Mixing Your Moves

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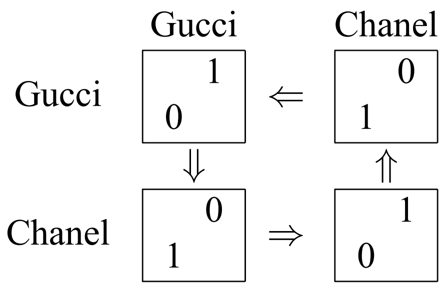
(Based on notes by David Myatt)

Objective:

- Extend games to include the possibility of randomization.
- Mixed strategies.
- Finding mixed strategy Nash equilibria.
- Domination by a mixed strategy and never-best-response.
- Rationalisability.
- Mixed strategies with continuous action spaces.
- Population frequency interpretation of mixed equilibria.

Strategic Form Games: Recap

- Representation of games in strategic form.
- Possible interpretation of strategies as plans of action.
- Solution concepts:
  - – Dominant strategy equilibrium.
  - – Iterated deletion of strictly dominated strategies.
  - – Mutual best response and Nash equilibrium.
- Can we interpret strategies in different ways?
- What happens when there are no (pure strategies) equilibria? e.g.:



Matching Pennies

Two players simultaneously display coins. If their faces match, then the first player wins her opponent's coin. If they differ, then the second player wins.

Players: Two individuals $i \in \{1,2\}$.

Actions: Player 1 chooses $x \in \{H,T\}$ and Player 2 chooses $y \in \{H,T\}$.

Payoffs: The payoffs in the strategic form matrix are:

	Heads	Tails
Heads	-1 1	1 -1
Tails	1 -1	-1 1

- The best-response structure is:

$$\begin{tikzcd}
 & \text{Heads} & \text{Tails} \\
\text{Heads} & \boxed{\begin{smallmatrix} & -1 \\ 1 & \end{smallmatrix}} \arrow[r, Rightarrow] & \boxed{\begin{smallmatrix} & 1 \\ -1 & \end{smallmatrix}} \arrow[d, Rightarrow] \\
\text{Tails} & \boxed{\begin{smallmatrix} & 1 \\ -1 & \end{smallmatrix}} \arrow[u, Rightarrow] & \boxed{\begin{smallmatrix} & -1 \\ 1 & \end{smallmatrix}} \arrow[l, Rightarrow]
\end{tikzcd}$$

⇒ There is no pure strategy Nash equilibrium.

- This can be a stylised description of many sporting and military scenarios (as the fashion guru game).

Mixed Strategies

- With *pure strategies* players use definite plans of action.
- Games also admit *mixed strategies*, involving randomisation between actions or pure strategies.
- A player might proceed as follows:
  - She outlines two (or more) pure strategies.
  - She then tosses a coin (or rolls a dice).
  - She chooses her pure strategy contingent on the realisation of the randomisation device.
- When a player adopts a mixed strategy her actions become *unpredictable*.
- Can we justify the consideration of mixed strategies?
  - A tennis player might randomise between serves to forehand and to backhand, so that the receiver cannot predict where the ball will land.
  - But it might be argued that people rarely randomise when choosing their actions.
  - A player might make a pure choice contingent on some variable that her opponent cannot observe. To the opponent, the action *appears* to be random.
- Population frequencies are also used as a justification (this is the original interpretation of Nash).

Mixing Scenarios

- **Matching Pennies:** Two players simultaneously display coins. If their faces match, then the first player wins her opponent's coin. If they differ, then the second player wins.
- **Tennis Match:** Martina Hingis and Monica Seles are playing tennis. Seles is about to attempt a passing shot — she can hit down the line or crosscourt. Hingis must choose to defend one of these.
- **Penalty Kicks:** The kicker decides whether to kick left or right and the goalkeeper simultaneously decides whether to dive left or right.
- **Technology Adoption**: Two researchers simultaneously decide whether to adopt Windows or Linux as their operating system. There are benefits to compatibility with coworkers, so researchers would rather adopt the same system.
- **Investment Race:** Two firms simultaneously choose investment levels. The firm with the highest investment level captures the entire market. If they both choose the same investment level, they split the market.

Strategic Games with Mixed Strategies

The "mixed extension" of a *strategic form game* consists of:

Players: A finite set $I = \{1, 2, \ldots, n\}$, with members i.

Actions: For each player i, a nonempty set A_i of available actions or *pure strategies*.

Mixed Strategies: The strategy set of player i is the set of *mixed strategies* $\Delta(A_i)$, i.e. the set of all probability distributions defined over A_i. A mixed strategy for i, $\sigma_i : A_i \to [0,1]$, assigns to each pure strategy $a_i \in A_i$ a probability $\sigma_i(a_i)$, where $\sum_{a_i \in A_i} \sigma_i(a_i) = 1$.

Payoffs: For each player i, a preference relation on the set of outcomes $A = \times_{j \in I} A_j$. This will be represented by a von Neumann Morgenstern utility function $\pi_i : A \to \mathbb{R}$. Player i's payoff, given a profile of mixed strategies $\sigma = (\sigma_1, ..., \sigma_n)$, is her expected utility:

$$\begin{aligned}\Pi_i(\sigma) &\equiv \mathbb{E}_\sigma[\pi_i] = \sum_{a \in A} \Pr(a) \cdot \pi_i(a) \\ &= \sum_{a \in A} [\sigma_1(a_1) \cdot ... \cdot \sigma_n(a_n)]\, \pi_i(a).\end{aligned}$$

- Note that a pure strategy a_i' can be interpreted as a mixed strategy which assigns probability 0 to all other strategies $a_i \neq a_i'$.

- For example, in the matching pennies game:
  - The set of pure strategies for player i is $A_i = \{\text{H, T}\}$.
  - The set of outcomes is $A = \{\text{HH, HT, TH, TT}\}$.
  - A mixed strategy for player 1 is a couple $(x,\ 1-x)$ where x represents the probability that he plays Heads and $(1-x)$ is the probability that he plays Tails. A mixed strategy for player 2 is $(y,\ 1-y)$.
  - A mixed strategy profile is $\sigma = (x, y)$.
  - Given σ, the payoff of player 1 is:

$$\Pi_1(x,y) = \mathbb{E}_{(x,y)}[\pi_1] = \sum_{a \in A} \Pr(a) \cdot \pi_1(a)$$

$$= \Pr(\text{HH}) \cdot \pi_1(\text{HH}) + \Pr(\text{HT}) \cdot \pi_1(\text{HT}) + \Pr(\text{TH}) \cdot \pi_1(\text{TH}) + \Pr(\text{TT}) \cdot \pi_1(\text{TT})$$

$$= xy(1) + x(1-y)(-1) + (1-x)y(-1) + (1-x)(1-y)(1).$$

(We now require the use of vNM utilities. This enables the payoff when facing a mixed strategy (a randomisation) to be an expected utility.)

Mixed Strategy Nash Equilibrium

- **Definition**: A mixed strategy profile $(\sigma_1^*, \sigma_2^*, ..., \sigma_n^*)$ is a *Nash equilibrium* (in mixed strategies) if, for every player i,

$$\Pi_i\left(\sigma_i^*, \sigma_{-i}^*\right) \geq \Pi_i\left(\sigma_i, \sigma_{-i}^*\right), \quad \forall \sigma_i \in \Delta(A_i).$$

- A mixed strategy Nash equilibrium is a Nash equilibrium of the “extended” game.
- In a NE, mixed strategies are mutual best responses: NE is the intersection of the best response correspondences.

- In mixed strategy equilibrium, given the strategy of his opponent, a player is *indifferent between all the pure strategies* that he plays with positive probability — all such strategies yield the same expected payoff.

- **Theorem:** *Every finite game (finite action set, finite number of players) has at least one Nash equilibrium, possibly in mixed strategies.*
- **Proof:** see Fudenberg and Tirole.

Playing Tennis

Marco and Riccardo are playing tennis. Riccardo is about to attempt a passing shot — he can hit down the line or crosscourt. Marco must choose to defend one of these options. A player's payoff is the percentage probability that she wins the point.

Players: The players are Riccardo (Row) and Marco (Column).

Actions: Riccardo can hit down the line (D) or crosscourt (C). Marco can defend either of these.

Mixed Strategies: Let $x \in [0,1]$ be the probability that Riccardo chooses D, and $y \in [0,1]$ be the probability that Marco chooses D.

Payoffs: The payoffs in the strategic form matrix are:

	D (y)	C $(1-y)$
D (x)	50 50	20 80
C $(1-x)$	10 90	80 20

- This is a *constant sum* or *strictly competitive* game.
- There is no pure strategy Nash equilibrium.

Riccardo's Response

- Consider the payoffs of Riccardo (Column) when Marco chooses probability y:

	D (y)	C $(1-y)$	Riccardo's Expected Payoff
D	50	80	$50y + 80\,(1-y)$
C	90	20	$90y + 20\,(1-y)$

- What is Riccardo's best response to Marco's play?
  - He should hit down the line (or $x = 1$) whenever:

$$\mathbb{E}\left[\pi_R\left(\mathrm{D}\right)\right] > \mathbb{E}\left[\pi_R\left(\mathrm{C}\right)\right]$$

$$\Leftrightarrow \quad 50y + 80(1-y) > 90y + 20(1-y)$$

$$\Leftrightarrow \quad 60 > 100y \quad \Leftrightarrow \quad y < 0.6.$$

  - He should hit crosscourt (or $x = 0$) whenever:

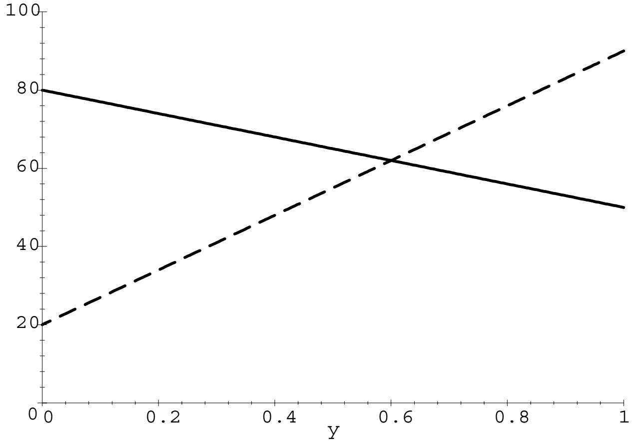
$$\mathbb{E}\left[\pi_R\left(\mathrm{D}\right)\right] < \mathbb{E}\left[\pi_R\left(\mathrm{C}\right)\right]$$

$$\Leftrightarrow \quad y > 0.6.$$

  - If $y = 0.6$, then *both* plays are optimal, and he is indifferent between *any* x.

	D (y)	C $(1-y)$	Riccardo's Expected Payoff
D	50	80	$50y + 80\,(1-y)$
C	90	20	$90y + 20\,(1-y)$

- Graphically, Riccardo's expected payoffs as a function of y are:



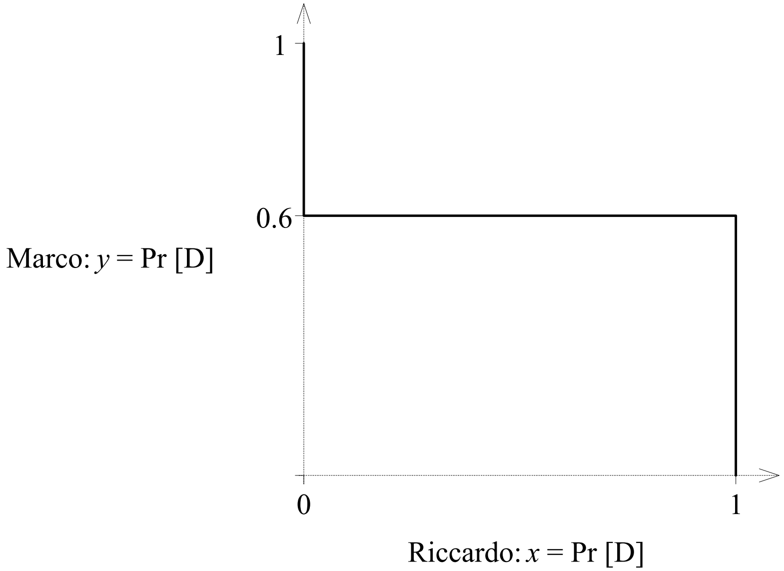
D (or $x = 1$): solid; C (or $x = 0$): dash

Riccardo's Reaction Function

- We can construct a reaction function for Riccardo:

$$x = \mathrm{BR}_R\,(y) = \begin{cases} 1 \text{ (play D)} & \text{if } y < 0.6 \\ \text{anything} & \text{if } y = 0.6 \\ 0 \text{ (play C)} & \text{if } y > 0.6 \end{cases}$$

- This is Riccardo's best response given his belief about Marco's play (i.e. y).
- Graphically:



(Best responses are now in the space of mixed stategies.)

Marco's Response

- Consider the payoffs of Marco (Column) when Riccardo chooses probability x:

	D	C
D (x)	50	20
C $(1-x)$	10	80
Marco's Expected Payoff	$50x + 10\,(1-x)$	$20x + 80\,(1-x)$

- What is Marco's best response to Riccardo's play?
  - He should defend down the line (or $y = 1$) whenever:

$$\mathbb{E}\left[\pi_M\left(\mathrm{D}\right)\right] > \mathbb{E}\left[\pi_M\left(\mathrm{C}\right)\right]$$

$$\Leftrightarrow \quad 50x + 10(1-x) > 20x + 80(1-x)$$

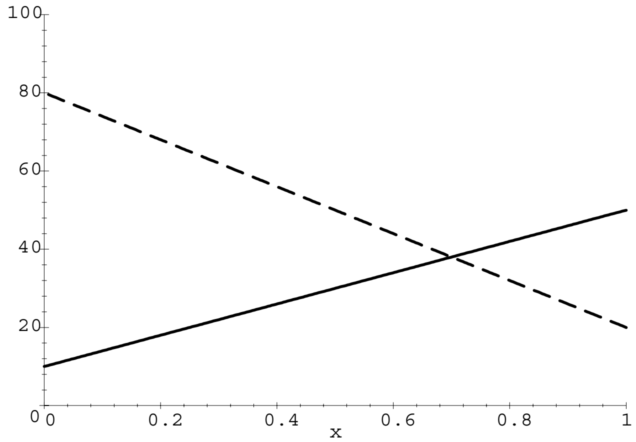
$$\Leftrightarrow \quad x > 0.7.$$

  - He should defend crosscourt (or $y = 0$) whenever:

$$\mathbb{E}\left[\pi_M\left(\mathrm{D}\right)\right] < \mathbb{E}\left[\pi_H\left(\mathrm{C}\right)\right] \quad \Leftrightarrow \quad x < 0.7.$$

  - If $x = 0.7$, then *both* plays are optimal, and he is indifferent between *any* y.

- Graphically:

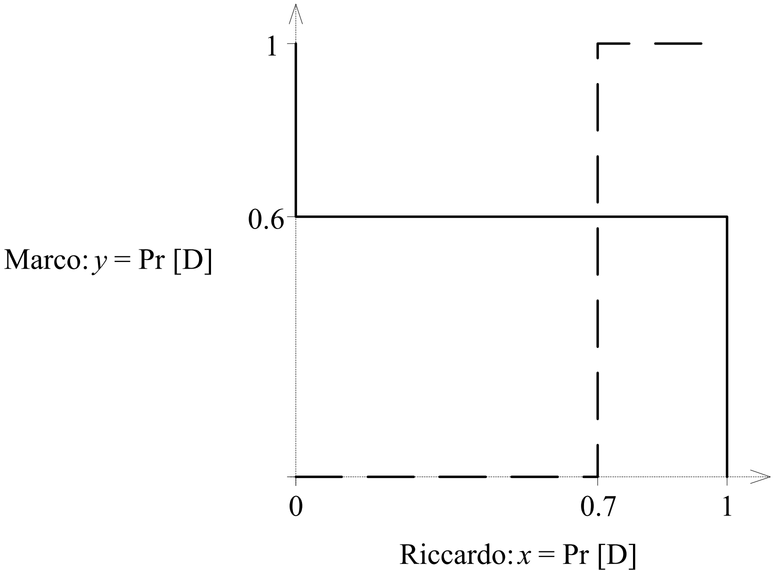


D (or $y = 1$): solid; C (or $y = 0$): dash

- We can construct a reaction function for Marco:

$$y = \mathrm{BR}_M\left(x\right) = \begin{cases} 0 \text{ (play C)} & \text{if } x < 0.7 \\ \text{anything} & \text{if } x = 0.7 \\ 1 \text{ (play D)} & \text{if } x > 0.7 \end{cases}$$

- To find a **mixed strategy Nash equilibrium** we need x and y such that:
  - For Riccardo, randomization with prob. x is a best response to Marco's randomization with prob. y.
  - For Marco, randomization with prob. y is a best response to Riccardo's randomization with prob. x.
- Hence, we need the intersection of the reaction functions.
- Add Marco's reaction function (dashed line) to obtain:



⇒ The unique (mixed strategy) Nash equilibrium is: $x = 0.7$ and $y = 0.6$.

- For these probabilities, players are indifferent between their available pure strategies.
- With this mixed equilibrium:
  - Each player randomizes between pure strategies.
  - The randomization makes the opponent indifferent.
  - Hence, due to indifference, players are happy to randomize.

- Questions:
  - Do players really randomize? Why should they bother?
  - Why would we expect indifferent players to randomize so precisely?

Coordination Revisited

Two friends, Chris and Patrick, need to meet up to discuss their love for economics. They can meet in either the pub or the cafe. They would both rather meet than miss each other, but have different preferences for where to meet.

Players: Patrick (row player) and Chris (column player).

Actions: The actions available to both players are Cafe and Pub.

Mixed Strategies: Patrick chooses Cafe with probability $x \in [0,1]$ and Chris chooses Cafe with probability $y \in [0,1]$.

Payoffs: Represent the payoffs in the strategic form matrix:

	Cafe (y)	Pub ($1-y$)	Expected
Cafe (x)	3 4	1 1	$4y+(1-y)$
Pub ($1-x$)	0 0	4 3	$3\,(1-y)$
Expected	$3x$	$x+4\,(1-x)$	

$\Rightarrow$ Chris should go to the Cafe ($y=1$) iff:

$$3x > x + 4(1-x) \quad \Leftrightarrow \quad 6x > 4$$
$$\Leftrightarrow \quad x > \frac{2}{3}.$$

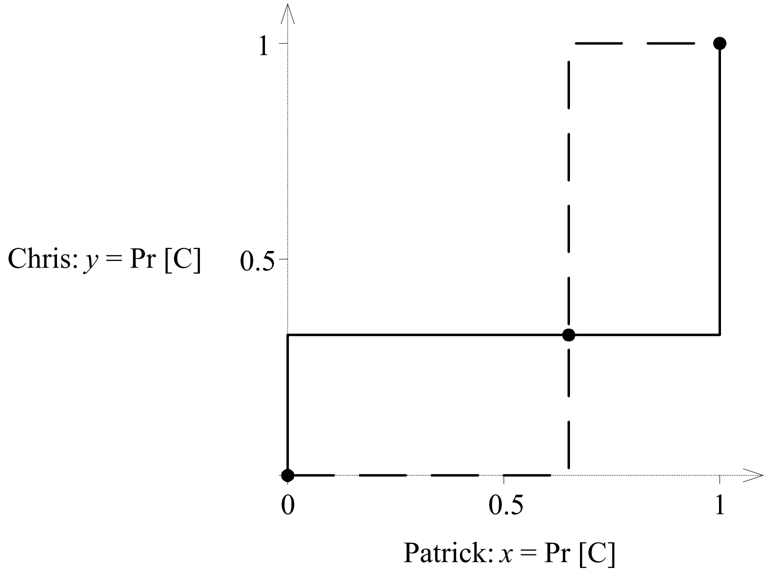
- For $x = \frac{2}{3}$, Chris is indifferent between going to the Cafe and going to the Pub.

$\Rightarrow$ Patrick should go to the Cafe ($x=1$) iff:

$$4y + (1-y) > 3(1-y) \quad \Leftrightarrow \quad 6y > 2$$
$$\Leftrightarrow \quad y > \frac{1}{3}.$$

- For $y = \frac{1}{3}$, Patrick is indifferent between going to the Cafe and going to the Pub.

- Plot the reaction functions and look for intersections:



- Again we have 2 equilibria in pure strategies.
- There is also a third Nash equilibrium in mixed strategies.

- **Proposition**: *Generically finite games have a finite and odd number of equilibria.*

  (In our examples reaction function are continuous. Hence, by a “fixed point” theorem they must always intersect.)

Deletion of Dominated Strategies: Recap

- Strategy A *strictly dominates* strategy B if it produces a *strictly higher payoff*, irrespective of the strategy choices made by others.
- We say that strategy B is *strictly dominated*.
- A rational player will never choose a strictly dominated strategy — so we can eliminate it in from the game.
- And we can then proceed to delete further strategies that are strictly dominated.
- Restricting to strategies which are not strictly dominated leaves us with a “smaller” game.

- We can extend this procedure to include mixed strategies.

Domination by a Mixed Strategy

- Consider the following game:

	Left	Middle	Right
Top	4 10	3 0	1 3
Bottom	0 0	2 10	10 3

- No strategy is strictly dominated by a pure strategy.
- But suppose that the column player chooses either L or M with probability $1/2$ each. Then his expected payoff is 5, whatever the row player does.
- This is strictly better than the payoff of 3 from playing R.

⇒ Strategy R is strictly dominated by a mix of strategies L and M, so we can eliminate it.

	L	M
T	4 10	3 0
B	0 0	2 10

⟶

	L	M
T	4 10	3 0

⟶

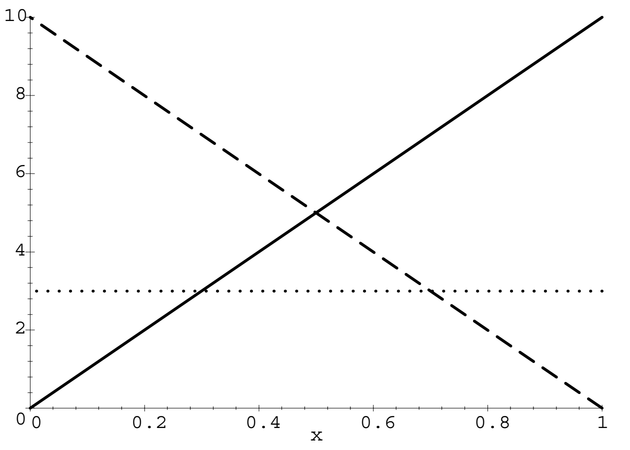
	L
T	4 10

- The game is dominance solvable, leaving us with (T, L).

- Let x for the probability that the row player chooses Top, and $(1-x)$ the probability that he chooses Bottom:

	L	M	R
T (x)	4 10	3 0	1 3
B $(1-x)$	0 0	2 10	10 3
Column's Expected Payoff	$10x$	$10(1-x)$	3

- Graphically, column's expected payoffs are:



Left: solid; Middle: dash; Right: dots

⇒ Irrespective of x, R is never a best response for column (it never yields a higher payoff than both L and M).

- In fact, a strategy is strictly dominated if and only if it is never a best response.

Rationalisability

- A weaker notion than Nash equilibrium is *rationalisability*:
- **Definition**: *A pure strategy is rationalisable if it is a best response to a set of beliefs held by a player about the strategies of his opponents.*

- A player must believe than an opponent will only play rationalisable strategies.
- Hence everyone optimises, and believes that all others optimise:
  - Players only adopt best responses.
  - These must be best responses to other potential best responses.
- In two players games, rationalisable strategies are those remaining after iterated deletion of strictly dominated strategies.
- Nash equilibria are always rationalisable (since it involves mutual best response).

Penalty Kicks

Penalty kicks may be represented as a strategic form game:

Players: Kicker (row) and Goalkeeper (column).

Actions: Kicker aims Left or Right. Goalkeeper dives Left or Right.

Mixed Strategies: Kicker kicks left with probability k, Goalkeeper dives left with probability g.

Payoffs: Kicker wishes to score, Goalkeeper wishes to prevent him:

$$\begin{tikzcd}
 & \text{Dive Left} & & \text{Dive Right} \\
\text{Kick Left} & \boxed{\text{Save}} \arrow[d, Rightarrow] & & \boxed{\text{Goal}} \arrow[ll, Rightarrow] \\
\text{Kick Right} & \boxed{\text{Goal}} \arrow[rr, Rightarrow] & & \boxed{\text{Save}} \arrow[u, Rightarrow]
\end{tikzcd}$$

- Based on 1,417 English, Italian and German premier league penalties, scoring probabilities are:

	Left	Right
Left	58.30%	94.97%
Right	92.91%	69.92%

- So, expressing the payoffs as percentages:

$$\begin{array}{lccc} & \text{Left } (g) & & \text{Right } (1-g) \\ \text{Left } (k) & \boxed{58.3 \quad 41.7} & \Leftarrow & \boxed{95.0 \quad 5.0} \\ & \Downarrow & & \Uparrow \\ \text{Right } (1-k) & \boxed{92.9 \quad 7.1} & \Rightarrow & \boxed{69.9 \quad 30.1} \end{array}$$

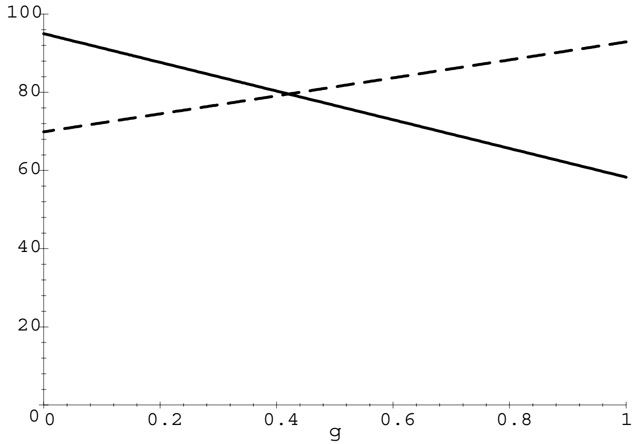
⇒ There is no pure strategy Nash equilibrium.

(“Kicking right” actually means “kicking in the favoured direction”.)

- Calculate expected payoffs for the kicker:

	L (g)	R $(1-g)$	Expected
L	58.3	95.0	$58.3g + 95\,(1-g)$
R	92.9	69.9	$92.9g + 69.9\,(1-g)$

- Graphically:



- The reaction function is:

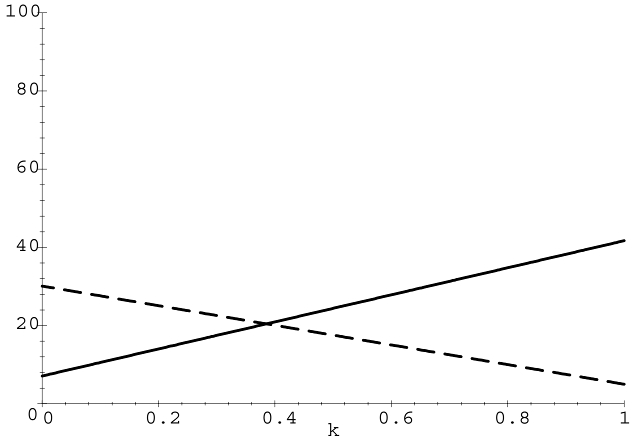
$$k = \text{BR}_K\,(g) = \begin{cases} 1 & \text{if } g < 0.4199 \\ \text{any} & \text{if } g = 0.4199 \\ 0 & \text{if } g > 0.4199 \end{cases}$$

⇒ Diving left with probability $g = 0.4199$ makes the kicker indifferent between Left or Right.

- Calculate expected payoffs for the goalkeeper:

	L	R
L (k)	41.7	5.0
R $(1-k)$	7.1	30.1
Expected	$41.7k + 7.1(1-k)$	$5k + 30.1(1-k)$

- Graphically:

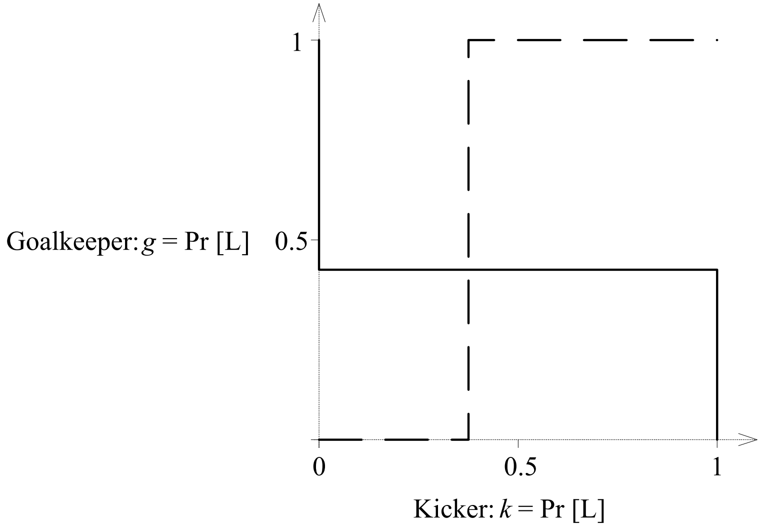


- Reaction function is:

$$g = \text{BR}_G(k) = \begin{cases} 0 & \text{if } k < 0.3854 \\ \text{any} & \text{if } k = 0.3854 \\ 1 & \text{if } k > 0.3854 \end{cases}$$

⇒ Kicking left with probability $k = 0.3854$ makes the goalkeeper indifferent between Left or Right.

- Put reaction functions together to find a Nash equilibrium:



⇒ There is a unique Nash Equilibrium in mixed strategies: $k = 0.3854$ and $g = 0.4199$.

- Compare to real mixing probabilities from the 1,417 observations:

	g	$1-g$	k	$1-k$
Predicted	41.99%	58.01%	38.54%	61.46%
Actual	42.31%	57.69%	39.98%	60.02%

⇒ Professionals play Nash! (see Palacios-Huerta: `http://www.econ.brown.edu/~iph/research.html`)

The Investment Race

Two firms simultaneously choose investment levels from the unit interval. The firm with the highest investment wins the market, which has unit value. If they both choose the same investment level, they split the market 50:50.

Players: Two firms $i \in \{1, 2\}$.

Actions: Investment levels $x \in [0, 1]$ and $y \in [0, 1]$.

Mixed Strategies: The probability distributions on the unit interval, $F(x)$ and $G(y)$ respectively.[1]

Payoffs: The firms' expected profits:

$$\pi_1(x, y) = \begin{cases} 1 - x & \text{if } x > y \\ \frac{1}{2} - x & \text{if } x = y \\ -x & \text{if } x < y \end{cases}$$

and

$$\pi_2(x, y) = \begin{cases} 1 - y & \text{if } y > x \\ \frac{1}{2} - y & \text{if } y = x \\ -y & \text{if } y < x \end{cases}$$

- There are no pure strategy equilibria:
  - If $y < 1$, then firm 1 would choose $x = y + \epsilon$.
  - If $x = y = 1$, then firm 1 would choose $x = 0$.
- But there is a mixed strategy Nash equilibrium.

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[1] Where $F(x) = \Pr[X < x]$.

- There can be no atoms in the mixing distribution:
  - Suppose that firm 1 chooses a particular value x with strictly positive probability.
  - Then firm 2 would never choose $y \in (x - \epsilon, x]$, but rather set $y > x$.
  - Hence firm 1 would do better to lower x slightly.
- A similar argument eliminates "gaps" in the distribution.

⇒ Firms mix *continuously* over the unit interval.

- For a mixed strategies equilibrium, a firm must be indifferent between the pure strategies available.
- Assume firm 2 adopts mixed strategy G.
- For firm 1, indifference between the strategies $x' = k$ and $x'' = k + \triangle$ yields:

$$\mathbb{E}\left[\pi_1\left(x'\right)\right] = \mathbb{E}\left[\pi_1\left(x''\right)\right]$$

$$\Leftrightarrow \quad 1 \cdot \Pr\left[1 \text{ wins with } x'\right] - x' = 1 \cdot \Pr\left[1 \text{ wins with } x''\right] - x''$$

$$\Leftrightarrow \quad \Pr\left[y < k\right] - k = \Pr\left[y < k + \triangle\right] - \left(k + \triangle\right)$$

$$\Leftrightarrow \quad G\left(k\right) - k = G\left(k + \triangle\right) - \left(k + \triangle\right)$$

$$\Leftrightarrow \quad G(k + \triangle) - G(k) = \triangle \quad \Leftrightarrow \quad G(k) = k.$$

- Hence, 1 is indifferent between his pure strategies if 2 mixes according to $G\left(y\right) = y$: the uniform distribution.
- By symmetry, 2 is indifferent between his pure strategies if 1 mixes according to $F\left(x\right) = x$.

⇒ The mixed strategy Nash equilibrium involves uniform mixing.

- The expected payoff is:

$$\begin{aligned}\mathbb{E}\left[\pi_1\right] &= \left(1 - x\right)\Pr\left[y < x\right] - x\Pr\left[y > x\right] \\ &= \left(1 - x\right)x - x\left(1 - x\right) = 0.\end{aligned}$$

Problems:

1. Guess the mixed strategy equilibrium of “matching pennies”. Check your answer.

2. Consider the following symmetric 2×2 “technology adoption” game:

	Windows	Linux
Windows	5 5	2 4
Linux	4 2	6 6

(i) By inspection, what are the pure strategy Nash equilibria?

(ii) Which one is the Pareto-optimal equilibrium?

(iii) If you had to play this game, would you choose Windows or Linux? Why?

(iv) Find the additional mixed strategy equilibrium by using the fact that if a player is willing to mix between two or more strategies, she will be indifferent between them in equilibrium.

(v) Draw the best-response correspondences. Where do they intersect?