B. REPRESENTATION OF DIFFERENTIAL QUANTITIES IN ORTHOGONAL COORDINATE SYSTEMS

For the analysis of any particular flow problem, points in the spatial domain will be located by reference to some convenient system of coordinates. The choice of a coordinate system usually depends on the configuration of the solid boundaries, or of the streamlines of some known part of the flow, such as the flow outside a boundary layer. Usually the coordinates will be orthogonal, so that three surfaces, on each of which one coördinate is constant, meet at right angles at any given point. Let the coordinates be called α, β, and γ. At the point of intersection there will be an orthogonal triad of unit vectors, $(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma})$, with $\boldsymbol{\alpha}$ pointing along the intersection of surfaces of constant $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$, in the direction in which α increases, and so on.

In general, the coordinates will not themselves be distances; the local distance between coordinate surfaces $\alpha = \alpha_1$ and $\alpha = \alpha_1 + d\alpha$ is given by the formula $dr_\alpha = h_\alpha \, d\alpha$. This introduces the *metric factor*, h_α. A general infinitesimal displacement will be represented as

$$d\mathbf{r} = (h_\alpha \, d\alpha)\boldsymbol{\alpha} + (h_\beta \, d\beta)\boldsymbol{\beta} + (h_\gamma \, d\gamma)\boldsymbol{\gamma} \tag{A29}$$

Variability of the unit vectors when the coordinate surfaces are curved. In general, the unit vectors $(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma})$, and the metric coefficients $(h_\alpha, h_\beta, h_\gamma)$, all vary from point to point. Consider, Fig. A3, an infinitesimal patch of the surface $\gamma = \text{constant}$. For the moment, we imagine the patch to be plane, and seen in true size. You can see that $\boldsymbol{\beta}$ at point B differs from $\boldsymbol{\beta}$ at point A, because it has rotated through a small angle $d\theta$. In fact, $(\partial\boldsymbol{\beta}/\partial\beta)\, d\beta = -d\theta\boldsymbol{\alpha}$. Looking at the other small triangle that involves $d\theta$, you can also see that $(h_\alpha \, d\alpha)\, d\theta = (\partial(h_\beta \, d\beta)/\partial\alpha)\, d\alpha$. Eliminating $d\theta$ between these equations, we find that $\partial\boldsymbol{\beta}/\partial\beta = -(\boldsymbol{\alpha}/h_\alpha)(\partial h_\beta/\partial\alpha)$. However, this is not the whole story for the general case, because then the path from A to B may also be curved in the surface $\alpha = \text{constant}$, which we see on edge. Looking down the normal to that surface, we see a picture just like the one considered above, and deduce that

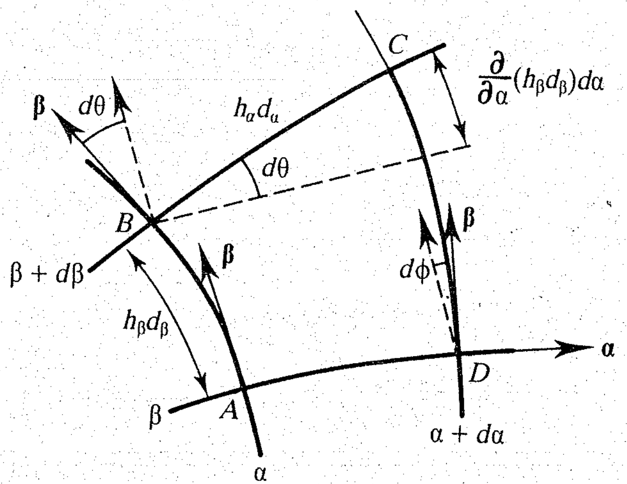

FIGURE A3

$\partial\boldsymbol{\beta}/\partial\beta = -(\boldsymbol{\gamma}/h_\gamma)(\partial h_\beta/\partial\gamma)$. In the general case, both the surface of constant α and that of constant γ are curved, and the partial effects add. Thus,

$$\partial\boldsymbol{\beta}/\partial\beta = -(\boldsymbol{\alpha}/h_\alpha)(\partial h_\beta/\partial\alpha) - (\boldsymbol{\gamma}/h_\gamma)(\partial h_\beta/\partial\gamma) \tag{A30}$$

and two corresponding equations obtained by cyclic permutations of α, β, and γ.

We see also that $\boldsymbol{\beta}$ at point D differs from $\boldsymbol{\beta}$ at point A, because it has rotated through a small angle $d\phi$. A further analysis of the sketch produces the results

$$\partial\boldsymbol{\beta}/\partial\alpha = (\boldsymbol{\alpha}/h_\beta)(\partial h_\alpha/\partial\beta) \quad \text{and} \quad \partial\boldsymbol{\alpha}/\partial\beta = (\boldsymbol{\beta}/h_\alpha)(\partial h_\beta/\partial\alpha) \tag{A31}$$

This time, these results are complete; views of the other coordinate surfaces only produce the companion relationships

$$\partial\boldsymbol{\gamma}/\partial\beta = (\boldsymbol{\beta}/h_\gamma)(\partial h_\beta/\partial\gamma), \qquad \partial\boldsymbol{\beta}/\partial\gamma = (\boldsymbol{\gamma}/h_\beta)(\partial h_\gamma/\partial\beta),$$
$$\partial\boldsymbol{\alpha}/\partial\gamma = (\boldsymbol{\gamma}/h_\alpha)(\partial h_\gamma/\partial\alpha), \qquad \partial\boldsymbol{\gamma}/\partial\alpha = (\boldsymbol{\alpha}/h_\gamma)(\partial h_\alpha/\partial\gamma).$$

These results affirm that the change in any of the unit vectors that results from a displacement in a surface normal to that vector is parallel to the displacement.[2]

The gradient operator. Let the small volume involved in Eq. (A19) be bounded by the coordinate surfaces $\alpha = \alpha_1$, $\alpha = \alpha_1 + d\alpha$, $\beta = \beta_1$, $\beta = \beta_1 + d\beta$, $\gamma = \gamma_1$, $\gamma = \gamma_1 + d\gamma$. The integral in (A19) is, to the first order of the infinitesimal quantities $d\alpha$, $d\beta$, and $d\gamma$, equal to

$$\begin{aligned}\iint &= [s\boldsymbol{\alpha}h_\beta\, d\beta h_\gamma\, d\gamma]_{\alpha 1+d\alpha} - [s\boldsymbol{\alpha}h_\beta\, d\beta h_\gamma\, d\gamma]_{\alpha 1}\\ &\quad + [s\boldsymbol{\beta}h_\gamma\, d\gamma h_\alpha\, d\alpha]_{\beta 1+d\beta} - [s\boldsymbol{\beta}h_\gamma\, d\gamma h_\alpha\, d\alpha]_{\beta 1}\\ &\quad + [s\boldsymbol{\gamma}h_\alpha\, d\alpha h_\beta\, d\beta]_{\gamma 1+d\gamma} - [s\boldsymbol{\gamma}h_\alpha\, d\alpha h_\beta\, d\beta]_{\gamma 1}\end{aligned}$$

Using the first term of a Taylor series to approximate each line, we get

$$d\alpha(s\boldsymbol{\alpha}h_\beta d\beta h_\gamma\, d\gamma)_{,\alpha} + d\beta(s\boldsymbol{\beta}h_\gamma\, d\gamma h_\alpha\, d\alpha)_{,\beta} + d\gamma(s\boldsymbol{\gamma}h_\alpha\, d\alpha h_\beta d\beta)_{,\gamma}$$

The corresponding approximation to the volume is $V = h_\alpha h_\beta h_\gamma\, d\alpha\, d\beta\, d\gamma$, so we get, initially,

$$\begin{aligned}\boldsymbol{\nabla}s &= (h_\alpha h_\beta h_\gamma)^{-1}[(\boldsymbol{\alpha}sh_\beta h_\gamma)_{,\alpha} + (\boldsymbol{\beta}sh_\gamma h_\alpha)_{,\beta} + (\boldsymbol{\gamma}sh_\alpha h_\beta)_{,\gamma}]\\ &= [(\boldsymbol{\alpha}/h_\alpha)s_{,\alpha} + (\boldsymbol{\beta}/h_\beta)s_{,\beta} + (\boldsymbol{\gamma}/h_\gamma)s_{,\gamma}]\\ &\quad + s(h_\alpha h_\beta h_\gamma)^{-1}[(\boldsymbol{\alpha}h_\beta h_\gamma)_{,\alpha} + (\boldsymbol{\beta}h_\gamma h_\alpha)_{,\beta} + (\boldsymbol{\gamma}h_\alpha h_\beta)_{,\gamma}]\end{aligned}$$

However, the cofactor of s in the last line is just $\boldsymbol{\nabla}(1)$, and the gradient

---

[2] To see this, insert toothpicks in a small but noticeably curved part of the surface of a piece of fruit or vegetable, taking care to get each one accurately normal to the surface. Then look straight down one of toothpicks. The others will all appear to lean directly away from that one.

of any constant is zero. This leaves

$$\boldsymbol{\nabla} s = (\boldsymbol{\alpha}/h_\alpha)s,_\alpha + (\boldsymbol{\beta}/h_\beta)s,_\beta + (\boldsymbol{\gamma}/h_\gamma)s,_\gamma \tag{A32}$$

Inserting this result and Eq. (A29) into (A18), the indirect definition of $\boldsymbol{\nabla} s$, we obtain

$$d\mathbf{r} \cdot \boldsymbol{\nabla} s = s,_\alpha \, d\alpha + s,_\beta \, d\beta + s,_\gamma \, d\gamma = ds$$

verifying that the Chain Rule is satisfied.

This last analysis suggests that the differential operator, $\boldsymbol{\nabla}$, can usefully be represented as

$$\boldsymbol{\nabla} = (\boldsymbol{\alpha}/h_\alpha)\, \partial/\partial\alpha + (\boldsymbol{\beta}/h_\beta)\, \partial/\partial\beta + (\boldsymbol{\gamma}/h_\gamma)\, \partial/\partial\gamma \tag{A33}$$

In many textbooks, this equation is taken as the definition of the gradient operator, and our definitions are derived as consequences.

Divergence of a vector. Evaluating the integral in (A20) with the same choice of bounding surfaces, and noting that $\mathbf{n} \cdot \mathbf{v} = v_\alpha$ on the surface $\alpha = \alpha_1 + d\alpha$, while $\mathbf{n} \cdot \mathbf{v} = -v_\alpha$ on the surface $\alpha = \alpha_1$ and so on, we find that

$$\boldsymbol{\nabla} \cdot \mathbf{v} = (h_\alpha h_\beta h_\gamma)^{-1}[(v_\alpha h_\beta h_\gamma),_\alpha + (v_\beta h_\gamma h_\alpha),_\beta + (v_\gamma h_\alpha h_\beta),_\gamma] \tag{A34}$$

Curl of a vector. It is easiest to calculate one component at a time, using (A22). To get the α-component, let the curve C lie in the surface $\alpha =$ constant, and be composed of the four segments shown in Fig. A.4. Note that on the segment AB $d\mathbf{r} = h_\beta \, d\beta \boldsymbol{\beta}$, and that $\mathbf{v} \cdot d\mathbf{r} = (v_\beta h_\beta \, d\beta)_{\gamma=\gamma 1}$. The line integral is, to the first order in infinitesimal quantities,

$$\int = (v_\beta h_\beta \, d\beta)_{\gamma=\gamma 1} + (v_\gamma h_\gamma \, d\gamma)_{\beta=\beta 1 + d\beta} - (v_\beta h_\beta \, d\beta)_{\gamma=\gamma 1 + d\gamma} - (v_\gamma h_\gamma \, d\gamma)_{\beta=\beta 1}$$

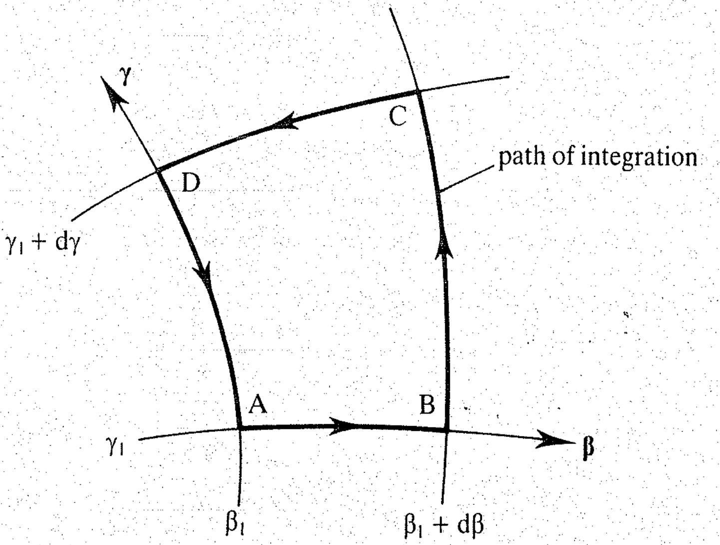

FIGURE A4

The area A is $h_\beta h_\gamma \, d\beta \, d\gamma$, and the Taylor-series expansion leads to the final result

$$\boldsymbol{\alpha} \cdot (\nabla \times \mathbf{v}) = (h_\beta h_\gamma)^{-1}[(v_\gamma h_\gamma),_\beta - (v_\beta h_\beta),_\gamma]$$

The results for all three components can conveniently be collected in the formula

$$\nabla \times \mathbf{v} = (h_\alpha h_\beta h_\gamma)^{-1} \begin{vmatrix} h_\alpha \boldsymbol{\alpha} & h_\beta \boldsymbol{\beta} & h_\gamma \boldsymbol{\gamma} \\ \partial/\partial\alpha & \partial/\partial\beta & \partial/\partial\gamma \\ h_\alpha v_\alpha & h_\beta v_\beta & h_\gamma v_\gamma \end{vmatrix} \tag{A35}$$

Gradient of a vector. If the direct definition, (A24), is analyzed as were (A19) and (A20), the preliminary result is

$$\nabla; \mathbf{v} = (h_\alpha h_\beta h_\gamma)^{-1}[(\boldsymbol{\alpha}; \mathbf{v} h_\beta h_\gamma),_\alpha + (\boldsymbol{\beta}; \mathbf{v} h_\gamma h_\alpha),_\beta + (\boldsymbol{\gamma}; \mathbf{v} h_\alpha h_\beta),_\gamma]$$

Again this can be simplified by expanding the derivatives of products and noting that $\nabla(1) = 0$. The result is

$$\nabla; \mathbf{v} = (\boldsymbol{\alpha}/h_\alpha); \partial\mathbf{v}/\partial\alpha + (\boldsymbol{\beta}/h_\beta); \partial\mathbf{v}/\partial\beta + (\boldsymbol{\gamma}/h_\gamma); \partial\mathbf{v}/\partial\gamma.$$

When differentiating $\mathbf{v}$ with respect to α, we must remember to differentiate the unit vectors as well as the scalar components. Thus

$$\begin{aligned} \partial\mathbf{v}/\partial\alpha &= \boldsymbol{\alpha}\, \partial v_\alpha/\partial\alpha + v_\alpha\, \partial\boldsymbol{\alpha}/\partial\alpha + \boldsymbol{\beta}\, \partial v_\beta/\partial\alpha + v_\beta\, \partial\boldsymbol{\beta}/\partial\alpha + \boldsymbol{\gamma}\, \partial v_\gamma/\partial\alpha + v_\gamma\, \partial\boldsymbol{\gamma}/\partial\alpha \\ &= \boldsymbol{\alpha}[\partial v_\alpha/\partial\alpha + (v_\beta/h_\beta)\, \partial h_\alpha/\partial\beta + (v_\gamma/h_\gamma)\, \partial h_\alpha/\partial\gamma] + \boldsymbol{\beta}[\ \ ] + \boldsymbol{\gamma}[\ \ ] \end{aligned}$$

where the terms in [ ] after $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$ follow by cyclic permutation of indices. When all these results are pulled together, the final result is

$$\begin{aligned} \nabla; \mathbf{v} = \operatorname{grad} \mathbf{v} = {} & \frac{1}{h_\alpha}\left(\frac{\partial v_\alpha}{\partial\alpha} + \frac{v_\beta}{h_\beta}\frac{\partial h_\alpha}{\partial\beta} + \frac{v_\gamma}{h_\gamma}\frac{\partial h_\alpha}{\partial\gamma}\right)(\boldsymbol{\alpha}; \boldsymbol{\alpha}) \\ & + \frac{1}{h_\alpha}\left(\frac{\partial v_\beta}{\partial\alpha} - \frac{v_\alpha}{h_\beta}\frac{\partial h_\alpha}{\partial\beta}\right)(\boldsymbol{\alpha}; \boldsymbol{\beta}) + \frac{1}{h_\alpha}\left(\frac{\partial v_\gamma}{\partial\alpha} - \frac{v_\alpha}{h_\gamma}\frac{\partial h_\alpha}{\partial\gamma}\right)(\boldsymbol{\alpha}; \boldsymbol{\gamma}) \\ & + \frac{1}{h_\beta}\left(\frac{\partial v_\alpha}{\partial\beta} - \frac{v_\beta}{h_\alpha}\frac{\partial h_\beta}{\partial\alpha}\right)(\boldsymbol{\beta}; \boldsymbol{\alpha}) + \frac{1}{h_\beta}\left(\frac{\partial v_\beta}{\partial\beta} + \frac{v_\gamma}{h_\gamma}\frac{\partial h_\beta}{\partial\gamma} + \frac{v_\alpha}{h_\alpha}\frac{\partial h_\beta}{\partial\alpha}\right)(\boldsymbol{\beta}; \boldsymbol{\beta}) \\ & + \frac{1}{h_\beta}\left(\frac{\partial v_\gamma}{\partial\beta} - \frac{v_\beta}{h_\gamma}\frac{\partial h_\beta}{\partial\gamma}\right)(\boldsymbol{\beta}; \boldsymbol{\gamma}) + \frac{1}{h_\gamma}\left(\frac{\partial v_\alpha}{\partial\gamma} - \frac{v_\gamma}{h_\alpha}\frac{\partial h_\gamma}{\partial\alpha}\right)(\boldsymbol{\gamma}; \boldsymbol{\alpha}) \\ & + \frac{1}{h_\gamma}\left(\frac{\partial v_\beta}{\partial\gamma} - \frac{v_\gamma}{h_\beta}\frac{\partial h_\gamma}{\partial\beta}\right)(\boldsymbol{\gamma}; \boldsymbol{\beta}) + \frac{1}{h_\gamma}\left(\frac{\partial v_\gamma}{\partial\gamma} + \frac{v_\alpha}{h_\alpha}\frac{\partial h_\gamma}{\partial\alpha} + \frac{v_\beta}{h_\beta}\frac{\partial h_\gamma}{\partial\beta}\right)(\boldsymbol{\gamma}; \boldsymbol{\gamma}) \end{aligned} \tag{A36}$$

Divergence of a tensor. Again the direct definition yields the computationally most useful form. Note that the surface $\alpha = \alpha_1 + d\alpha$ contributes $\boldsymbol{\alpha} \mathsf{A} h_\beta h_\gamma \, d\beta \, d\gamma$ to the surface integral; then use (A8) and remember that $\boldsymbol{\alpha}(\boldsymbol{\alpha}; \boldsymbol{\alpha}) = \boldsymbol{\alpha}$, $\boldsymbol{\alpha}(\boldsymbol{\alpha}; \boldsymbol{\beta}) = \boldsymbol{\beta}$, and $\boldsymbol{\alpha}(\boldsymbol{\alpha}; \boldsymbol{\gamma}) = \boldsymbol{\gamma}$, while $\boldsymbol{\alpha}$ times any dyadic in which the first member is $\boldsymbol{\beta}$ or $\boldsymbol{\gamma}$ equals zero. The final result is

$$\begin{aligned}\nabla\cdot\mathrm{A}=\operatorname{div}\mathrm{A}=\frac{\boldsymbol{\alpha}}{h_\alpha h_\beta h_\gamma}&\left\{\frac{\partial}{\partial\alpha}(h_\beta h_\gamma A_{\alpha\alpha})+\frac{\partial}{\partial\beta}(h_\gamma h_\alpha A_{\beta\alpha})+\frac{\partial}{\partial\gamma}(h_\alpha h_\beta A_{\gamma\alpha})\right.\\&\left.+h_\gamma\left(A_{\alpha\beta}\frac{\partial h_\alpha}{\partial\beta}-A_{\beta\beta}\frac{\partial h_\beta}{\partial\alpha}\right)+h_\beta\left(A_{\alpha\gamma}\frac{\partial h_\alpha}{\partial\gamma}-A_{\gamma\gamma}\frac{\partial h_\gamma}{\partial\alpha}\right)\right\}\\&+\boldsymbol{\beta}\{\ \}\boldsymbol{\gamma}\{\ \}\end{aligned}\tag{A37}$$

where the terms in $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$ come from those in $\boldsymbol{\alpha}$ by cyclic permutation of indices.

Convective rate of change. This is given by the scalar operator $(\mathbf{u}\cdot\nabla)$, where $\mathbf{u}$ is the fluid velocity. Its representation is simply

$$(\mathbf{u}\cdot\nabla)=(u_\alpha/h_\alpha)\,\partial/\partial\alpha+(u_\beta/h_\beta)\,\partial/\partial\beta+(u_\gamma/h_\gamma)\,\partial/\partial\gamma\tag{A38}$$

Convective acceleration. Knowing how to evaluate $\partial\mathbf{u}/\partial\alpha$, etc, we can quickly get the result

$$\begin{aligned}\mathbf{u}\operatorname{grad}\mathbf{u}=\boldsymbol{\alpha}&\left\{\frac{u_\alpha}{h_\alpha}\left(\frac{\partial u_\alpha}{\partial\alpha}+\frac{u_\beta}{h_\beta}\frac{\partial h_\alpha}{\partial\beta}+\frac{u_\gamma}{h_\gamma}\frac{\partial h_\alpha}{\partial\gamma}\right)+\frac{u_\beta}{h_\beta}\left(\frac{\partial u_\alpha}{\partial\beta}-\frac{u_\beta}{h_\alpha}\frac{\partial h_\beta}{\partial\alpha}\right)\right.\\&\left.+\frac{u_\gamma}{h_\gamma}\left(\frac{\partial u_\alpha}{\partial\gamma}-\frac{u_\gamma}{h_\alpha}\frac{\partial h_\gamma}{\partial\alpha}\right)\right\}+\boldsymbol{\beta}\{\ \}+\boldsymbol{\gamma}\{\ \}\end{aligned}\tag{A39}$$

Calculation of the metric coefficients. Most of the commonly used coordinate systems are related to a simple rectangular Cartesian (x, y, z) by transformation equations

$$x=x(\alpha,\beta,\gamma),\qquad y=y(\alpha,\beta,\gamma),\qquad z=z(\alpha,\beta,\gamma)$$

An infinitesimal displacement can be represented in either coordinate system; the square of its length is thus

$$dr^2=dx^2+dy^2+dz^2=h_\alpha^2\,d\alpha^2+h_\beta^2\,d\beta^2+h_\gamma^2\,d\gamma^2$$

From the Chain Rule, $dx=x,_\alpha\,d\alpha+x,_\beta\,d\beta+x,_\gamma\,d\gamma$, and so on. If $d\beta$ and $d\gamma$ are zero, we get

$$dx^2+dy^2+dz^2=(x,_\alpha^2+y,_\alpha^2+z,_\alpha^2)\,d\alpha^2=h_\alpha^2\,d\alpha^2$$

From this, and the corresponding special results if $d\gamma$ and $d\alpha$ are zero, of if $d\alpha$ and $d\beta$ are zero, it follows that

$$\begin{aligned}h_\alpha^2&=x,_\alpha^2+y,_\alpha^2+z,_\alpha^2\\h_\beta^2&=x,_\beta^2+y,_\beta^2+z,_\beta^2\\h_\gamma^2&=x,_\gamma^2+y,_\gamma^2+z,_\gamma^2\end{aligned}\tag{A40}$$

For example, in spherical polar (R, θ, ϕ) coordinates, $x=R\sin\theta\cos\phi$,

$y = R\sin\theta\sin\phi$, $z = R\cos\theta$, so $h_\phi^2 = (-R\sin\theta\sin\phi)^2 + (R\sin\theta\cos\phi)^2 + (0)^2 = R^2\sin^2\theta$.

Metric coefficients for popular coordinate systems

Rectangular Cartesian:

$$h_x = 1, \qquad h_y = 1, \qquad h_z = 1 \tag{A41}$$

Cylindrical polar, (r, θ, z): $x = r\cos\theta$, $y = r\sin\theta$, $z = z$;

$$h_r = 1, \qquad h_\theta = r, \qquad h_z = 1 \tag{A42}$$

Spherical polar, (R, θ, ϕ): $x = R\sin\theta\cos\phi$, $y = R\sin\theta\sin\phi$, $z = R\cos\theta$;

$$h_R = 1, \qquad h_\theta = R, \qquad h_\phi = R\sin\theta \tag{A43}$$

Plane elliptical, (Φ, Ψ, z): $x/L = \sinh\Phi\cos\Psi$, $y/L = \cosh\Phi\sin\Psi$, $z = z$.

$$h_\Phi = h_\Psi = L(\cosh^2\Phi - \sin^2\Psi)^{1/2}, \qquad h_z = 1 \tag{A44}$$

Plane parabolic, (ξ, η, z): $x/L = \xi^2 - \eta^2 + 1$, $y/L = 2\xi\eta$, $z = z$;

$$h_\xi = h_\eta = 2L(\xi^2 + \eta^2)^{1/2}, \qquad h_z = 1 \tag{A45}$$

C. EQUATIONS OF MOTION IN RECTANGULAR, CYLINDRICAL, AND SPHERICAL COORDINATES

With the formulas presented in A and B above, you should be able to work out the form of the equations of motion in any of the popular coordinate systems. Some of the most frequently used results are given here for easy reference. They all embody the assumption that ρ and μ are constants.

The representation of the viscous force per unit mass is worked out from the formula $\mathbf{f}_v = -\nu \operatorname{curl} \mathbf{\Omega}$, which is convenient in many idealized analyses in which there is only one scalar component of the vorticity. The formulas in other textbooks can be found from these by use of the fact that $\operatorname{div}\mathbf{u} = 0$. For the general orthogonal coordinate system, this representation is

$$\mathbf{f}_v = -\nu \operatorname{curl}\mathbf{\Omega} = -\nu(h_\alpha h_\beta h_\gamma)^{-1}\begin{vmatrix} h_\alpha \boldsymbol{\alpha} & h_\beta \boldsymbol{\beta} & h_\gamma \boldsymbol{\gamma} \\ \partial/\partial\alpha & \partial/\partial\beta & \partial/\partial\gamma \\ h_\alpha\Omega_\alpha & h_\beta\Omega_\beta & h_\gamma\Omega_\gamma \end{vmatrix}$$

Rectangular Cartesian Coordinates

Symbols for velocity components:

$$\mathbf{u} = u(x, y, z, t)\mathbf{e}_x + v(x, y, z, t)\mathbf{e}_y + w(x, y, z, t)\mathbf{e}_z$$

Vorticity:

$$\mathbf{\Omega} = \xi(x, y, z, t)\mathbf{e}_x + \eta(x, y, z, t)\mathbf{e}_y + \zeta(x, y, z, t)\mathbf{e}_z$$

where

$$\xi = w,_y - v,_z \qquad \eta = u,_z - w,_x \qquad \zeta = v,_x - u,_y \tag{A46}$$

Continuity equation:

$$\operatorname{div} \mathbf{u} = u,_x + v,_y + w,_z = 0 \tag{A47}$$

x-momentum:

$$u,_t + uu,_x + vu,_y + wu,_z + \Pi,_x = \nu(\eta,_z - \zeta,_y) \tag{A48}$$

y-momentum:

$$v,_t + uv,_x + vv,_y + wv,_z + \Pi,_y = \nu(\zeta,_x - \xi,_z) \tag{A49}$$

z-momentum:

$$w,_t + uw,_x + vw,_y + ww,_z + \Pi,_z = \nu(\xi,_y - \eta,_x) \tag{A50}$$

Viscous stresses:

$$\begin{gathered} \tau_{xx} = 2\mu u,_x \qquad \tau_{yy} = 2\mu v,_y \qquad \tau_{zz} = 2\mu w,_z \\ \tau_{xy} = \tau_{yx} = \mu(u,_y + v,_x) \qquad \tau_{yz} = \tau_{zy} = \mu(v,_z + w,_y) \\ \tau_{zx} = \tau_{xz} = \mu(w,_x + u,_z) \end{gathered} \tag{A51}$$

Cylindrical Polar Coordinates

Symbols for velocity components:

$$\mathbf{u} = u(r, \theta, z, t)\mathbf{e}_r + v(r, \theta, z, t)\mathbf{e}_\theta + w(r, \theta, z, t)\mathbf{e}_z$$

Vorticity:

$$\mathbf{\Omega} = \xi(r, \theta, z, t)\mathbf{e}_r + \eta(r, \theta, z, t)\mathbf{e}_\theta + \zeta(r, \theta, z, t)\mathbf{e}_z$$

where

$$\xi = r^{-1}w,_\theta - v,_z \qquad \eta = u,_z - w,_r \qquad \zeta = r^{-1}[(rv),_r - u,_\theta] \tag{A52}$$

Continuity equation:

$$\operatorname{div} \mathbf{u} = r^{-1}[(ru),_r + v,_\theta] + w,_z = 0 \tag{A53}$$

r-momentum:

$$u,_t + uu,_r + r^{-1}vu,_\theta + wu,_z - r^{-1}v^2 + \Pi,_r = \nu(\eta,_z - r^{-1}\zeta,_\theta) \tag{A54}$$

θ-momentum:

$$v,_t + uv,_r + r^{-1}vv,_\theta + wv,_z + r^{-1}uv + r^{-1}\Pi,_\theta = \nu(\zeta,_r - \xi,_z) \tag{A55}$$

z-momentum:

$$w,_t + uw,_r + r^{-1}vw,_\theta + ww,_z + \Pi,_z = \nu r^{-1}[\xi,_\theta - (r\eta),_r] \tag{A56}$$

Viscous stresses:

$$\begin{gathered} \tau_{rr} = 2\mu u,_r \qquad \tau_{\theta\theta} = 2\mu r^{-1}(v,_\theta + u) \qquad \tau_{zz} = 2\mu w,_z \\ \tau_{r\theta} = \tau_{\theta r} = \mu[r^{-1}w,_\theta + v,_z] \qquad \tau_{\theta z} = \tau_{z\theta} = \mu[r(v/r),_r + r^{-1}u,_\theta] \\ \tau_{zr} = \tau_{rz} = \mu[w,_r + u,_z] \end{gathered} \tag{A57}$$

Spherical Polar Coordinates

Symbols for velocity components:

$$\mathbf{u} = u(R, \theta, \phi, t)\mathbf{e}_R + v(R, \theta, \phi, t)\mathbf{e}_\theta + w(R, \theta, \phi, t)\mathbf{e}_\phi$$

Vorticity:

$$\mathbf{\Omega} = \xi(R, \theta, \phi, t)\mathbf{e}_R + \eta(R, \theta, \phi, t)\mathbf{e}_\theta + \zeta(R, \theta, \phi, t)\mathbf{e}_\phi$$

where

$$\xi = (R \sin\theta)^{-1}[(w \sin\theta)_{,\theta} - v_{,\phi}] \qquad \eta = (R \sin\theta)^{-1}u_{,\phi} - R^{-1}(Rw)_{,R}$$
$$\zeta = R^{-1}[(Rv)_{,R} - u_{,\theta}] \tag{A58}$$

Continuity equation:

$$\operatorname{div}\mathbf{u} = R^{-2}(R^2u)_{,R} + (R \sin\theta)^{-1}[(v \sin\theta)_{,\theta} + w_{,\phi}] = 0 \tag{A59}$$

R-momentum:

$$u_{,t} + uu_{,R} + R^{-1}vu_{,\theta} + (R \sin\theta)^{-1}wu_{,\phi} - R^{-1}(v^2 + w^2) - \Pi_{,R}$$
$$= \nu(R^2 \sin\theta)^{-1}[(R\eta)_{,\phi} - (R\zeta \sin\theta)_{,\theta}] \tag{A60}$$

θ-momentum:

$$v_{,t} + uv_{,R} + R^{-1}vv_{,\theta} + (R \sin\theta)^{-1}wv_{,\phi} + R^{-1}uv - (R \tan\theta)^{-1}w^2 + R^{-1}\Pi_{,\theta}$$
$$= \nu(R \sin\theta)^{-1}[(R\zeta \sin\theta)_{,R} - \xi_{,\phi}] \tag{A61}$$

ϕ-momentum:

$$w_{,t} + uw_{,R} + R^{-1}vw_{,\theta} + (R \sin\theta)^{-1}ww_{,\phi} + R^{-1}uw - (R \tan\theta)^{-1}vw$$
$$+ (R \sin\theta)^{-1}\Pi_{,\phi} = \nu R^{-1}[\xi_{,\theta} - (R\eta)_{,R}] \tag{A62}$$

Viscous stresses

$$\tau_{RR} = 2\mu u_{,R} \qquad \tau_{\theta\theta} = 2\mu R^{-1}(v_{,\theta} + u)$$
$$\tau_{\phi\phi} = 2\mu(R \sin\theta)^{-1}(w_{,\phi} + u \sin\theta + v \cos\theta)$$
$$\tau_{R\theta} = \tau_{\theta R} = \mu[R(v/R)_{,R} + R^{-1}u_{,\theta}]$$
$$\tau_{\theta\phi} = \tau_{\phi\theta} = \mu R^{-1}[\sin\theta(w \csc\theta)_{,\theta} + (\sin\theta)^{-1}v_{,\phi}]$$
$$\tau_{\phi R} = \tau_{R\phi} = \mu[(R \sin\theta)^{-1}u_{,\phi} + R(w/R)_{,R}] \tag{A63}$$