equivalent to four vectors, the three that are the essence of its symmetric part, and the one that is the essence of its skew-symmetric part.

Multiplication of second-rank tensors. Two kinds of multiplication of second-rank tensors are important in viscous-flow theory. Let A and C be the tensors.

The *ordinary product,* $\mathsf{P} = \mathsf{AC}$, is another second-rank tensor, which acts on any vector, $\mathbf{v}$, as follows

$$\mathbf{v}\mathsf{P} = \mathbf{v}(\mathsf{AC}) = (\mathbf{v}\mathsf{A})\mathsf{C} \tag{A15}$$

When $\mathbf{v}, \mathsf{A}, \mathsf{C}$ and P are represented by (A5) and (A8), and coefficients of v_α, v_β, and v_γ are matched, we find that

$$\begin{aligned} P_{\alpha\alpha} &= A_{\alpha\alpha}C_{\alpha\alpha} + A_{\alpha\beta}C_{\beta\alpha} + A_{\alpha\gamma}C_{\gamma\alpha} \\ P_{\alpha\beta} &= A_{\alpha\alpha}C_{\alpha\beta} + A_{\alpha\beta}C_{\beta\beta} + A_{\alpha\gamma}C_{\gamma\beta} \\ P_{\alpha\gamma} &= A_{\alpha\alpha}C_{\alpha\gamma} + A_{\alpha\beta}C_{\beta\gamma} + A_{\alpha\gamma}C_{\gamma\gamma} \end{aligned} \tag{A16}$$

with six similar formulas for the remaining components. These formulas follow the general rule for the multiplication of square matrices.

The *scalar product* or *inner product,* $S = \mathsf{A} \cdot \mathsf{C}$, may be represented as follows, using an arbitrary triad of orthogonal unit vectors

$$\begin{aligned} S = \mathsf{A}\cdot\mathsf{C} &= (\boldsymbol{\alpha}\mathsf{A})\cdot(\boldsymbol{\alpha}\mathsf{C}) + (\boldsymbol{\beta}\mathsf{A})\cdot(\boldsymbol{\beta}\mathsf{C}) + (\boldsymbol{\gamma}\mathsf{A})\cdot(\boldsymbol{\gamma}\mathsf{C}) \\ &= A_{\alpha\alpha}C_{\alpha\alpha} + A_{\alpha\beta}C_{\alpha\beta} + A_{\alpha\gamma}C_{\alpha\gamma} \\ &\quad + A_{\beta\alpha}C_{\beta\alpha} + A_{\beta\beta}C_{\beta\beta} + A_{\beta\gamma}C_{\beta\gamma} \\ &\quad + A_{\gamma\alpha}C_{\gamma\alpha} + A_{\gamma\beta}C_{\gamma\beta} + A_{\gamma\gamma}C_{\gamma\gamma} \end{aligned} \tag{A17}$$

Note that if A and C are both symmetric, there will be only six independent terms in this sum. If they are not only symmetric, but share the same principal axes, the sum can be reduced to three terms by referring to the special triad $(\boldsymbol{\alpha}', \boldsymbol{\beta}', \boldsymbol{\gamma}')$.

Note also that if A is symmetric and C is skew-symmetric, or vice versa, $\mathsf{A}\cdot\mathsf{C} = 0$.

Differential Properties of Vector and Tensor Fields

If a scalar, a vector, or a tensor is defined at every point in some spatial domain, we can speak of a *scalar field,* a *vector field,* or a *tensor field.* The following local measures of the spatial nonuniformity of such a field are important in the study of fluid mechanics.

The gradient of a scalar, s. Consider two neighboring points, P and Q, separated by an infinitesimal displacement $d\mathbf{r} = \mathbf{r}(Q) - \mathbf{r}(P)$. The gradient of s at point P, a vector that we call ∇s or grad s, may be defined indirectly, without reference to any particular set of unit vectors, by the equation

$$ds = d\mathbf{r}\cdot\nabla s \tag{A18}$$

Alternatively, it may be defined directly by the equation

$$\nabla s = \lim_{V\to 0}\left(V^{-1}\iint_A \mathbf{n}s\,dA\right) \tag{A19}$$

The integral is executed over the bounding surface, A, of a small volume, V, that encloses the point P, as shown in Fig. A2. It multiplies each element of surface area by the local value of s, and assigns to the product the direction of the local outward normal unit vector, $\mathbf{n}$. The integral will thus be a vector quantity directed away from P toward the part of the surface A where s has the largest values. In the limit as $V\to 0$, it specifies the direction in which s increases most rapidly near P, and evaluates the spatial rate of that change.

The divergence of a vector, $\mathbf{v}$. The most natural definition of this quantity is analogous to the second definition of ∇s, and is written as

$$\operatorname{div}\mathbf{v} = \nabla\cdot\mathbf{v} = \lim_{V\to 0}\left(V^{-1}\iint_A \mathbf{n}\cdot\mathbf{v}\,dA\right) \tag{A20}$$

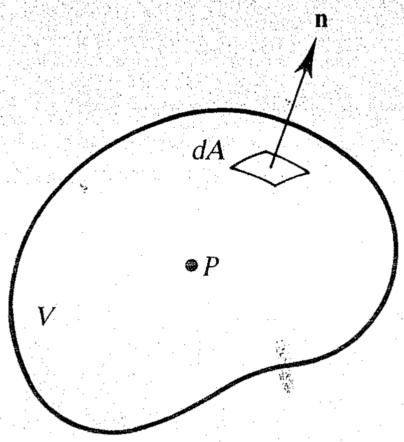

FIGURE A2

In this case, the integrand weights each infinitesimal element of boundary area with the outward normal component of the vector, $\mathbf{n} \cdot \mathbf{v}$. Thus the integral evaluates the net rate at which the vector field diverges away from point P.

The curl of a vector, $\boldsymbol{v}$. There are again two natural definitions, one utilizing the surface integral, in the form

$$\operatorname{curl} \mathbf{v} = \boldsymbol{\nabla} \times \mathbf{v} = \lim_{V \to 0} \left(V^{-1} \iint_A \mathbf{n} \times \mathbf{v} \, dA \right) \tag{A21}$$

In this case, the integrand weights each infinitesimal element of boundary area with the tangential component of the vector, $\mathbf{n} \times \mathbf{v}$. Thus the integral evaluates the net extent to which the vector field curls around P. Curl $\mathbf{v}$ is evidently a vector, and the alternative definition evaluates an arbitrary component of that vector. Let the unit vector $\boldsymbol{\alpha}$ be normal to an infinitesimal area A, which is surrounded by a closed curve C. The α-component of curl $\mathbf{v}$ is then given by the limit of a line integral around C, to wit

$$\boldsymbol{\alpha} \cdot \operatorname{curl} \mathbf{v} = \lim_{A \to 0} \left(A^{-1} \int_C \mathbf{v} \cdot d\mathbf{r} \right) \tag{A22}$$

Here $d\mathbf{r}$ is an element of the arclength, directed so that an observer standing on the surface A so that $\boldsymbol{\alpha}$ points up, and facing in the direction of $d\mathbf{r}$, would look left to see the surface. This formula is very convenient for calculation, as will soon be shown.

The gradient of a vector, $\boldsymbol{v}$. Consider again two neighboring points, P and Q, separated by an infinitesimal displacement $d\mathbf{r} = \mathbf{r}(Q) - \mathbf{r}(P)$. The gradient of $\mathbf{v}$ at point P, a second-rank tensor that we call $\boldsymbol{\nabla}; \mathbf{v}$ or grad $\mathbf{v}$, may be defined indirectly by the equation

$$d\mathbf{v} = d\mathbf{r}(\boldsymbol{\nabla}; \mathbf{v}) = (d\mathbf{r} \cdot \boldsymbol{\nabla})\mathbf{v} \tag{A23}$$

Alternatively, it may be defined directly by the equation

$$\operatorname{grad} \mathbf{v} \equiv \nabla; \mathbf{v} = \lim_{V\to 0} \left(V^{-1} \iint_A \mathbf{n}; \mathbf{v}\, dA \right) \tag{A24}$$

The second definition brings out the tensor character of $\nabla; \mathbf{v}$, but the first is generally more useful for calculation. Closely associated with $\operatorname{grad} \mathbf{v}$ is its *transpose*

$$(\operatorname{grad} \mathbf{v})^* \equiv (\nabla; \mathbf{v})^* = \lim_{V\to 0} \left(V^{-1} \iint_A \mathbf{v}; \mathbf{n}\, dA \right) \tag{A25}$$

and its decomposition into symmetric and skew-symmetric parts:

$$\operatorname{def} \mathbf{v} \equiv (1/2)[\operatorname{grad} \mathbf{v} + (\operatorname{grad} \mathbf{v})^*] \tag{A26}$$

$$\operatorname{rot} \mathbf{v} \equiv (1/2)[\operatorname{grad} \mathbf{v} - (\operatorname{grad} \mathbf{v})^*] \tag{A27}$$

The divergence of a second-rank tensor, T. For applications to fluid mechanics, the natural definition involves the surface integral

$$\operatorname{div} \mathsf{T} = \nabla \cdot \mathsf{T} = \lim_{V\to 0} \left(V^{-1} \iint_A \mathbf{n}\mathsf{T}\, dA \right) \tag{A28}$$

Since $\mathbf{n}\mathsf{T}$ is a vector, the integral evaluates an average value of that vector over the surface A.

A collection of useful identities. Differential operations must often be performed on a product of scalars and/or vectors, or performed repeatedly. Here are some useful results, in which s is a scalar, $\mathbf{a}$ and $\mathbf{b}$ are vectors, A is a second-rank tensor with transpose A^*, B is a skew-symmetric tensor, and I is the unit tensor.

$$\operatorname{grad}(\mathbf{a}\cdot\mathbf{b}) = \mathbf{b}(\operatorname{grad}\mathbf{a})^* + \mathbf{a}(\operatorname{grad}\mathbf{b})^*$$

$$\operatorname{div}(\mathbf{b}\mathsf{A}) = \mathbf{b}\cdot\operatorname{div}\mathsf{A}^* + \operatorname{grad}\mathbf{b}\cdot\mathsf{A}^*$$

$$\operatorname{curl}(\mathbf{a}s) = -\mathbf{a}\times\operatorname{grad}s + s\operatorname{curl}\mathbf{a}$$

$$\operatorname{div}(\mathbf{b};\mathbf{a}) = \mathbf{b}\operatorname{grad}\mathbf{a} + \mathbf{a}\operatorname{div}\mathbf{b}$$

$$\operatorname{curl}(\mathbf{a}\times\mathbf{b}) = \operatorname{div}[(\mathbf{b};\mathbf{a}) - (\mathbf{a};\mathbf{b})] = \mathbf{b}\operatorname{grad}\mathbf{a} + \mathbf{a}\operatorname{div}\mathbf{b} \underset{-}{\cancel{+}} \mathbf{a}\operatorname{grad}\mathbf{b} - \mathbf{b}\operatorname{div}\mathbf{a}$$

$$\operatorname{div}(s\mathsf{I}) = \operatorname{grad}s$$

$$\mathsf{I}\cdot\operatorname{grad}\mathbf{a} = \operatorname{div}\mathbf{a}$$

$$\mathbf{a}\operatorname{grad}\mathbf{a} \equiv (\mathbf{a}\cdot\nabla)\mathbf{a} = (1/2)\operatorname{grad}(\mathbf{a}\cdot\mathbf{a}) + (\operatorname{curl}\mathbf{a})\times\mathbf{a}$$

$$2\operatorname{div}(\operatorname{def}\mathbf{a}) = 2\operatorname{grad}(\operatorname{div}\mathbf{a}) - \operatorname{curl}(\operatorname{curl}\mathbf{a})$$

∇

$$\nabla^2 s \equiv \operatorname{div}(\operatorname{grad}s)$$

$$\nabla^2\mathbf{a} \equiv \operatorname{div}(\operatorname{grad}\mathbf{a}) = \operatorname{grad}(\operatorname{div}\mathbf{a}) - \operatorname{curl}(\operatorname{curl}\mathbf{a})$$

The last two lines define and evaluate the *Laplacian* operator, applied to a scalar or a vector.

$2 \operatorname{curl}\operatorname{curl} \underline{a} = \operatorname{div} 2 \operatorname{rot} \underline{a} = \operatorname{div}[\nabla; a - a; \nabla]$