Signalling Your Strenght

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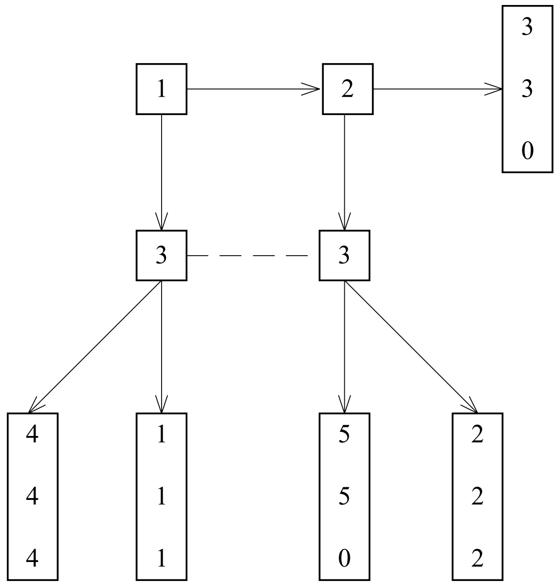
(Based on Notes by David Myatt)

Objective:

- Extensive form games with imperfect information.
- Incomplete information, and initial moves by nature.
- Sequential rationality.
- Forward induction.
- Prospects.
- Consistency of beliefs.
- Perfect Bayesian Equilibrium.
- Signalling: separating and pooling equilibria.
- Refinements: the intuitive criterion.

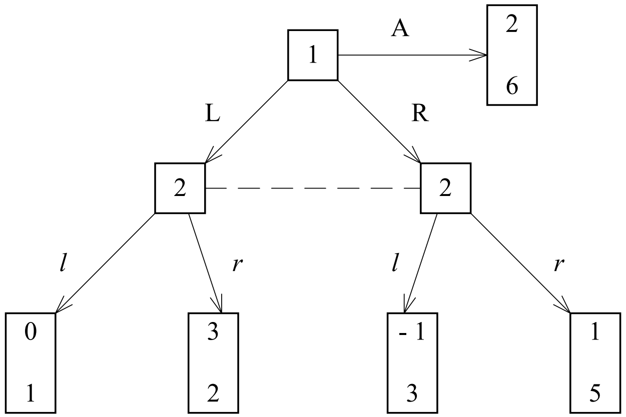
Subgame Perfection with Imperfect Information

- A subgame must always start at a singleton node.
- Hence, at the start of a subgame, all players know everything they need to know: beliefs do not rely on earlier actions.
- Games with incomplete information (and imperfect information) may have no subgames. Then the subgame perfection concept coincides with the Nash equilibrium concept.
- In this case, backward induction cannot take place, e.g.:



Sequential Rationality: Beyond Subgame Perfection

- Consider the following extensive form game with imperfect information:



- There is no proper subgames, so all Nash equilibria are subgame perfect — we cannot employ backward induction.
- Write the game in strategic form:

	l		r	
A	2	6	2	6
L	0	1	3	2
R	-1	3	1	5

- There are 2 Nash Equilibria: $\{$A; $l\}$ with payoffs (2,6) and $\{$L; $r\}$ with payoffs (3,2).

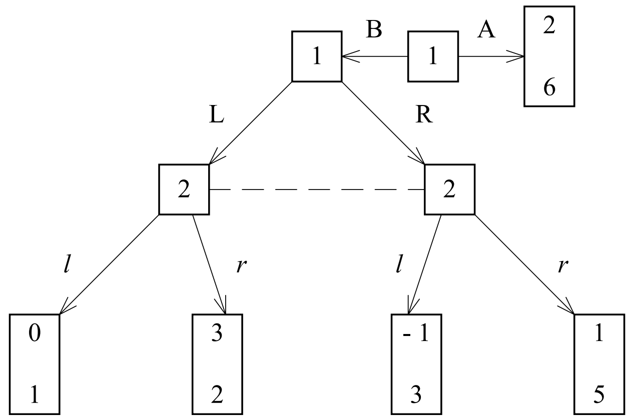
- But suppose Player 2 gets to move. Then it is optimal for her to play r.
- Therefore, Player 1 is better off playing L instead of A or R, obtaining a payoff of 3 instead of 2 or 1.

$\Rightarrow$ We expect the equilibrium to be $\{$L; $r\}$.

- We have required *sequential rationality* of strategies — a natural extension to backward induction.
- **Definition**: A strategy profile is **sequentially rational** if each player's strategy specifies an optimal action at every point in the game tree, given the player's beliefs.

Modifying the Extensive Form

- Consider re-writing the previous game in order to create a subgame:

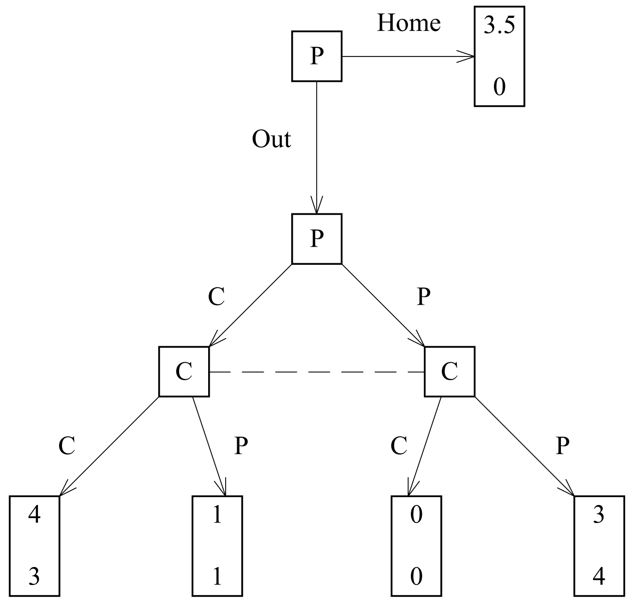


- We can now use backward induction.
- Consider the subgame starting at the leftmost node of Player 1.
- There is a unique NE of {L; r} in this subgame.
- Therefore, in the first node 1 is better off playing B.
- Hence {B–L; r} is the unique SPNE: the same equilibrium obtained before.

Coordination with Outside Option

Chris and Patrick, need to meet up to discuss their love for economics. They can meet in either the pub or the cafe. If they go out, they would both rather meet than not, but have different preferences for where to meet. Patrick, however, can choose not to go out: he can stay home and read a book. In fact, he even prefers reading a book to meeting Chris at the pub.

- Represent this game in extensive form:



⇒ There are two pure strategy subgame perfect equilibria: {Out–Cafe, Cafe} and {Home–Pub, Pub}.

Forward Induction

- In the pure coordination game, Chris does not know where to go.
- But here if Chris gets to move, he realises that Patrick has decided to go out.
- Then Chris concludes that Patrick must be expecting to meet him at the Cafe, otherwise he would have stayed home (since Patrick prefers staying home to meeting Chris at the Pub).
- So Chris should go to the Cafe (and Patrick prefers to go out).

⇒ {Out–Cafe, Cafe} is the only equilibrium that survives *forward induction*.

- This required us to consider the *beliefs* of Chris at his information set.

- Subgame perfection requires players at the start of a game to make choices based upon what rational players will do *later* in the game.
- Forward induction requires players at some late stage of a game to think about where rational play (by themselves and other) in *previous* stages of the game would have led.

Perfect Bayesian Equilibrium

- A *prospect* for a game consists of:

  Strategy Profile: A mapping from information sets to actions for each player.

  Beliefs: A probability distribution over each node in each information set for each player.

- Recall that a Nash equilibrium requires:

  Optimality: Players must optimise given their beliefs.

  Consistency: Players expect the same strategy profile.

- A *Perfect Bayesian Equilibrium* requires:

  (i) Each player to act optimally, given the opponents' strategies and her beliefs, at each information set.

  (ii) Beliefs must be mutually consistent, and consistent with the posited strategy profile. They must evolve via Bayes' Rule whenever applicable.

- **Def.**: A PBE is a set of beliefs μ and strategies σ s.t.:

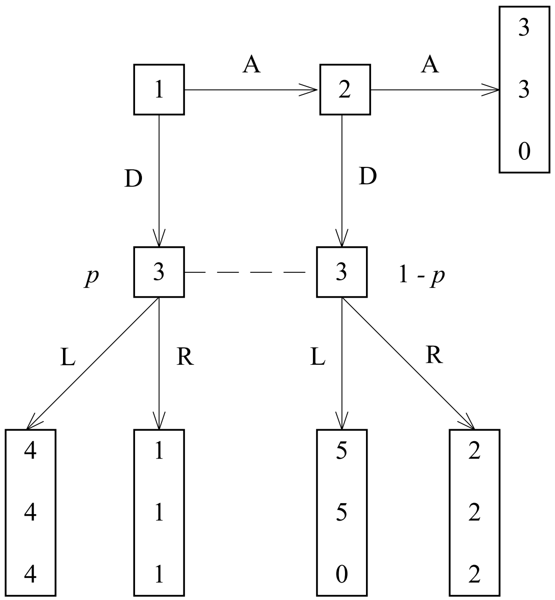
  (i) σ is sequentially rational given μ, and

  (ii) μ is consistent with σ.

  (In equilibrium, players have correct beliefs about their opponents' strategies, on the equilibrium path.)

Example

- Consider the following game:



- This is a *prospect* where:
  - A *strategy profile* is an action for each of the three players.
  - A *belief profile* is a probability distribution $(p, 1-p)$ over the two nodes in Player 3's information set.

Optimality Given Beliefs

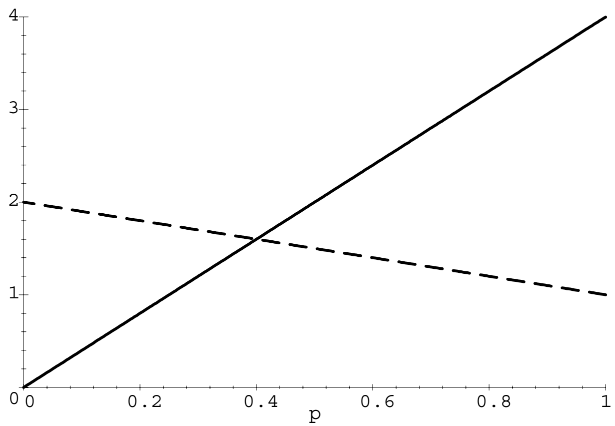
- Let p be Player 3's belief that she is at the left node of her information set.
- Then her payoffs from choosing L and R are:

$$\mathbb{E}\left[\pi\left(\mathrm{L}\right)\right] = 4p \quad \text{and} \quad \mathbb{E}\left[\pi\left(\mathrm{R}\right)\right] = p + 2(1-p)$$

- Player 3 chooses L iff:

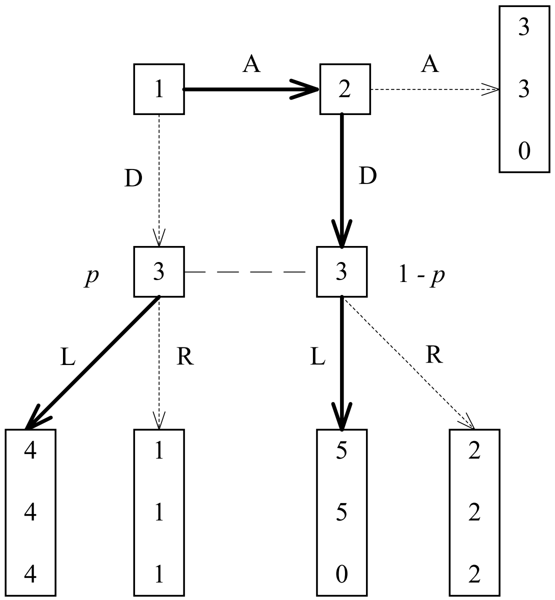
$$\mathbb{E}\left[\pi\left(\mathrm{L}\right)\right] > \mathbb{E}\left[\pi\left(\mathrm{R}\right)\right] \quad \Leftrightarrow \quad p > \tfrac{2}{5}.$$

- Graphically:



Inconsistency: Setting $p > \frac{2}{5}$

- Set $p > 2/5$, and compute best responses:

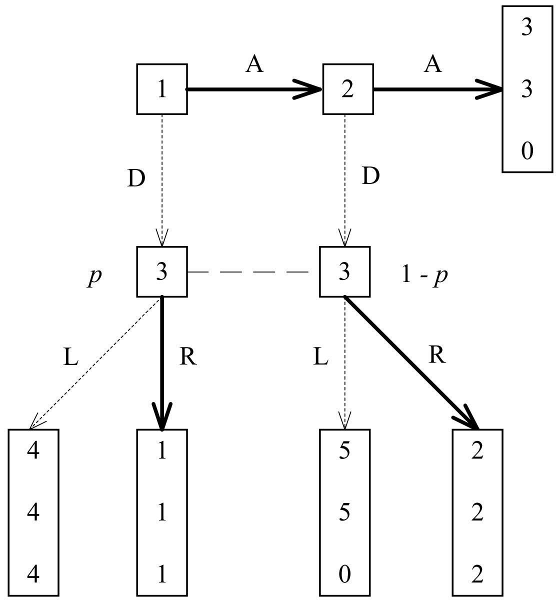


- Player 3, if she gets to move, chooses L.
- Hence Player 2, if she gets to move, chooses D (yielding a payoff of 5 instead of 3).
- Therefore, Player 1 chooses A.

⇒ The belief $p > 2/5$ is inconsistent with the play of the game — given this belief, 1 does not choose D and hence p should be 0.

Consistency: Setting $p < \frac{2}{5}$

- Set $p < 2/5$, and compute best responses:



- Player 3 chooses R. Hence Player 2, if she gets to move, chooses A.
- Therefore, Player 1 chooses A.

⇒ Any belief $p < 2/5$ is consistent with the play of the game, since 3's information set is never actually reached — a Perfect Bayesian Equilibrium (but not uniquely defined).

Refinements

- In a PBE, no restriction is placed on beliefs off the equilibrium path: any belief can be assigned at information sets that are not reached with positive probability.
- Extensive form refinements deal with this issue:
- **(Strong) Perfect Bayesian**: Players act optimally at all information sets, given their beliefs. Beliefs are formed using the posited strategy profile and Bayes' Rule where possible. Off the equilibrium path, beliefs can be arbitrary, but must follow Bayes from then on if possible.
- **Sequential:** As above, plus an extra restriction on beliefs off the equilibrium path. There must be a sequence of completely mixed strategy profiles that converge to the equilibrium strategy profile, generating a sequence of beliefs converging to equilibrium beliefs.
- **Trembling Hand Perfect:** There must be a fully mixed profile arbitrarily close to the posited profile that has a best response that is also arbitrarily close.

Trembling Hand Perfect

$\Downarrow$

Sequential

$\Downarrow$

Perfect Bayesian

- But these concepts often do not yield unique equilibria. Further refinements include:
- **Passive Beliefs**: Off the equilibrium path, players retain their prior beliefs.
- **Intuitive Criterion**: Players form off-path beliefs that justify the rationality of the off-equilibrium move.

The Beer-Quiche Game

Marco and Jack go to the pub. Marco likes starting fights. but only if his opponent is weaker. Jack prefers to avoid conflict. Marco does not know whether Jack is *strong* or *weak*. Before Marco decides whether to pick a fight with Jack, he observes Jack's order at the bar: either *beer* or *quiche*. Jack prefers to drink beer if he is strong and to eat quiche if he is weak. He would rather pick his preferred refreshment, but also wants to avoid conflict.

- **Players:** Nature, Marco and Jack.
- **Order of Play**: Play proceeds as follows:

  (1) Nature decides whether Jack is weak (with prob. 0.2) or strong (with prob. 0.8).

  (2) Jack observes his type, and chooses either "beer" or "quiche".

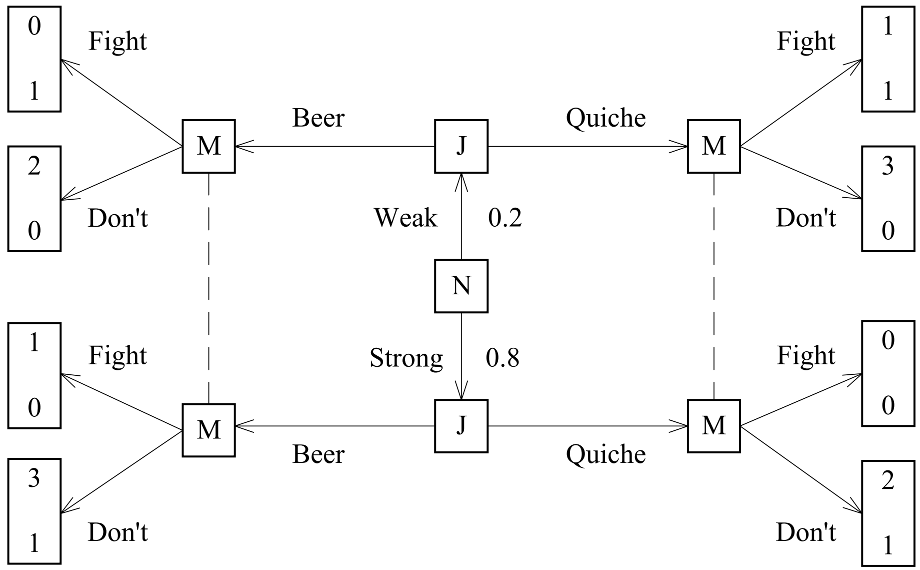
  (3) Marco observes Jack's choice, but not his type, and chooses "fight" or "don't fight".

- **Payoffs:**

  (a) Jack gets 1 if she consumes his favourite refreshment, and 2 for avoiding a fight.

  (b) Marco gets 1 if he fights a weak opponent, or avoids fighting a strong one, and 0 otherwise.

- Consider the extensive form of the game:

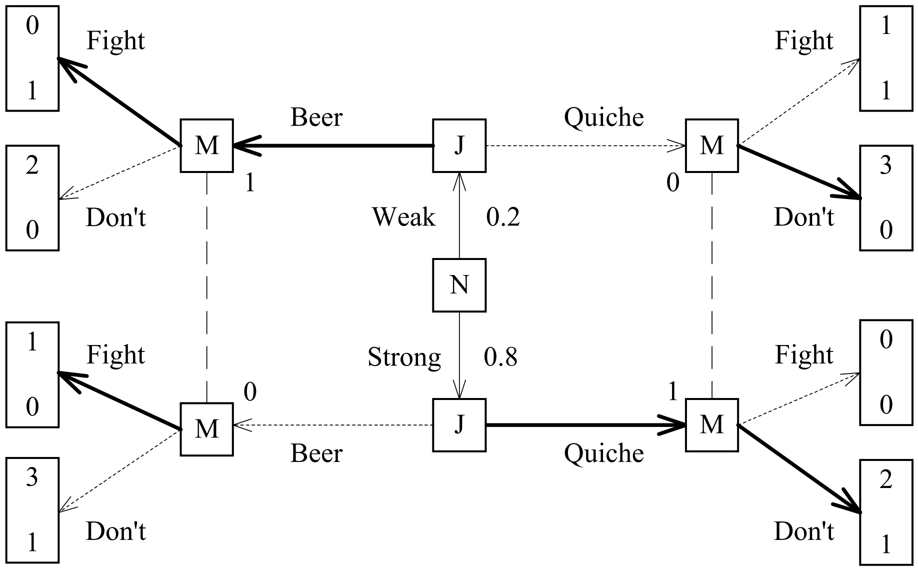


- This is a **signalling game**: after Nature selects his type, Jack sends a signal to Marco.
- There are two types of equilibrium:

  (i) A **separating** equilibrium in which different types send different signals, and hence are found out.

  (ii) A **pooling** equilibrium in which all types send the same signal, and hence are not found out.

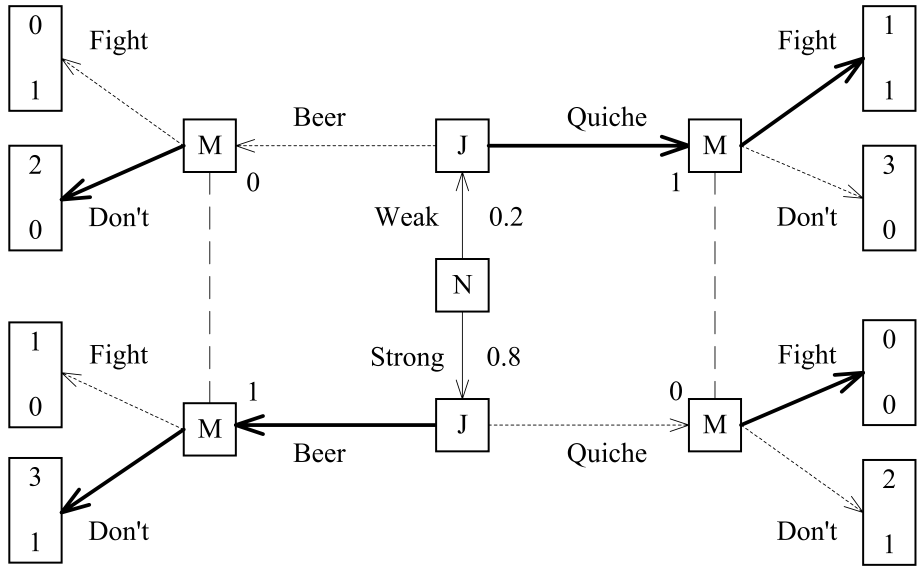
Separating I: Choosing the Wrong Refreshment



- Suppose that Jack chooses beer if he is weak and quiche if he is strong.
- Then Marco's optimal response is to fight beer-drinkers, and don't fight quiche-eaters.
- But then a weak Jack would imitate a strong one and eat quiche to avoid a fight.

⇒ This is not a PBE.

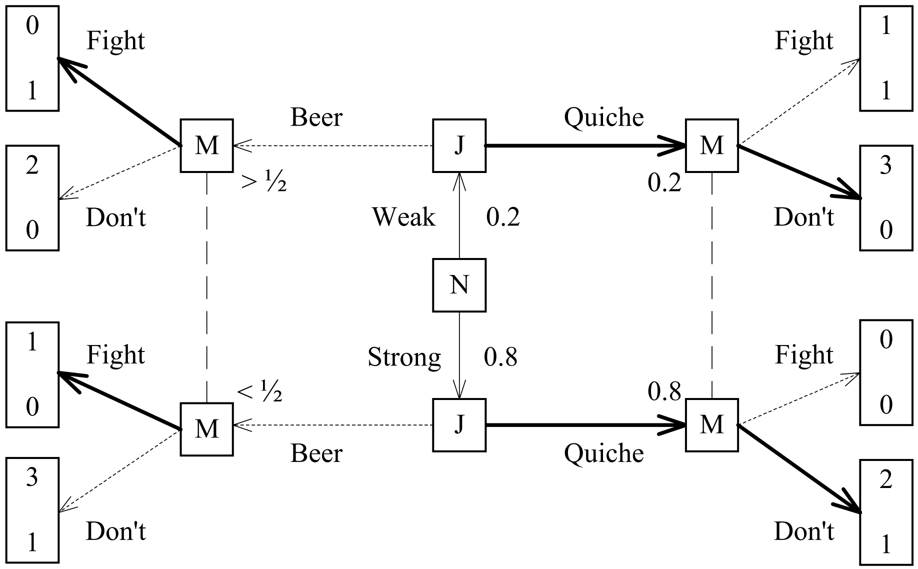
Separating II: Choosing the Right Refreshment



- Suppose that Jack chooses beer if he is strong and quiche if he is weak.
- Then Marco's optimal response is to fight quiche-eaters, and don't fight beer-drinkers.
- But then a weak Jack would imitate a strong one, drinking beer to avoid a fight.

⇒ There is no separating PBE.

Pooling I: Everyone Eats Quiche

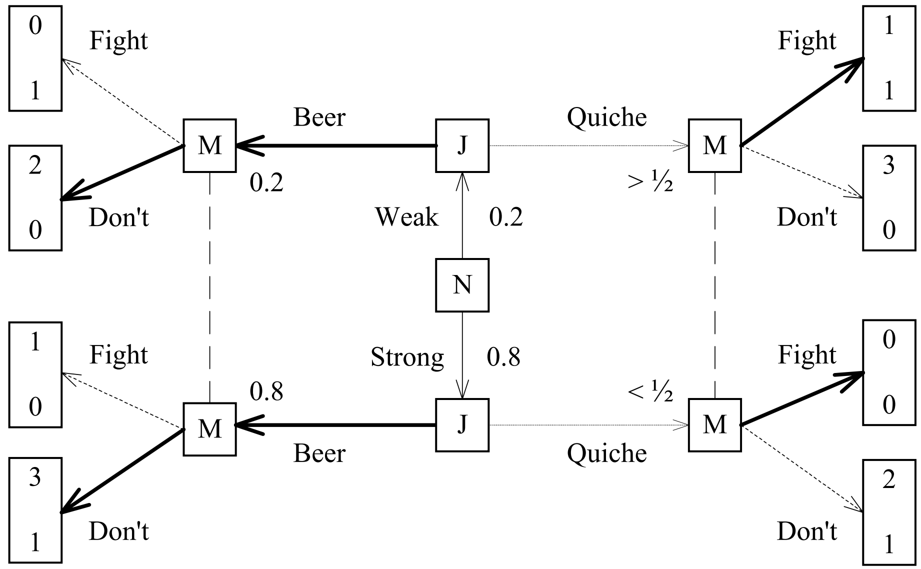


- Suppose Jack always eats quiche.
- Marco must believe with probability $p > \frac{1}{2}$ that Jack is weak when he drinks beer, so that Marco fights him in this case (otherwise a strong Jack prefers to choose beer).

⇒ This is a pooling PBE.

- A strong Jack distorts his choice to hide his type and avoid a fight.

Pooling II: Everyone Drinks Beer

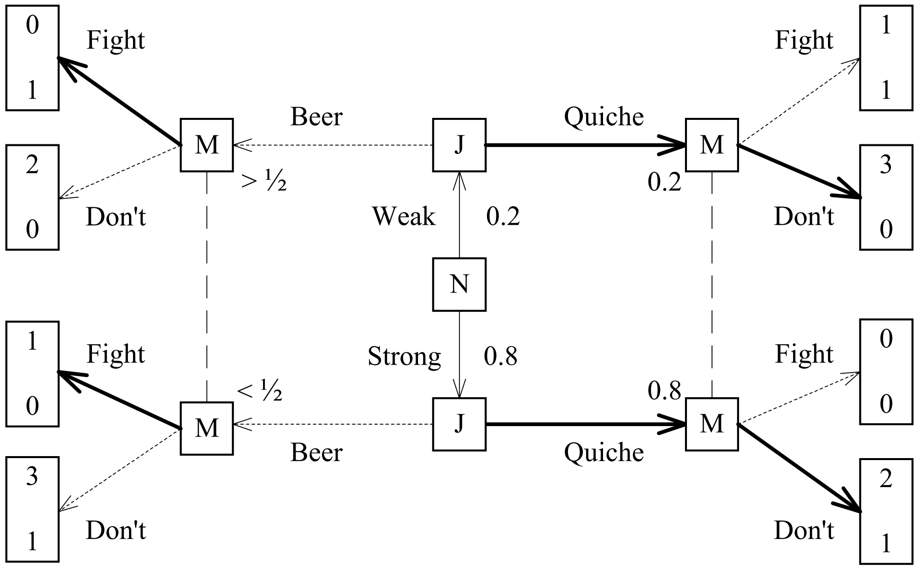


- Suppose Jack always drinks beer.
- Marco must believe with probability $p > \frac{1}{2}$ that Jack is weak when he eats quiche, so that Marco fights him in this case (otherwise a weak Jack prefers to choose quiche).

⇒ This is a pooling PBE.

- A weak Jack distorts his choice to hide his type and avoid a fight.

• But consider the first pooling equilibrium:



• The belief that if Jack drinks beer he is weak with prob. $> \frac{1}{2}$ is not convincing: it is never optimal for a weak Jack to deviate from equilibrium and drink beer.

⇒ If Marco observes Jack drinking beer, he should not fight.

• We eliminate this equilibrium by the *intuitive criterion*.

"I am having beer, which ought to convince you that I am strong. The only benefit to me of having beer comes if I am strong. I would never wish to have beer if I were weak, but if I am strong and this message is convincing, then I benefit from having beer."

Dominated Choices

• **Definition**: An action $a' \in A$ is a *strictly dominated choice* for type θ if there is an action $a'' \in A$ such that:

$$\min_{s\in S} u_\theta\left(a'', s\right) > \max_{s\in S} u_\theta\left(a', s\right).$$

(Alternatively, s can be restricted to belong to $S^*(a)$, the set of the opponents' equilibrium strategies given that type θ chooses a.)

• **Definition**: An action $a' \in A$ is *equilibrium dominated* for type θ in a Perfect Bayesian Equilibrium if:

$$u_\theta^* \geq \max_{s\in S^*(a')} u_\theta\left(a', s\right)$$

where u_θ^* is the equilibrium payoff of type θ.

• After observing an action which is equilibrium dominated for type θ, the intuitive criterion requires player's beliefs to assign zero probability to θ.

• Beer is equilibrium dominated for a weak Jack, since he obtains 3 in equilibrium while playing beer he obtains at most 2. So Marco should not expect him to play it.