

Matrici a gradini

Definizione. Sia $A \in \mathbb{R}_{m,n}$ una matrice e sia $\mathbf{a}_i$ una sua riga non nulla. Il primo termine non nullo di $\mathbf{a}_i$ si dice **pivot** di $\mathbf{a}_i$.

Più precisamente:

a_{ij} è il pivot di $\mathbf{a}_i \Leftrightarrow a_{ij} \neq 0$ e $a_{ik} = 0 \ \forall k < j$.

Una matrice si dice **a gradini** (o **a scalini**) se valgono i seguenti due fatti:

- il pivot di ogni riga non nulla è più a sinistra dei pivot delle righe non nulle successive;
- le righe nulle sono tutte alla fine.

Operazioni elementari

- Scambio di due righe:

$$\mathbf{a}_i \leftrightarrow \mathbf{a}_j$$

- Moltiplicazione di una riga per uno scalare non nullo:

$$h\mathbf{a}_i \rightarrow \mathbf{a}_i$$

- Sostituzione di una riga con la somma della riga stessa con un'altra:

$$\mathbf{a}_i + \mathbf{a}_j \rightarrow \mathbf{a}_i$$

Definizione. Due matrici si dicono
equivalenti

se si possono ottenere l'una dall'altra con una
sequenza di trasformazioni elementari.

Osservazione. L'operazione

$$h\mathbf{a}_i + k\mathbf{a}_j \rightarrow \mathbf{a}_i, \quad \text{con } \underline{h \neq 0}, \underline{i \neq j},$$

pur non essendo elementare, trasforma matrici in matrici equivalenti, perché equivale alla sequenza di operazioni elementari:

$$k\mathbf{a}_j \rightarrow \mathbf{a}_j, \quad h\mathbf{a}_i \rightarrow \mathbf{a}_i, \quad \mathbf{a}_i + \mathbf{a}_j \rightarrow \mathbf{a}_i, \quad \frac{1}{k}\mathbf{a}_j \rightarrow \mathbf{a}_j.$$

Procedimento di riduzione a scalini

- Si individui una riga non nulla che abbia il pivot più a sinistra possibile.
- Se la riga trovata non è la prima, si scambi la riga trovata con la prima riga.
- Detto a_{1j} il pivot della prima riga, si scelgano due numeri h e k , con $h \neq 0$, tali che

$$\frac{k}{h} = \frac{a_{2j}}{a_{1j}}.$$

Quindi si effettui la trasformazione:

$$\underline{a_2} \longrightarrow h\underline{a_2} - k\underline{a_1}.$$

- Si proceda ancora allo stesso modo per tutte le righe successive.
- Si ripeta tutto dall'inizio, applicandolo però solo alla matrice ottenuta “dimenticando” la prima riga.

$$\begin{pmatrix} 1 & -2 & 1 & 4 \\ 3 & 1 & -4 & -2 \\ 1 & -9 & 8 & 18 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -2 & 1 & 4 \\ 3 & 1 & -4 & -2 \\ 1 & -9 & 8 & 18 \end{pmatrix} \quad \mathbf{a}_2 - 3\mathbf{a}_1 \rightarrow \mathbf{a}_2$$

$$\left(\begin{array}{cccc} 1 & -2 & 1 & 4 \\ 3 & 1 & -4 & -2 \\ 1 & -9 & 8 & 18 \end{array} \right) \quad \begin{array}{l} \mathbf{a}_2 - 3\mathbf{a}_1 \rightarrow \mathbf{a}_2 \\ \mathbf{a}_3 - \mathbf{a}_1 \rightarrow \mathbf{a}_3 \end{array}$$

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$$\mathbf{a}_3 + \mathbf{a}_2 \rightarrow \mathbf{a}_3$$

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$$\left(\begin{array}{ccccc} 1 & 1 & 1 & 1 & 4 \\ 0 & 1 & 1 & 2 & 4 \\ 0 & -2 & -2 & -6 & \end{array} \right)$$

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$$\left(\begin{array}{ccccc} 1 & 1 & 1 & 1 & 4 \\ 0 & 1 & 1 & 2 & 4 \\ 0 & -2 & -2 & -6 & -10 \end{array} \right)$$

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$$\left(\begin{array}{ccccc} 1 & 1 & 1 & 1 & 4 \\ 0 & 1 & 1 & 2 & 4 \\ 0 & -2 & -2 & -6 & -10 \\ 0 & 1 & 0 & -1 & \end{array} \right)$$

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$$\left(\begin{array}{ccccc} 1 & 1 & 1 & 1 & 4 \\ 0 & 1 & 1 & 2 & 4 \\ 0 & -2 & -2 & -6 & -10 \\ 0 & 1 & 0 & -1 & 0 \end{array}\right) \quad \begin{array}{l} \mathbf{a}_3 + 2\mathbf{a}_2 \rightarrow \mathbf{a}_3 \\ \mathbf{a}_4 - \mathbf{a}_2 \rightarrow \mathbf{a}_4 \end{array}$$

$$\left(\begin{array}{ccccc} 1 & 1 & 1 & 1 & 4 \\ 0 & 1 & & & \end{array}\right)$$

$$\left(\begin{array}{ccccc} 1 & 1 & 1 & 1 & 4 \\ 1 & 2 & 2 & 3 & 8 \\ 2 & 0 & 0 & -4 & -2 \\ 1 & 2 & 1 & 0 & 4 \end{array} \right) \quad \begin{array}{l} \mathbf{a}_2 - \mathbf{a}_1 \rightarrow \mathbf{a}_2 \\ \mathbf{a}_3 - 2\mathbf{a}_1 \rightarrow \mathbf{a}_3 \\ \mathbf{a}_4 - \mathbf{a}_1 \rightarrow \mathbf{a}_4 \end{array}$$

$$\left(\begin{array}{ccccc} 1 & 1 & 1 & 1 & 4 \\ 0 & 1 & 1 & 2 & 4 \\ 0 & -2 & -2 & -6 & -10 \\ 0 & 1 & 0 & -1 & 0 \end{array} \right) \quad \begin{array}{l} \mathbf{a}_3 + 2\mathbf{a}_2 \rightarrow \mathbf{a}_3 \\ \mathbf{a}_4 - \mathbf{a}_2 \rightarrow \mathbf{a}_4 \end{array}$$

$$\left(\begin{array}{ccccc} 1 & 1 & 1 & 1 & 4 \\ 0 & 1 & 1 & & \end{array} \right)$$

$$\left(\begin{array}{ccccc} 1 & 1 & 1 & 1 & 4 \\ 1 & 2 & 2 & 3 & 8 \\ 2 & 0 & 0 & -4 & -2 \\ 1 & 2 & 1 & 0 & 4 \end{array} \right) \quad \begin{array}{l} \mathbf{a}_2 - \mathbf{a}_1 \rightarrow \mathbf{a}_2 \\ \mathbf{a}_3 - 2\mathbf{a}_1 \rightarrow \mathbf{a}_3 \\ \mathbf{a}_4 - \mathbf{a}_1 \rightarrow \mathbf{a}_4 \end{array}$$

$$\left(\begin{array}{ccccc} 1 & 1 & 1 & 1 & 4 \\ 0 & 1 & 1 & 2 & 4 \\ 0 & -2 & -2 & -6 & -10 \\ 0 & 1 & 0 & -1 & 0 \end{array} \right) \quad \begin{array}{l} \mathbf{a}_3 + 2\mathbf{a}_2 \rightarrow \mathbf{a}_3 \\ \mathbf{a}_4 - \mathbf{a}_2 \rightarrow \mathbf{a}_4 \end{array}$$

$$\left(\begin{array}{ccccc} 1 & 1 & 1 & 1 & 4 \\ 0 & 1 & 1 & 2 & \end{array} \right)$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 4 \\ 1 & 2 & 2 & 3 & 8 \\ 2 & 0 & 0 & -4 & -2 \\ 1 & 2 & 1 & 0 & 4 \end{pmatrix} \quad \begin{array}{l} \mathbf{a}_2 - \mathbf{a}_1 \rightarrow \mathbf{a}_2 \\ \mathbf{a}_3 - 2\mathbf{a}_1 \rightarrow \mathbf{a}_3 \\ \mathbf{a}_4 - \mathbf{a}_1 \rightarrow \mathbf{a}_4 \end{array}$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 4 \\ 0 & 1 & 1 & 2 & 4 \\ 0 & -2 & -2 & -6 & -10 \\ 0 & 1 & 0 & -1 & 0 \end{pmatrix} \quad \begin{array}{l} \mathbf{a}_3 + 2\mathbf{a}_2 \rightarrow \mathbf{a}_3 \\ \mathbf{a}_4 - \mathbf{a}_2 \rightarrow \mathbf{a}_4 \end{array}$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 4 \\ 0 & 1 & 1 & 2 & 4 \end{pmatrix}$$

$$\left(\begin{array}{ccccc} 1 & 1 & 1 & 1 & 4 \\ 1 & 2 & 2 & 3 & 8 \\ 2 & 0 & 0 & -4 & -2 \\ 1 & 2 & 1 & 0 & 4 \end{array} \right) \begin{array}{l} \mathbf{a}_2 - \mathbf{a}_1 \rightarrow \mathbf{a}_2 \\ \mathbf{a}_3 - 2\mathbf{a}_1 \rightarrow \mathbf{a}_3 \\ \mathbf{a}_4 - \mathbf{a}_1 \rightarrow \mathbf{a}_4 \end{array}$$

$$\left(\begin{array}{ccccc} 1 & 1 & 1 & 1 & 4 \\ 0 & 1 & 1 & 2 & 4 \\ 0 & -2 & -2 & -6 & -10 \\ 0 & 1 & 0 & -1 & 0 \end{array} \right) \begin{array}{l} \mathbf{a}_3 + 2\mathbf{a}_2 \rightarrow \mathbf{a}_3 \\ \mathbf{a}_4 - \mathbf{a}_2 \rightarrow \mathbf{a}_4 \end{array}$$

$$\left(\begin{array}{ccccc} 1 & 1 & 1 & 1 & 4 \\ 0 & 1 & 1 & 2 & 4 \\ 0 & & & & \end{array} \right)$$

$$\left(\begin{array}{ccccc} 1 & 1 & 1 & 1 & 4 \\ 1 & 2 & 2 & 3 & 8 \\ 2 & 0 & 0 & -4 & -2 \\ 1 & 2 & 1 & 0 & 4 \end{array} \right) \quad \begin{array}{l} \mathbf{a}_2 - \mathbf{a}_1 \rightarrow \mathbf{a}_2 \\ \mathbf{a}_3 - 2\mathbf{a}_1 \rightarrow \mathbf{a}_3 \\ \mathbf{a}_4 - \mathbf{a}_1 \rightarrow \mathbf{a}_4 \end{array}$$

$$\left(\begin{array}{ccccc} 1 & 1 & 1 & 1 & 4 \\ 0 & 1 & 1 & 2 & 4 \\ 0 & -2 & -2 & -6 & -10 \\ 0 & 1 & 0 & -1 & 0 \end{array} \right) \quad \begin{array}{l} \mathbf{a}_3 + 2\mathbf{a}_2 \rightarrow \mathbf{a}_3 \\ \mathbf{a}_4 - \mathbf{a}_2 \rightarrow \mathbf{a}_4 \end{array}$$

$$\left(\begin{array}{ccccc} 1 & 1 & 1 & 1 & 4 \\ 0 & 1 & 1 & 2 & 4 \\ 0 & 0 & & & \end{array} \right)$$

$$\left(\begin{array}{ccccc} 1 & 1 & 1 & 1 & 4 \\ 1 & 2 & 2 & 3 & 8 \\ 2 & 0 & 0 & -4 & -2 \\ 1 & 2 & 1 & 0 & 4 \end{array}\right) \begin{array}{l} \mathbf{a}_2 - \mathbf{a}_1 \rightarrow \mathbf{a}_2 \\ \mathbf{a}_3 - 2\mathbf{a}_1 \rightarrow \mathbf{a}_3 \\ \mathbf{a}_4 - \mathbf{a}_1 \rightarrow \mathbf{a}_4 \end{array}$$

$$\left(\begin{array}{ccccc} 1 & 1 & 1 & 1 & 4 \\ 0 & 1 & 1 & 2 & 4 \\ 0 & -2 & -2 & -6 & -10 \\ 0 & 1 & 0 & -1 & 0 \end{array}\right) \begin{array}{l} \mathbf{a}_3 + 2\mathbf{a}_2 \rightarrow \mathbf{a}_3 \\ \mathbf{a}_4 - \mathbf{a}_2 \rightarrow \mathbf{a}_4 \end{array}$$

$$\left(\begin{array}{ccccc} 1 & 1 & 1 & 1 & 4 \\ 0 & 1 & 1 & 2 & 4 \\ 0 & 0 & 0 & & \end{array}\right)$$

$$\left(\begin{array}{ccccc} 1 & 1 & 1 & 1 & 4 \\ 1 & 2 & 2 & 3 & 8 \\ 2 & 0 & 0 & -4 & -2 \\ 1 & 2 & 1 & 0 & 4 \end{array} \right) \quad \begin{array}{l} \mathbf{a}_2 - \mathbf{a}_1 \rightarrow \mathbf{a}_2 \\ \mathbf{a}_3 - 2\mathbf{a}_1 \rightarrow \mathbf{a}_3 \\ \mathbf{a}_4 - \mathbf{a}_1 \rightarrow \mathbf{a}_4 \end{array}$$

$$\left(\begin{array}{ccccc} 1 & 1 & 1 & 1 & 4 \\ 0 & 1 & 1 & 2 & 4 \\ 0 & -2 & -2 & -6 & -10 \\ 0 & 1 & 0 & -1 & 0 \end{array} \right) \quad \begin{array}{l} \mathbf{a}_3 + 2\mathbf{a}_2 \rightarrow \mathbf{a}_3 \\ \mathbf{a}_4 - \mathbf{a}_2 \rightarrow \mathbf{a}_4 \end{array}$$

$$\left(\begin{array}{ccccc} 1 & 1 & 1 & 1 & 4 \\ 0 & 1 & 1 & 2 & 4 \\ 0 & 0 & 0 & -2 & \end{array} \right)$$

$$\left(\begin{array}{ccccc} 1 & 1 & 1 & 1 & 4 \\ 1 & 2 & 2 & 3 & 8 \\ 2 & 0 & 0 & -4 & -2 \\ 1 & 2 & 1 & 0 & 4 \end{array} \right) \quad \begin{array}{l} \mathbf{a}_2 - \mathbf{a}_1 \rightarrow \mathbf{a}_2 \\ \mathbf{a}_3 - 2\mathbf{a}_1 \rightarrow \mathbf{a}_3 \\ \mathbf{a}_4 - \mathbf{a}_1 \rightarrow \mathbf{a}_4 \end{array}$$

$$\left(\begin{array}{ccccc} 1 & 1 & 1 & 1 & 4 \\ 0 & 1 & 1 & 2 & 4 \\ 0 & -2 & -2 & -6 & -10 \\ 0 & 1 & 0 & -1 & 0 \end{array} \right) \quad \begin{array}{l} \mathbf{a}_3 + 2\mathbf{a}_2 \rightarrow \mathbf{a}_3 \\ \mathbf{a}_4 - \mathbf{a}_2 \rightarrow \mathbf{a}_4 \end{array}$$

$$\left(\begin{array}{ccccc} 1 & 1 & 1 & 1 & 4 \\ 0 & 1 & 1 & 2 & 4 \\ 0 & 0 & 0 & -2 & -2 \end{array} \right)$$

$$\left(\begin{array}{ccccc} 1 & 1 & 1 & 1 & 4 \\ 1 & 2 & 2 & 3 & 8 \\ 2 & 0 & 0 & -4 & -2 \\ 1 & 2 & 1 & 0 & 4 \end{array} \right) \quad \begin{array}{l} \mathbf{a}_2 - \mathbf{a}_1 \rightarrow \mathbf{a}_2 \\ \mathbf{a}_3 - 2\mathbf{a}_1 \rightarrow \mathbf{a}_3 \\ \mathbf{a}_4 - \mathbf{a}_1 \rightarrow \mathbf{a}_4 \end{array}$$

$$\left(\begin{array}{ccccc} 1 & 1 & 1 & 1 & 4 \\ 0 & 1 & 1 & 2 & 4 \\ 0 & -2 & -2 & -6 & -10 \\ 0 & 1 & 0 & -1 & 0 \end{array} \right) \quad \begin{array}{l} \mathbf{a}_3 + 2\mathbf{a}_2 \rightarrow \mathbf{a}_3 \\ \mathbf{a}_4 - \mathbf{a}_2 \rightarrow \mathbf{a}_4 \end{array}$$

$$\left(\begin{array}{ccccc} 1 & 1 & 1 & 1 & 4 \\ 0 & 1 & 1 & 2 & 4 \\ 0 & 0 & 0 & -2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right)$$

$$\left(\begin{array}{ccccc} 1 & 1 & 1 & 1 & 4 \\ 1 & 2 & 2 & 3 & 8 \\ 2 & 0 & 0 & -4 & -2 \\ 1 & 2 & 1 & 0 & 4 \end{array} \right) \quad \begin{array}{l} \mathbf{a}_2 - \mathbf{a}_1 \rightarrow \mathbf{a}_2 \\ \mathbf{a}_3 - 2\mathbf{a}_1 \rightarrow \mathbf{a}_3 \\ \mathbf{a}_4 - \mathbf{a}_1 \rightarrow \mathbf{a}_4 \end{array}$$

$$\left(\begin{array}{ccccc} 1 & 1 & 1 & 1 & 4 \\ 0 & 1 & 1 & 2 & 4 \\ 0 & -2 & -2 & -6 & -10 \\ 0 & 1 & 0 & -1 & 0 \end{array} \right) \quad \begin{array}{l} \mathbf{a}_3 + 2\mathbf{a}_2 \rightarrow \mathbf{a}_3 \\ \mathbf{a}_4 - \mathbf{a}_2 \rightarrow \mathbf{a}_4 \end{array}$$

$$\left(\begin{array}{ccccc} 1 & 1 & 1 & 1 & 4 \\ 0 & 1 & 1 & 2 & 4 \\ 0 & 0 & 0 & -2 & -2 \\ 0 & 0 & & & \end{array} \right)$$

$$\left(\begin{array}{ccccc} 1 & 1 & 1 & 1 & 4 \\ 1 & 2 & 2 & 3 & 8 \\ 2 & 0 & 0 & -4 & -2 \\ 1 & 2 & 1 & 0 & 4 \end{array} \right) \begin{array}{l} \mathbf{a}_2 - \mathbf{a}_1 \rightarrow \mathbf{a}_2 \\ \mathbf{a}_3 - 2\mathbf{a}_1 \rightarrow \mathbf{a}_3 \\ \mathbf{a}_4 - \mathbf{a}_1 \rightarrow \mathbf{a}_4 \end{array}$$

$$\left(\begin{array}{ccccc} 1 & 1 & 1 & 1 & 4 \\ 0 & 1 & 1 & 2 & 4 \\ 0 & -2 & -2 & -6 & -10 \\ 0 & 1 & 0 & -1 & 0 \end{array} \right) \begin{array}{l} \mathbf{a}_3 + 2\mathbf{a}_2 \rightarrow \mathbf{a}_3 \\ \mathbf{a}_4 - \mathbf{a}_2 \rightarrow \mathbf{a}_4 \end{array}$$

$$\left(\begin{array}{ccccc} 1 & 1 & 1 & 1 & 4 \\ 0 & 1 & 1 & 2 & 4 \\ 0 & 0 & 0 & -2 & -2 \\ 0 & 0 & -1 & & \end{array} \right)$$

$$\left(\begin{array}{ccccc} 1 & 1 & 1 & 1 & 4 \\ 1 & 2 & 2 & 3 & 8 \\ 2 & 0 & 0 & -4 & -2 \\ 1 & 2 & 1 & 0 & 4 \end{array} \right) \quad \begin{array}{l} \mathbf{a}_2 - \mathbf{a}_1 \rightarrow \mathbf{a}_2 \\ \mathbf{a}_3 - 2\mathbf{a}_1 \rightarrow \mathbf{a}_3 \\ \mathbf{a}_4 - \mathbf{a}_1 \rightarrow \mathbf{a}_4 \end{array}$$

$$\left(\begin{array}{ccccc} 1 & 1 & 1 & 1 & 4 \\ 0 & 1 & 1 & 2 & 4 \\ 0 & -2 & -2 & -6 & -10 \\ 0 & 1 & 0 & -1 & 0 \end{array} \right) \quad \begin{array}{l} \mathbf{a}_3 + 2\mathbf{a}_2 \rightarrow \mathbf{a}_3 \\ \mathbf{a}_4 - \mathbf{a}_2 \rightarrow \mathbf{a}_4 \end{array}$$

$$\left(\begin{array}{ccccc} 1 & 1 & 1 & 1 & 4 \\ 0 & 1 & 1 & 2 & 4 \\ 0 & 0 & 0 & -2 & -2 \\ 0 & 0 & -1 & -3 & \end{array} \right)$$

$$\left(\begin{array}{ccccc} 1 & 1 & 1 & 1 & 4 \\ 1 & 2 & 2 & 3 & 8 \\ 2 & 0 & 0 & -4 & -2 \\ 1 & 2 & 1 & 0 & 4 \end{array} \right) \quad \begin{array}{l} \mathbf{a}_2 - \mathbf{a}_1 \rightarrow \mathbf{a}_2 \\ \mathbf{a}_3 - 2\mathbf{a}_1 \rightarrow \mathbf{a}_3 \\ \mathbf{a}_4 - \mathbf{a}_1 \rightarrow \mathbf{a}_4 \end{array}$$

$$\left(\begin{array}{ccccc} 1 & 1 & 1 & 1 & 4 \\ 0 & 1 & 1 & 2 & 4 \\ 0 & -2 & -2 & -6 & -10 \\ 0 & 1 & 0 & -1 & 0 \end{array} \right) \quad \begin{array}{l} \mathbf{a}_3 + 2\mathbf{a}_2 \rightarrow \mathbf{a}_3 \\ \mathbf{a}_4 - \mathbf{a}_2 \rightarrow \mathbf{a}_4 \end{array}$$

$$\left(\begin{array}{ccccc} 1 & 1 & 1 & 1 & 4 \\ 0 & 1 & 1 & 2 & 4 \\ 0 & 0 & 0 & -2 & -2 \\ 0 & 0 & -1 & -3 & -4 \end{array} \right)$$

$$\left(\begin{array}{ccccc} 1 & 1 & 1 & 1 & 4 \\ 1 & 2 & 2 & 3 & 8 \\ 2 & 0 & 0 & -4 & -2 \\ 1 & 2 & 1 & 0 & 4 \end{array} \right) \begin{array}{l} \mathbf{a}_2 - \mathbf{a}_1 \rightarrow \mathbf{a}_2 \\ \mathbf{a}_3 - 2\mathbf{a}_1 \rightarrow \mathbf{a}_3 \\ \mathbf{a}_4 - \mathbf{a}_1 \rightarrow \mathbf{a}_4 \end{array}$$

$$\left(\begin{array}{ccccc} 1 & 1 & 1 & 1 & 4 \\ 0 & 1 & 1 & 2 & 4 \\ 0 & -2 & -2 & -6 & -10 \\ 0 & 1 & 0 & -1 & 0 \end{array} \right) \begin{array}{l} \mathbf{a}_3 + 2\mathbf{a}_2 \rightarrow \mathbf{a}_3 \\ \mathbf{a}_4 - \mathbf{a}_2 \rightarrow \mathbf{a}_4 \end{array}$$

$$\left(\begin{array}{ccccc} 1 & 1 & 1 & 1 & 4 \\ 0 & 1 & 1 & 2 & 4 \\ 0 & 0 & 0 & -2 & -2 \\ 0 & 0 & -1 & -3 & -4 \end{array} \right) \mathbf{a}_3 \leftrightarrow \mathbf{a}_4$$

$$\left(\begin{array}{ccccc} 1 & 1 & 1 & 1 & 4 \\ 0 & 1 & 1 & 2 & 4 \\ 0 & -2 & -2 & -6 & -10 \\ 0 & 1 & 0 & -1 & 0 \end{array} \right) \begin{array}{l} \mathbf{a}_3 + 2\mathbf{a}_2 \rightarrow \mathbf{a}_3 \\ \mathbf{a}_4 - \mathbf{a}_2 \rightarrow \mathbf{a}_4 \end{array}$$

$$\left(\begin{array}{ccccc} 1 & 1 & 1 & 1 & 4 \\ 0 & 1 & 1 & 2 & 4 \\ 0 & 0 & 0 & -2 & -2 \\ 0 & 0 & -1 & -3 & -4 \end{array} \right) \mathbf{a}_3 \leftrightarrow \mathbf{a}_4$$

$$\left(\begin{array}{ccccc} 1 & 1 & 1 & 1 & 4 \\ 0 & 1 & 1 & 2 & 4 \\ 0 & 0 & -1 & -3 & -4 \\ 0 & 0 & 0 & -2 & -2 \end{array} \right)$$