From polynomial interpolation to Spline functions

Basis meaning

In Linear Algebra, a basis is a set of vectors satisfying:

- Linear combination of the basis can represent every vector in a given vector space;
- No element of the basis set can be represented as a linear combination of the others.

In Function Space, a basis is a set of ***basis functions such that:***

- Each function in the function space can be represented as a linear combination of the basis functions.

Example: Quadratic Polynomial bases {1,t,t^2}

How to represent by ***basis functions***?

We need flexible method for constructing a function *f(t)* that can track local curvature.

Fixed a system of *K* *basis* functions $\varphi_i(t)$, $i=1,\ldots,k$, and call this the *basis* for *f(t)*:

f(t) can be express as a weighted sum of these basis functions:

$$f(t) = a_1\varphi_1(t) + a_2\varphi_2(t) + \ldots + a_K\varphi_K(t)$$

The coefficients $a_1, \ldots, a_K$ determine the shape of the function.

What do we want from basis functions?

- Fast computation of individual basis functions.
- Flexible: can exhibit the required curvature where needed, but also be nearly linear when appropriate.
- Fast computation of coefficients a_k: possible if matrices of values are diagonal, banded or sparse.
- Differentiable as required: We make lots of use of derivatives in functional data analysis.
- Constrained as required, such as periodicity, positivity, monotonicity, asymptotes and etc.

What are some commonly used basis functions?

- Powers: *1, t,* t^2, and so on. They are the basis functions for polynomials. These are not very flexible, and are used only for simple problems.
- Fourier series: *1, sin(ωt), cos(ωt), sin(2ωt), cos(2ωt),* and so on for a fixed known frequency *ω*. These are used for periodic functions.
- Spline functions: particular piecewise polynomials with suitable regularization properties.

Higher order polynomial interpolation could be a bad idea

Example

$$f(x) = \frac{1}{1+25x^2}$$

Table : Six equidistantly spaced points in [-1, 1]

x	$y = \frac{1}{1+25x^2}$
-1.0	0.038461
-0.6	0.1
-0.2	0.5
0.2	0.5
0.6	0.1
1.0	0.038461

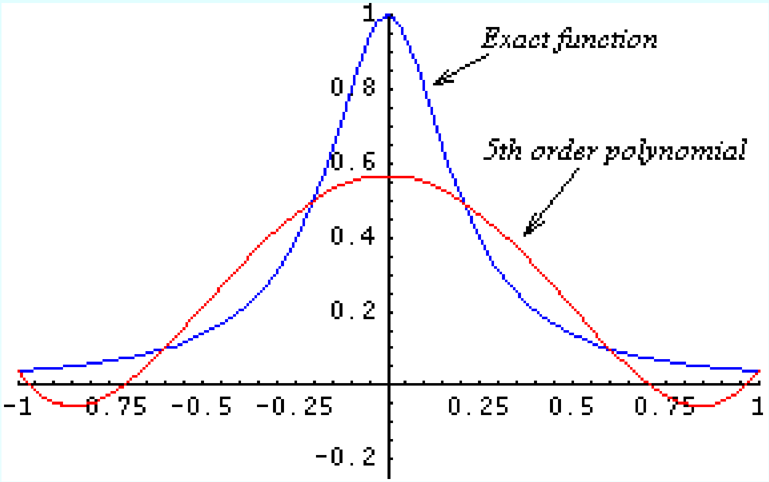


Figure : 5th order polynomial vs. exact function

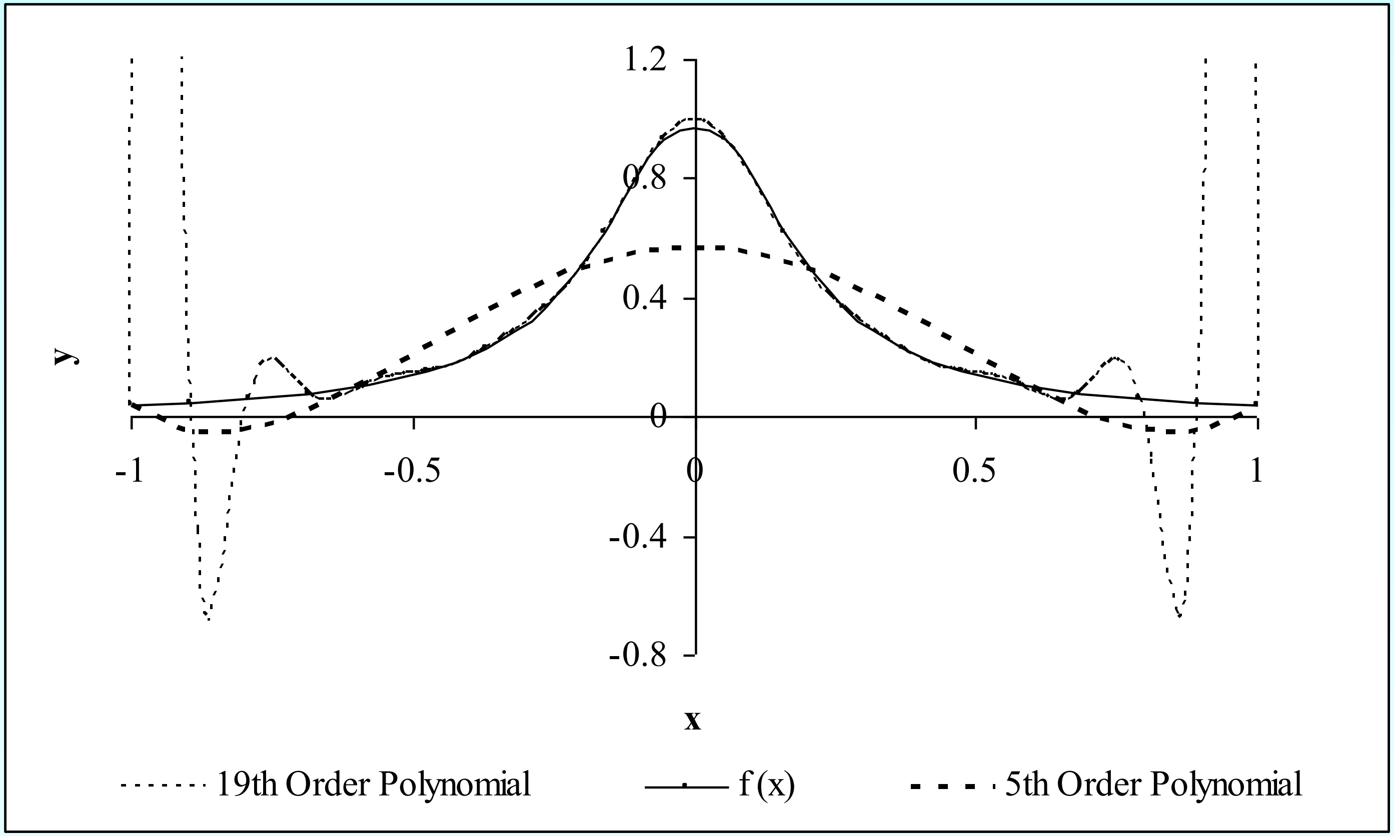


Figure : Higher order polynomial interpolation is a bad idea

Piecewise interpolation

Polynomials are the most common choice of interpolants because they are easy to:

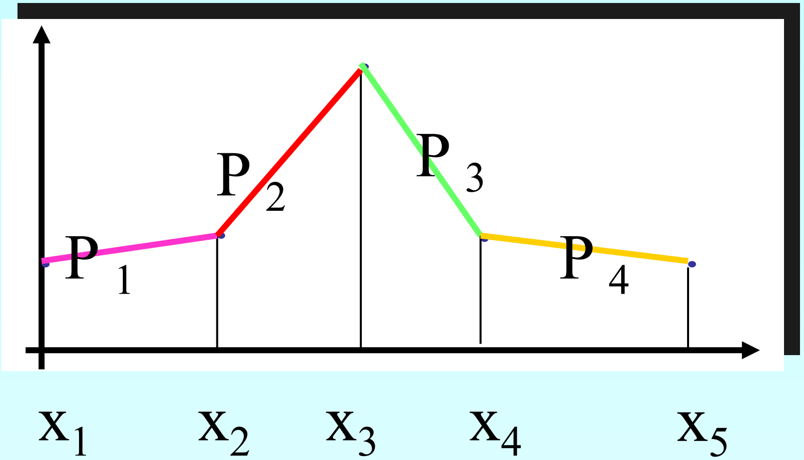
- Evaluate
- Differentiate
- Integrate.

The main drawback is:

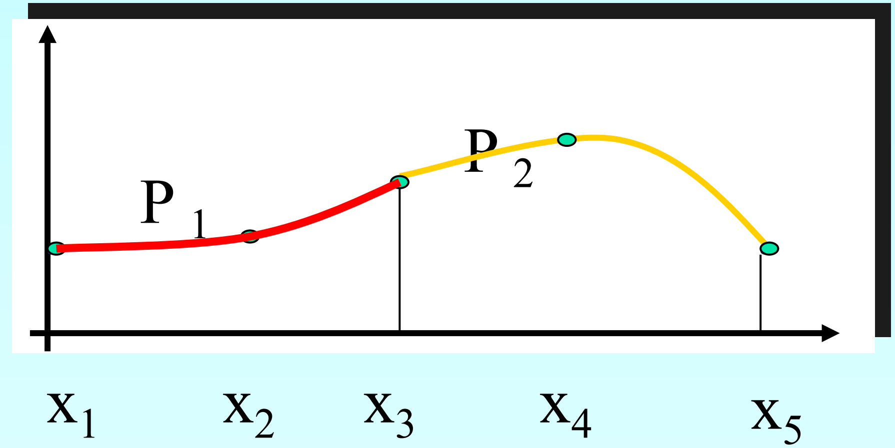
as the number of nodes increases, the oscillations increase

Idea:

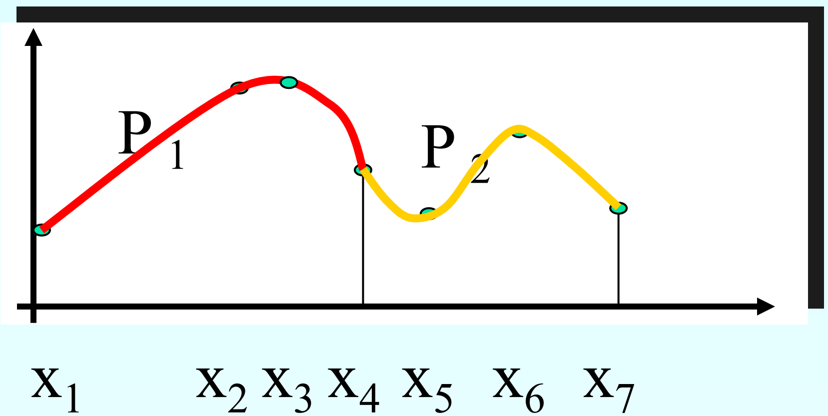
To reduce the degree of the interpolating polynomial built on all the nodes, several ***low-level interpolating polynomials are constructed relative to contiguous subsets of the set of nodes***



First-degree piecewise interpolating polynomial built on 5 knots



2nd-degree piecewise interpolating polynomial built on 5 knots



3°-degree piecewise interpolating polynomial built on 7 knots

Drawback :

the piecewise interpolating functions do not present oscillations

BUT : They may have irregularities in the nodes

From Piecewise Polynomials to Splines

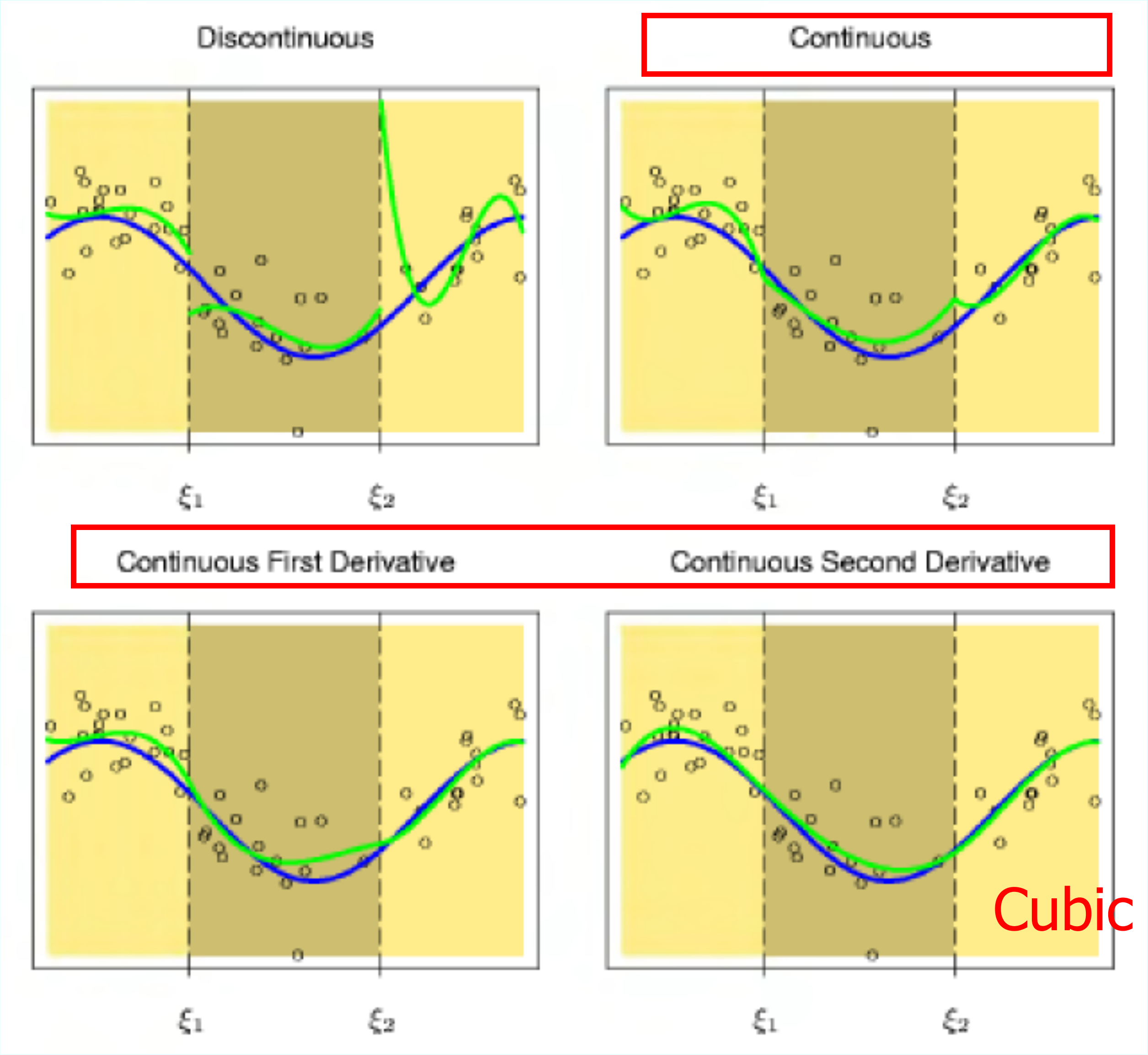
Spline:

- In Mathematics, a spline is a ***special function*** defined piecewise by polynomials, with a suitable number of continuous derivatives in the internal points;
- In Computer Science, the term spline more frequently refers to a piecewise polynomial (parametric) curve.

Advantages:

Simple construction, ease and accuracy of evaluation, capacity to approximate complex shapes through curve fitting and interactive curve design.

Piecewise Cubic Polynomial



Cubic spline